Fractals, Multifractals and Subadditive Thermodynamic Formalism

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Thermodynamic Formalism: Non-additive Aspects and Related Topics

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Thermodynamic formalism

Applies notions from classical thermodynamics to dynamical systems: Analogues of entropy, pressure, inverse temperatures, Gibbs measures, variational principle, bounded distortion, ...

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Map $f : D \to D$, potential $\psi : D \to \mathbb{R}$. Let

$$\psi_n(x) = \psi(x) + \psi(fx) + \cdots + \psi(f^{n-1}x)$$

Note: $\psi_{n+m}(x) = \psi_n(x) + \psi_m(f^n x)$.
Define the pressure

$$P(f, \psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{fix} f^n} \exp \psi_n(x).$$

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Get Bowen's formula for a repeller E of an expanding conformal f:

$$\dim_H E = s \quad \text{where} \quad P(-s \log |f'|) = 0.$$

- Bowen (1979) Hausdorff dimension of quasicircles
- Ruelle (1982) Repellers for real analytic maps

Sub-additive thermodynamic formalism

Stems from an attempt to extend these ideas to affine and general non-conformal differentiable maps.

Map $f : D \to D$, a family of potential functions $\psi_n : D \to \mathbb{R}$ such that

$$\psi_{n+m}(x) \leq \psi_n(x) + \psi_m(f^n x).$$

Define the sub-additive pressure as before:

$$P(f, \{\psi_n\}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{fix} f^n} \exp \psi_n(x).$$

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Many notions carry over to the sub-additive setting.

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A subadditive thermodynamic formalism for mixing repellers

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Received 27 April 1988

Iterated function systems

A family $S_1, \dots, S_m$ of contractions on $D \subseteq \mathbb{R}^N$, i.e.

$$|S_i(x) - S_i(y)| \leq c_i |x - y| \quad x, y \in D, \quad c_i < 1$$

is called an iterated function system (IFS).

Given an IFS there exists a unique, non-empty compact set E satisfying

$$E = \bigcup_{i=1}^{m} S_i(E),$$

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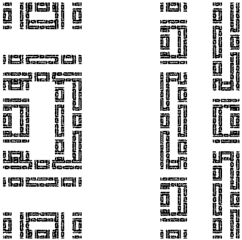
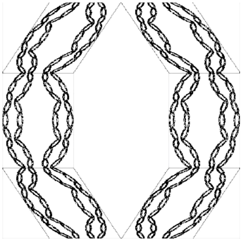
If the S_i are similarities E is called a self-similar set.

If the S_i are conformal maps E is called a self-conformal set.

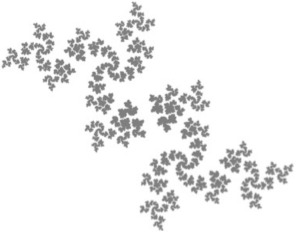
If the $S_i = T_i + \omega_i$ are affine contractions on $\mathbb{R}^N$, where the T_i are non-singular contracting linear mappings on $\mathbb{R}^N$ and $\omega_i \in \mathbb{R}^N$ are translation vectors, E is a self-affine set.



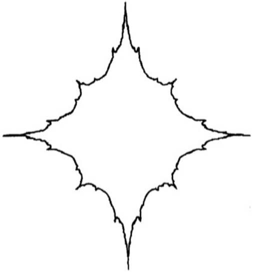
self-similar



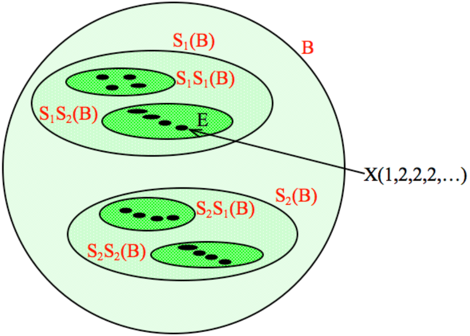
self-affine



self-conformal



nonlinear, nonconformal



Taking a (large) initial domain B we get an iterated construction of E:

$$E = \bigcap_{k=0}^{\infty} \bigcup_{i_1,\ldots,i_k} S_{i_1} \circ \cdots \circ S_{i_k}(B).$$

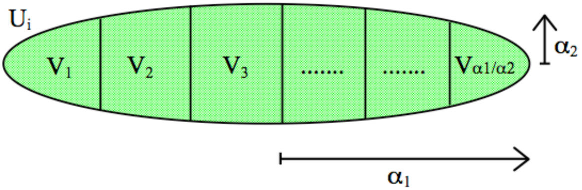
We also get a coding of points of E: if $\mathbf{i} = (i_1, i_2, \ldots) \in \{1, 2, \ldots, m\}^{\mathbb{N}}$, let $x(\mathbf{i}) \equiv x(i_1, i_2, \ldots) = \lim_{k\to\infty} S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}(0)$.

Then

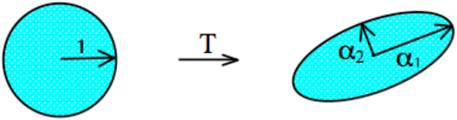
$$E = \bigcup_{i_1,i_2,\ldots} x(i_1, i_2, \ldots).$$

To find the Hausdorff dimension of E we need to consider sums $\sum |U_i|^s$ where $\{U_i\}$ is a cover of E.
Suppose some covering set U_i is 'long and thin', e.g. an ellipse with semi-axes $\alpha_1 \geq \alpha_2$.
The contribution to $\sum |U_i|^s$ from U_i is $\approx \alpha_1^s$.
OR we can cut U_i into α_1/α_2 pieces $\{V_j\}_{j=1}^{\alpha_1/\alpha_2}$ that are roughly square with side α_2, to replace $|U_i|^s$ by

$$\sum_{j=1}^{\alpha_1/\alpha_2} |V_j|^s \approx \frac{\alpha_1}{\alpha_2}\alpha_2^s = \alpha_1\alpha_2^{s-1} \quad \text{which is } \ll \alpha_1^s \text{ if } s > 1.$$



Thus we define the singular values $\alpha_1 \geq \alpha_2 \geq 0$ of a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the semi-axis lengths of T(unit ball):



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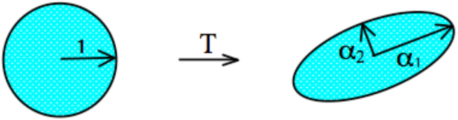

Equivalently α_i are the +ve square roots of the eigenvalues of TT^*. We define the singular value function of T by

$$\phi^s(T) = \begin{cases} \alpha_1^s & (0 \leq s \leq 1) \\ \alpha_1 \alpha_2^{s-1} & (1 \leq s \leq 2) \end{cases}$$

More generally for T on $\mathbb{R}^N$, $\phi^s(T) = \alpha_1 \ldots \alpha_{p-1} \alpha_p^{s-p+1}$ where $p - 1 \leq s \leq p$.

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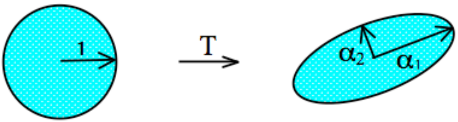

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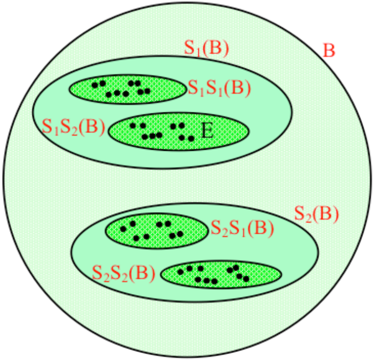
1. ϕ^s is submultiplicative:

$$\phi^s(T_1 T_2) \leq \phi^s(T_1)\phi^s(T_2).$$

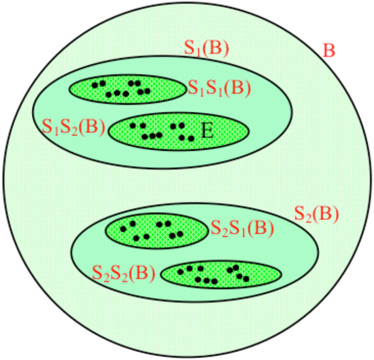
2. If T is contracting, $\phi^s(T)$ is decreasing in s.



Let $S_i(x) = T_i(x) + \omega_i$ be an affine IFS with attractor E. For each k:

$$E \subseteq \bigcup_{i_1 \dots i_k} S_{i_1} \circ \cdots \circ S_{i_k}(B).$$

Each $S_{i_1} \circ \cdots \circ S_{i_k}(B)$ is an ellipse, with semi-axes the singular values of $T_{i_1} \circ \cdots \circ T_{i_k}$



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Thus, for k large enough,

$$\mathcal{H}^s_\delta(E) \leq \begin{cases} \sum_{i_1\ldots i_k} \alpha_1(T_{i_1} \circ \cdots \circ T_{i_k})^s & (0 \leq s \leq 1) \\ \sum_{i_1\ldots i_k} \alpha_1(T_{i_1} \circ \cdots \circ T_{i_k})\alpha_2(T_{i_1} \circ \cdots \circ T_{i_k})^{s-1} & (1 \leq s \leq 2) \end{cases}$$

$$= \sum_{i_1\ldots i_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}).$$

Hence, writing

$$\Phi_k^s \equiv \sum_{i_1 \ldots i_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k})$$

we get $\mathcal{H}_\delta^s(E) \leq \Phi_k^s$ for large k.

By submultiplicativity, $\Phi_{k+l}^s \leq \Phi_k^s \Phi_l^s$ so

$$\lim_{k\to\infty} (\Phi_k^s)^{1/k} \equiv \Phi^s \quad \text{exists.}$$

[Note that (assuming strong separation) defining $f(x) = T_i^{-1}(x)$ if $x \in S_i(B)$, $\psi_k^s(x) := \log \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}) \equiv \log \phi^s((D_x f^k)^{-1})$, is a sub-additive potential.]

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Thus if $\Phi^s < 1$ then $\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E) \leq \lim_{k\to\infty} \left(\Phi^s_k\right) = 0$, so

$$\dim_H E \leq \left(\overline{\dim}_B E\right) \leq s \;\; \text{where } s \text{ satisfies } \;\Phi^s = 1.$$

The value of s satisfying $\Phi^s(T_1, \cdots, T_m) \equiv \Phi^s = 1$ is the affinity dimension $\dim_{\mathrm{aff}} E$ of the self-affine set E. Thus $\dim_{\mathrm{aff}} E$ is always an upper bound for $\dim_H E$ and and also for $\dim_B E$.

Q: When does $\dim_H E = \dim_B = E \dim_{\mathrm{aff}} E$?

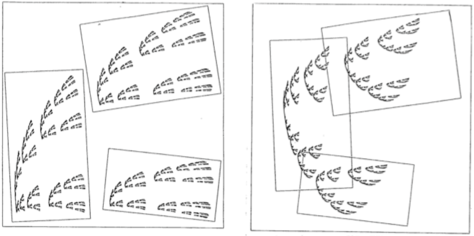
Self-affine sets - Generic results

Let $S_i(x) = T_i(x) + \omega_i$ be affine maps on $\mathbb{R}^N$ where the T_i are linear and ω_i are translations. Let E_ω be the self-affine attractor where $\omega \equiv (\omega_1, \ldots, \omega_m)$. Recall $\dim_{\text{aff}} E = s$ where $\Phi^s = \Phi^s(T_1, \ldots, T_m) = 1$.

Theorem (F 1988, Solomyak 1998) If $\|T_i\| < \frac{1}{2}$ for all i then

$$\dim_H E_\omega = \dim_B E_\omega = \min\{\dim_{\text{aff}} E, N\} \qquad (1)$$

for almost all $(\omega_1, \ldots, \omega_m) \in \mathbb{R}^{Nm}$ w.r.t. Nm-dimensional Lebesgue measure.



Conditions that ensure $\dim E = \dim_{\mathrm{aff}} E$

One seeks conditions that guarantee that the dimension of E equals the affinity dimension. Here is an early result.

Theorem (F 1992)

Let $S_i(x) = T_i(x) + \omega_i$ be an IFS of affine contractions on $\mathbb{R}^2$ with attractor E. Suppose that

(a) the open set condition holds,

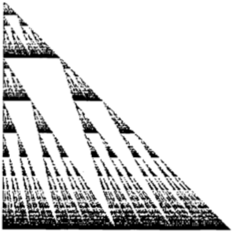
(b) there is a $c > 0$ such that the Lebesgue measure of the projection of E onto every line is $\geq c$.

Then

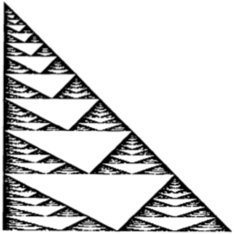
$$\dim_B E = \dim_{\mathrm{aff}} E.$$

Similar conclusions hold for affine IFSs on $\mathbb{R}^N$.

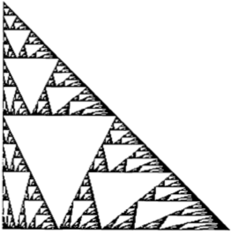
Examples: Generalised Sierpínski triangles where $\dim_B E = \dim_{\mathrm{aff}} E$:



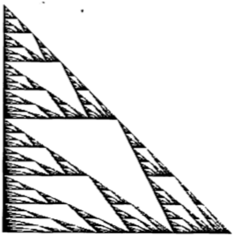
1. $\alpha = 0.75$, $b = 0.5$ (case (i))



2. $a = 0.25$, $b = 0.5$ (case (ii))

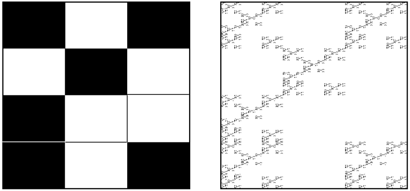


3. $a = 0.5$, $b = 0.25$

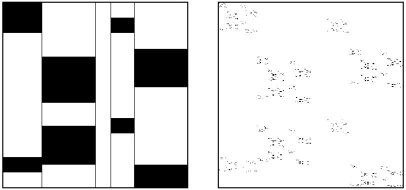


4. $a = 0.5$, $b = 0.75$

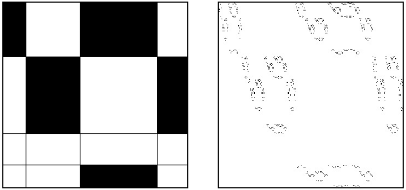
Self-affine sets where the dimension is not given by $\dim_{\mathrm{aff}} E$



Bedford-McMullen carpet (1984)



Gatzouras-Lalley carpet (1992)



Barański carpet (2007)

Conditions that ensure $\dim E = \dim_{\mathrm{aff}} E$

These examples suggest that it is the alignment of the components in the self-affine construction that lead to the dimensions of the sets being less than the affinity dimension.

Over the last 10 years ergodic theory methods have been employed to obtain increasingly general conditions that ensure $\dim_H E = \dim_{\mathrm{aff}} E$ for identifiable self-affine E. (Käenmäki, Bárány, Kempton, F, Hochman, Rapaport, Shmerkin, Morris,...)

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Theorem (Bárány, Hochman & Rapaport 2019) Let $E \subset \mathbb{R}^2$ be self-affine satisfying strong separation such that no finite union of lines through 0 is preserved by all of $T_1, \ldots, T_m$. Then

$$\dim_H E = \dim_B E = \dim_{\mathrm{aff}} E.$$

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Theorem (Morris + Rapaport, Recent) The analogous result for self-affine $E \subset \mathbb{R}^3$ holds.

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Theorem (Morris + Rapaport, Recent) The analogous result for self-affine $E \subset \mathbb{R}^3$ holds.

Q: Obtain analogues for self-affine $E \subset \mathbb{R}^N$ for $N \geq 4$.

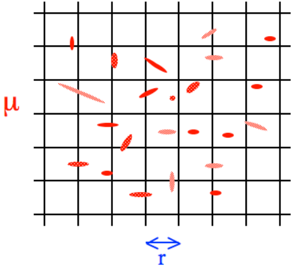
Q: Show these are valid with a weaker separation condition.

Q: Show that the box dimension $\dim_B E$ exists for all self-affine E.

Moment sums and L^q-dimensions

Let $\mathcal{M}_r$ be the mesh of side r.
Define the q-th power moment sum of a measure μ on $\mathbb{R}^n$ by

$$M_r(q) = \sum_{C \in \mathcal{M}_r} \mu(C)^q. \quad (1)$$



Then the L^q-dimension or generalised q dimension of μ is given by

$$D_q(\mu) = \frac{1}{q-1} \lim_{r \to 0} \frac{\log M_r(q)}{\log r} \qquad (q > 0, q \neq 1).$$

(or lim inf, lim sup). Equivalently we may replace (1) by a moment integral

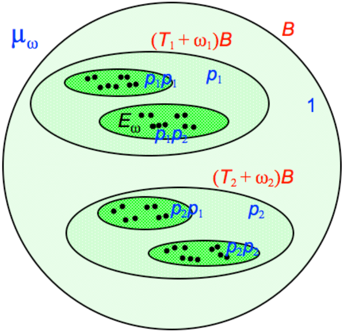
$$M_r(q) = \int \mu(B(x,r))^{q-1} d\mu(x) \qquad (q > 0, q \neq 1).$$

Let E_ω be the self affine set defined by the IFS $\{T_1 + \omega_1, \ldots, T_m + \omega_m\}$.

Let $p_1, \ldots, p_m$ be probabilities (so $0 < p_i < 1$ and $\sum p_i = 1$). Let μ be the Bernoulli probability measure on $\{1, \ldots, m\}^{\mathbb{N}}$ defined by

$$\mu(C_{\mathbf{i}}) = p_{i_1} p_{i_2} \ldots p_{i_k}$$

where $\mathbf{i} = (i_1, \ldots, i_k)$ and $C_{\mathbf{i}}$ is the corresponding cylinder.



Let μ_ω be the image measure of μ under X_ω, which is supported by E_ω. Thus $\mu_\omega\big((T_{i_1} + \omega_{i_1}) \cdots (T_{i_k} + \omega_{i_k})(B)\big) = p_{i_1} p_{i_2} \ldots p_{i_k}$.

We would like to find $D_q(\mu_\omega)$.

WIth a 'cutting up ellipses' argument, it is natural to introduce for $q > 0$

$$\Phi_q^s := \lim_{k\to\infty} \Big(\sum_{i_1 \dots i_k} \phi^s(T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k})^{1-q} \mu(C_{i_1,i_2,\dots,i_k})^q \Big)^{1/k}.$$

This exists by sub/supermultiplicativity, and $\log \Phi_q^s$ is a sub/superadditive pressure. We define s_q by $\Phi_q^{s_q} = 1$.

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Theorem (F 1999)

(a) For $q > 0$ the L^q-dimensions of μ_ω on the self-affine set E_ω satisfy

$$D_q(\mu_\omega) \leq \min\{s_q, n\}. \qquad (1)$$

(b) If $\|T_i\| < \frac{1}{2}$ for all i and $1 < q \leq 2$ then there is equality in (1) for almost all $\omega = \{\omega_1, \omega_2, \dots, \omega_m\}$.

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Q: What almost sure conclusions (if any) are there if $q > 2$?

Q: Find specific classes of sets with equality in (1). For $1 < q \leq 2$ one might hope this is the 'usually' the case for Benoulli measures on self-affine sets with equal Hausdorff and affinity dimension.

Q: What happens for $0 < q < 1$? The upper bound in (1) does not in general give the generic value (example by Barral & Feng 2013).

Almost self-affine sets

Can we remove the restrictions $\|T_i\| < \frac{1}{2}$ and $1 < q < 2$ in (b) above?
Recall for self-affine sets $E_\omega = \bigcup_{\mathbf{i} \in \{1,\ldots,m\}^{\mathbb{N}}} x_\omega(\mathbf{i})$ where

$$\begin{aligned} x_\omega(\mathbf{i}) &= \lim_{k\to\infty} (T_{i_1} + \omega_{i_1})(T_{i_2} + \omega_{i_2}) \cdots (T_{i_k} + \omega_{i_k})(0) \\ &= \omega_{i_1} + T_{i_1}\omega_{i_2} + T_{i_1} T_{i_2}\omega_{i_3} + \cdots \end{aligned}$$

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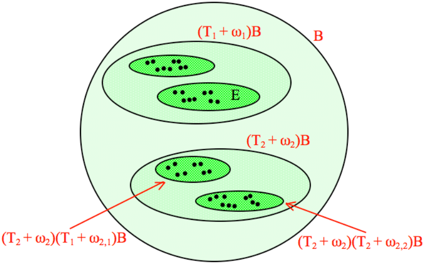
Introduce a random perturbation at each stage of the construction:

$$\begin{aligned} x_\omega(\mathbf{i}) &= \lim_{k\to\infty}(T_{i_1}+\omega_{i_1})(T_{i_2}+\omega_{i_1,i_2})(T_{i_3}+\omega_{i_1,i_2,i_3})\cdots(T_{i_k}+\omega_{i_1,i_2,\ldots i_k})(0) \\ &= \omega_{i_1} + T_{i_1}\omega_{i_1,i_2} + T_{i_1}T_{i_2}\omega_{i_1,i_2,i_3} + \cdots \end{aligned}$$

where $\omega_{i_1,i_2,\ldots,i_k}$ are i.i.d random ‘perturbations’.

We call $E_\omega = \bigcup_{\mathbf{i}} x_\omega(\mathbf{i})$ almost self-affine (Jordan, Pollicott & Simon 2007).



For μ_ω as above, so $\mu_\omega(C_{\mathbf{i}}) = p_{i_1} p_{i_2} \dots p_{i_k}$ for the cylinder $C_{\mathbf{i}}$, let

$$\Phi_q^s = \lim_{k\to\infty} \Big(\sum_{i_1 \dots i_k} \phi^s (T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k})^{1-q} \mu(C_{i_1,i_2,\dots,i_k})^q \Big)^{1/k},$$

Again the natural candidate for the L^q-dimensions of μ is the number s_q satisfying $\Phi_q^{s_q} = 1$.

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Let $\|T_i\| < 1$ for all i. Let μ_ω be the measure on the almost self-affine set E_ω such that $\mu_\omega(C_{\mathbf{i}}) = p_{i_1} p_{i_2} \dots p_{i_k}$ for cylinder $C_{\mathbf{i}}$.

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Q: How much randomness do we really need for these conclusions to remain valid?

Nonlinear analogue

The attractor of an IFS $\{S_1, \ldots, S_m\}$ may be thought of as a repeller of an expanding dynamical system f on defined by $f|_{S_i(D)} = S_i^{-1}$ for a suitable domain D. If the $S_i = T_i + \omega_i$ are affine, the pressure expression

$$\Phi^s \equiv \Phi^s(T_1, \ldots, T_m) = \lim_{k\to\infty}\Big(\sum_{i_1\ldots i_k} \phi^s(T_{i_1}\circ\cdots\circ T_{i_k})\Big)^{1/k}$$

can be written

$$\Phi^s = \lim_{k\to\infty}\left(\sum_{x\,\in\,\mathrm{fix} f^k} \phi^s\big(D_x f^k\big)^{-1}\right)^{1/k} \qquad (1)$$

where the sum is over the fixed points of f^k (there will be one such fixed point in each set $S_{i_1}\circ\cdots\circ S_{i_k}(D)$). As above the dimension of such a repeller is given by $\Phi^s = 1$ in various cases.

What if $f : D \to D$ is a C^1 expanding hyperbolic map? (1) might lead to a dimension formula for the repeller of f. If f is conformal, then this is Bowen's formula.

Theorem (F 1994)

Let D be a suitable domain in $\mathbb{R}^2$ and $f : D \to D$ as above be a $C^{1+\epsilon}$ strictly expanding and topologically mixing with repeller E. Define s by

$$\Phi^s := \lim_{k\to\infty} \Big(\sum_{x \in \mathrm{fix} f^k} \phi^s\big(D_x f^k\big)^{-1} \Big)^{1/k} = 1. \quad (1)$$

(a) Given one-bunched' condition $||(D_x f)^{-1}||^2||D_x f|| < 1$ all $x \in D$, then $\dim_H E \leq \dim_B E \leq s$.

(b) If also E has a connected component not contained in a line segment then $\dim_B E = s$.

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Recently Feng & Simon showed that the upper bound still holds without the one-bunched condition and with f just C^1. They also showed that for some prarmeterised families of IFSs, e.g. with lower triangular determinant matrix, $\dim_H E = \dim_B E = s$ for almost all parameters.

Q: Does $\dim_H E = \dim_B E = s$ where s is given by (1) hold in a 'generic' sense?

Nonlinear IFSs preserving vertical lines

Let $\{S_i\}_{i=1}^m$ be an iterated function system on $[0,1]^2$ of the form

$$S_i(x,y) = (f_i(x), g_i(x,y)), \qquad f_i, g_i \in C^{1+\epsilon},$$

so the S_i preserve vertical lines. The derivatives of the S_i have lower-triangular form:

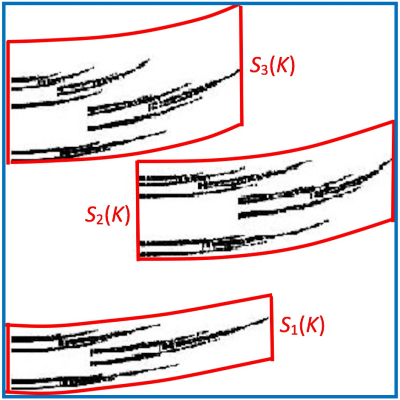
$$D_{\mathbf{a}} S_i = \begin{pmatrix} f_{i,x}(\mathbf{a}) & 0 \\ g_{i,x}(\mathbf{a}) & g_{i,y}(\mathbf{a}) \end{pmatrix}, \quad (\mathbf{a} \in [0,1]^2).$$

We assume:

- Rectangular open set condition: $\{\mathrm{int}\, S_i([0,1]^2)\}_{i=1}^m$ are disjoint;
- Domination condition:

There is more contraction in y-direction then the x-direction.

Example



$K = [0,1] \times [0,1]$

$$S_1(x,y) = \left(\frac{3x}{5} + \frac{3x^2}{40},\ \frac{x^2}{12} + \frac{y}{6}\right),$$

$$S_2(x,y) = \left(\frac{4x}{5} - \frac{4x^3}{30} + \frac{1}{3},\ \frac{x^2}{10} + \frac{y}{4} + \frac{17}{50}\right),$$

$$S_3(x,y) = \left(\frac{3x}{5},\ \frac{x^2}{10} + \frac{y}{5} + \frac{y^3}{9} + \frac{26}{45}\right).$$

Triangular iterated form

Recall our IFS $\{S_i\}_{i=1}^m$ on $[0,1]^2$:

$$S_i(x,y) = (f_i(x), g_i(x,y)) \text{ and } D_{\mathbf{a}}S_i = \begin{pmatrix} f_{i,x}(\mathbf{a}) & 0 \\ g_{i,x}(\mathbf{a}) & g_{i,y}(\mathbf{a}) \end{pmatrix}.$$

Iterate the IFS mappings and write $S_{\mathbf{i}} = S_{i_i} \circ \cdots \circ S_{i_k}$ where $\mathbf{i} = i_i, \ldots, i_k \in \mathcal{I}^k := \{1,2,\ldots,m\}^k$. Write

$$D_{\mathbf{a}}S_{\mathbf{i}} \equiv D_{\mathbf{a}}(S_{i_i} \circ \cdots \circ S_{i_k}) = \begin{pmatrix} f_{\mathbf{i},x}(\mathbf{a}) & 0 \\ g_{\mathbf{i},x}(\mathbf{a}) & g_{\mathbf{i},y}(\mathbf{a}) \end{pmatrix}, \quad (\mathbf{a} \in [0,1]^2).$$

for the derivatives of the iterates. Estimates using the domination condition give, uniformly in $\mathbf{i} \in \mathcal{I}^* \equiv \cup_{k=1}^{\infty} \mathcal{I}^k$ and $\mathbf{a}, \mathbf{b} \in [0,1]^2$,

$$\alpha_1(D_{\mathbf{a}}S_{\mathbf{i}}) \asymp |f_{\mathbf{i},x}(\mathbf{a})| \asymp |f_{\mathbf{i},x}(\mathbf{b})|$$

$$\alpha_2(D_{\mathbf{a}}S_{\mathbf{i}}) \asymp |g_{\mathbf{i},y}(\mathbf{a})| \asymp |g_{\mathbf{i},y}(\mathbf{b})|$$

$$|g_{\mathbf{i},x}(\mathbf{a})| \leq C|f_{\mathbf{i},x}(\mathbf{b})|.$$

Modified singular value function

Let μ be a Bernoulli measure on F defined by probabilities $\{p_i\}_{i=1}^m$. The projection $\pi(\mu)$ of μ onto the x-axis is a Bernoulli measure on the self-conformal set $\pi(F)$, so the L^q-spectrum of $\pi(\mu)$

$$\beta(q) := (q-1)D_q(\pi(\mu)),$$

exists for $q \geq 0$ (Peres & Solomyak). For $s \in \mathbb{R}$, $q \geq 0$ and $\mathbf{a} \in [0,1]^2$, define the *q-modified singular value function*, $\psi_{\mathbf{a}}^{s,q}$, by

$$\begin{aligned} \psi_{\mathbf{a}}^{s,q}(\mathbf{i}) &= p(\mathbf{i})^q \alpha_1(D_{\mathbf{a}}S_{\mathbf{i}})^{\beta(q)} \alpha_2(D_{\mathbf{a}}S_{\mathbf{i}})^{s-\beta(q)} \qquad (\mathbf{i} \in \mathcal{I}^* \equiv \cup_{k=1}^{\infty} \mathcal{I}^k) \\ &\asymp p(\mathbf{i})^q |f_{\mathbf{i},x}(\mathbf{a})|^{\beta(q)} |g_{\mathbf{i},y}(\mathbf{a})|^{s-\beta(q)}. \end{aligned}$$

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$$\psi_{\mathbf{a}}^{s,q}(\mathbf{i}) = p(\mathbf{i})^q \alpha_1(D_{\mathbf{a}}S_{\mathbf{i}})^{\beta(q)}\alpha_2(D_{\mathbf{a}}S_{\mathbf{i}})^{s-\beta(q)} \qquad (\mathbf{i} \in \mathcal{I}^* \equiv \cup_{k=1}^{\infty}\mathcal{I}^k)$$

$$\asymp p(\mathbf{i})^q |f_{\mathbf{i},x}(\mathbf{a})|^{\beta(q)}|g_{\mathbf{i},y}(\mathbf{a})|^{s-\beta(q)}.$$

Then for $s \in \mathbb{R}$, $q \geq 0$,

$$\psi_{\mathbf{a}}^{s,q}(\mathbf{ij}) \asymp \psi_{\mathbf{a}}^{s,q}(\mathbf{i})\psi_{\mathbf{a}}^{s,q}(\mathbf{j}).$$

uniformly in $\mathbf{i}, \mathbf{j} \in \mathcal{I}^*$ and $\mathbf{a}$. For each k let

$$\Psi_{\mathbf{a},k}^{s,q} = \sum_{\mathbf{i}\in\mathcal{I}^k} \psi_{\mathbf{a}}^{s,q}(\mathbf{i}),$$

then for $s \in \mathbb{R}$, $q \geq 0$,

$$\Psi_{\mathbf{a},k+\ell}^{s,q} \asymp \Psi_{\mathbf{a},k}^{s,q}\Psi_{\mathbf{a},\ell}^{s,q}$$

uniformly in $\mathbf{a}$, k and ℓ.

L^q-dimensions of μ

We can define a pressure function $P : \mathbb{R} \times [0, \infty) \to [0, \infty)$ by

$$P(s, q) = \lim_{k\to\infty} (\Psi^{s,q}_{\mathbf{a},k})^{1/k} \qquad \text{(independent of } \mathbf{a} \in [0, 1]^2\text{)}.$$

Define $\gamma : [0, \infty) \to \mathbb{R}$ by $P(\gamma(q), q) = 1$. Then γ is strictly decreasing, continuous and convex on $[0, \infty)$.

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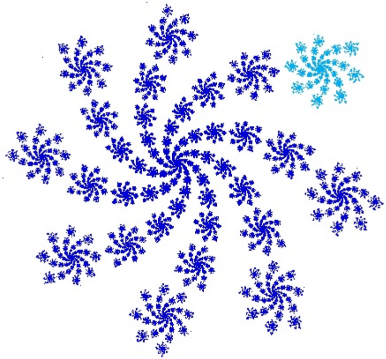
Define $\gamma : [0, \infty) \to \mathbb{R}$ by $P(\gamma(q), q) = 1$. Then γ is strictly decreasing, continuous and convex on $[0, \infty)$.

Theorem (Fraser, Lee, F, 2021) For a nonlinear IFS as above (domination condition, ROSC), and a Bernoulli measure μ on the attractor F, for $q \geq 0$,

$$D_q(\mu) = \frac{\gamma(q)}{q - 1}.$$

Thank you



Dziękuję