Typical self-affine sets with non-empty interior

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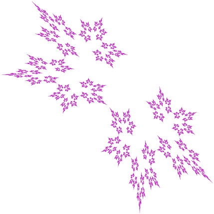
Main target

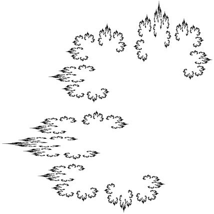
- To provide suitable sufficient conditions under which a "typical" self-affine set has non-empty interior.

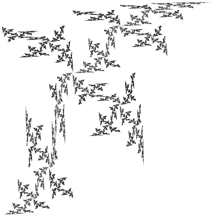
Preliminaries: self-similar and self-affine sets

- A finite family of contractive mappings $f_1, \ldots, f_m$ on $\mathbb{R}^d$ is called an **iterated function system** (IFS).
- Given an IFS, there exists a unique, nonempty compact $K \subset \mathbb{R}^d$ so that
$$K = \bigcup_{i=1}^{m} f_i(K).$$
The set K is called the **attractor** of the IFS.
- K is called **self-similar** if f_i are similarity maps, and **self-affine** if f_i are **affine maps**.

Pictures of some self-affine sets



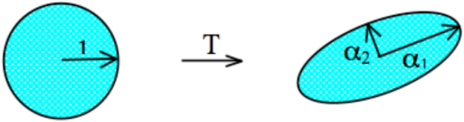




In 1988, Falconer introduced the concept of affinity dimension, and showed that it is a natural upper bound for the Hausdorff and box-counting dimensions of every self-affine set. Moreover, under a mild assumption, it is equal to the Hausdorff and box-counting dimensions of a typical self-affine set.

Notation: singular value function and affinity dimension

- For a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$, let $\alpha_1 \geq \alpha_2$ be the **singular values** of T, i.e. the semi-axis lengths of T(unit ball).



The **singular value function** of T is defined by

$$\phi^s(T) = \begin{cases} \alpha_1^s & \text{if } 0 \leq s \leq 1 \\ \alpha_1 \alpha_2^{s-1} & \text{if } 1 < s \leq 2 \\ (\alpha_1 \alpha_2)^{s/2} & \text{if } s > 2. \end{cases}$$

Notation: singular value function and affinity dimension

- More generally, for $T : \mathbb{R}^d \to \mathbb{R}^d$, define

$$\phi^s(T) = \begin{cases} \alpha_1 \dots \alpha_{j-1} \alpha_j^{s-j+1} & \text{if } j-1 < s \le j,\ j = 1, \dots, d, \\ (\alpha_1 \dots \alpha_d)^{s/d} & \text{if } s > d. \end{cases}$$

- (Sub-multiplicativity): $\phi^s(AB) \le \phi^s(A)\phi^s(B)$.
- $\mathcal{H}^s_\infty(T(\text{unit ball})) \approx_{s,d} \phi^s(T)$ for non-integer $s \in (0, d)$, where $\mathcal{H}^s_\infty$ stands for s-dimensional Hausdorff content.

Notation: singular value function and affinity dimension

- Let K be the self-affine set generated by an affine IFS $\{T_i x + a_i\}_{i=1}^m$ on $\mathbb{R}^d$.
- For $\mathbf{i} = i_1 \ldots i_n$, set $|\mathbf{i}| = n$ and set $T_{\mathbf{i}} = T_{i_1} \ldots T_{i_n}$.
- Define the affinity dimension of K (with respect to $T_1, \ldots, T_m$) by

$$\dim_{\mathrm{AFF}} K = \inf \left\{ s \geq 0 : \lim_{n \to \infty} \sum_{|\mathbf{i}|=n} \phi^s(T_{\mathbf{i}}) \leq 1 \right\}.$$

- Clearly, $\dim_{\mathrm{AFF}} K$ only depends on $T_1, \ldots, T_m$.
- $\dim_{\mathrm{AFF}} K$ is the unique s so that the topological pressure of the sub-additive potential $\{f_n(x) = \log \phi^s(T_{x|n})\}_{n=1}^\infty$ is zero.

Classical results on typical self-affine sets

Theorem

Let K be the self-affine set generated by an affine IFS $\{T_i x + a_i\}_{i=1}^m$ on $\mathbb{R}^d$. Assume that $\|T_i\| < 1/2$ for all i. Then

- (Falconer 1988, Solomyak 1998): *For $\mathcal{L}^{md}$-a.e. $(a_1, \ldots, a_m)$,*

$$\dim_H K = \dim_B K = \min\{d, \dim_{\mathrm{AFF}} K\}.$$

- (Jordan-Pollicott-Simon 2007): *If $\dim_{\mathrm{AFF}} K > d$, then for $\mathcal{L}^{md}$-a.e. $(a_1, \ldots, a_m)$, $\mathcal{L}^d(K) > 0$.*

Remark: By definition, $\dim_{\mathrm{AFF}} K > d \iff \sum_{i=1}^m |\det(T_i)| > 1$.

A natural question on typical self-affine sets

Let K be the self-affine set generated by an affine IFS $\{T_i x + a_i\}_{i=1}^m$ on $\mathbb{R}^d$.

Q: Under which assumptions on $(T_1, \ldots, T_m)$, K has non-empty interior for almost all $(a_1, \ldots, a_m)$?

Although this seems a rather fundamental question, it has hardly been studied.

- Define

$$\gamma(T_1, \ldots, T_m) = \inf \left\{ \gamma \geq 0 \colon \sup_{n \geq 1} \sum_{|I|=n} \alpha_d(T_I)^\gamma \cdot |\det(T_I)| \leq 1 \right\}.$$

- The quantity $\gamma(T_1, \ldots, T_m)$ is the unique γ so that the topological pressure of the super-additive potential

$$\left\{ f_n(x) = \log \left(\alpha_d(T_{x|n})^\gamma \cdot |\det(T_{x|n})| \right) \right\}_{n=1}^{\infty}$$

is zero.

Theorem A (F.-Feng, 2022). Assume that $\|T_i\| < 1/2$ for $1 \leq i \leq m$, and $\gamma(T_1, \ldots, T_m) > d$. Then K has non-empty interior for $\mathcal{L}^{md}$-a.e. $(a_1, \ldots, a_m)$.

Remark: By definition,

$$\gamma(T_1, \ldots, T_m) > d \iff \exists\, n \text{ such that } \sum_{|I|=n} \alpha_d(T_I)^d |\det(T_I)| > 1.$$

Our main results

Corollary (F.-Feng, 2022). Assume that $\|T_i\| < 1/2$ for $1 \le i \le m$. Then K has non-empty interior for $\mathcal{L}^{md}$-a.e. $(a_1, \ldots, a_m)$, provided that one of the following two conditions fulfills:

(i) $\sum_{i=1}^{m} \alpha_d(T_i)^d |\det(T_i)| > 1$.

(ii) All T_i are scalar multiples of orthogonal matrices, and $\sum_{i=1}^{m} |\det(T_i)|^2 > 1$.

Remark. Item (ii) corresponds to the self-similar case.

Our main results

Next we provide an improvement of Theorem A in the special case when the matrices $T_1, \ldots, T_m$ commute.

Theorem B (F.-Feng, 2022). Assume that $\|T_i\| < 1/2$ for $1 \le i \le m$. Moreover, suppose that

- $T_i T_j = T_j T_i$ for all $1 \le i, j \le m$, and
- $\sum_{i=1}^m |\det(T_i)|^2 > 1$.

Then K has non-empty interior for $\mathcal{L}^{md}$-a.e. $(a_1, \ldots, a_m)$.

Related work in the literature

- Shmerkin 2006 investigated the affine IFS $\Phi_{\alpha,\beta} = \{(\alpha x, \beta y),\ (\alpha x + 1, \beta y + 1)\}$ on $\mathbb{R}^2$. He constructed an open $V \subset \{(\alpha,\beta)\colon 0 < \alpha < \beta < 1,\ 2\alpha\beta > 1\}$ and showed that for almost all $(\alpha,\beta) \in V$, the attractor $K_{\alpha,\beta}$ of $\Phi_{\alpha,\beta}$ has non-empty interior.
- Dajani-Jiang-Kempton 2014: $\exists C \approx 1.05^{-1}$ such that $K_{\alpha,\beta}$ has non-empty interior for all $C < \alpha < \beta < 1$.
- Hare-Sidorov 2016, Hare-Sidorov 2017, Baker 2020: Further improvements of C.
- He-Lau-Rao 2003: provided an algorithm to check whether the attractor of an integral affine IFS $\{A^{-1}x + a_i\}_{i=1}^m$ on $\mathbb{R}^d$ has non-empty interior, here A is an integral expanding matrix, $a_i \in \mathbb{Z}^d$.
- Csörnyei-Jordan-Pollicott-Preiss-Solomyak 2006: constructed a rotation-free self-similar set in $\mathbb{R}^2$ which has positive Lebesgue measure, but has empty interior.

The idea of the proof of Theorem B

For simplicity, here we only consider the homogeneous case $\{Tx + a_i\}_{i=1}^m$ on $\mathbb{R}^d$, where

$$\|T\| < 1/2, \quad m \cdot \det(T)^2 > 1.$$

It is easily checked that

$$K = E + TE,$$

where E is the attractor of $\{T^2x + a_i\}_{i=1}^m$. By JPS07, $\mathcal{L}^d(E) > 0$ for a.e. $(a_1, \ldots, a_m)$. Hence by the Steinhaus theorem, K has non-empty interior for a.e. $(a_1, \ldots, a_m)$.

The idea of the proof of Theorem A

Here we use the method of Fourier transform. For a finite Borel measure η on $\mathbb{R}^d$ with compact support, the Fourier transform of η is defined by

$$\widehat{\eta}(\xi) = \int e^{-i\langle \xi, x\rangle}\, d\eta(x), \qquad \xi \in \mathbb{R}^d.$$

The following is a classical result (see e.g. Peres-Schlag 2000, Mattila 2015).

Lemma

Suppose that $\int_{\mathbb{R}^d} |\widehat{\eta}(\xi)|^2 |\xi|^\gamma d\xi < \infty$ for some $\gamma > d$. Then η is absolutely continuous with a continuous density, so its support has non-empty interior.

- For $\mathbf{a} = (a_1, \ldots, a_m)$, let $\pi^{\mathbf{a}} : \Sigma = \{1, \ldots, m\}^{\mathbb{N}} \to \mathbb{R}^d$ denote the coding map associated with the IFS $\{f_i(x) = T_i x + a_i\}_{i=1}^m$. That is,

$$\pi^{\mathbf{a}}(x) = \lim_{n\to\infty} f_{x_1} \circ f_{x_2} \circ \cdots \circ f_{x_n}(0), \quad x = (x_i)_{i=1}^{\infty}.$$

- For a Borel probability measure μ on Σ, let $\mu^{\mathbf{a}} := \mu \circ (\pi^{\mathbf{a}})^{-1}$ denote the projection of μ under $\pi^{\mathbf{a}}$.

- Suppose $\gamma(T_1, \ldots, T_m) > d$ and $\|T_i\| < 1/2$. Our purpose is to show that $\exists\, \mu$ on Σ and $\gamma > d$ such that

$$\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 |\xi|^\gamma \, d\xi d\mathbf{a} < \infty, \qquad \forall \rho > 0, \qquad (1)$$

which implies $\mathrm{supp}(\mu^{\mathbf{a}})$ has non-empty interior a.e. **a**.

- To prove (**??**), it suffices to show for every nonnegative $\psi \in C_0^\infty(\mathbb{R}^{md})$,

$$\int_{\mathbb{R}^{md}} \int_{\mathbb{R}^d} \psi(\mathbf{a}) |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 |\xi|^\gamma \, d\xi d\mathbf{a} < \infty. \qquad (2)$$

Notice that

$$\int_{\mathbb{R}^{md}} \int_{\mathbb{R}^d} \psi(\mathbf{a}) |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 |\xi|^\gamma \, d\xi d\mathbf{a}$$

$$= \int_{\mathbb{R}^{md}} \int_{\mathbb{R}^d} \int_\Sigma \int_\Sigma \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle} |\xi|^\gamma d\mu(x) d\mu(y) d\xi d\mathbf{a}$$

$$= \int_\Sigma \int_\Sigma \int_{\mathbb{R}^d} \int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle} |\xi|^\gamma d\mathbf{a} d\xi d\mu(x) d\mu(y)$$

Key inequality (I)

For $x, y \in \Sigma$, let $x \wedge y$ denote the common initial segment of x and y.

Proposition

Assume that $\delta := \max_{1 \le i \le m} \|T_i\| < 1/2$. *Let* $\psi \in C_0^\infty(\mathbb{R}^{md})$ *and* $N \in \mathbb{N}$. *Then there exists* $C = C(\psi, N, \delta) > 0$ *such that*

$$\left| \int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y) \rangle} \, d\mathbf{a} \right| \le C \left(1 + |T^*_{x\wedge y}\xi|\right)^{-N}$$

for all $\xi \in \mathbb{R}^d$ *and* $x, y \in \Sigma$ *with* $x \neq y$, *where* $T^*_{x\wedge y}$ *stands for the transpose of* $T_{x\wedge y}$.

The idea is to show that

$$|\nabla_{\mathbf{a}} \langle \xi/|\xi|, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y) \rangle| > \mathrm{Constant}, \text{ if } x_1 \neq y_1, \ \xi \neq 0,$$

in which we apply an argument of Solomyak (1998).

Key inequality (II)

Proposition

Let $d \in \mathbb{N}$, $\gamma \geq 0$ *and* $N > \gamma + d$. *Then*

$$\int_{\mathbb{R}^d} (1 + |Tx|)^{-N} |x|^{\gamma} \, dx \approx_{N,d,\gamma} \frac{1}{\alpha_d(T)^{\gamma} |\det(T)|}$$

for $T \in \mathrm{GL}(d, \mathbb{R})$.

Sketched proof of Theorem A

- Since $\gamma(T_1, \ldots, T_m) > d$, there exist a Borel probability measure μ on Σ, $\gamma > d$ and $r \in (0,1)$ such that

$$\mu([I]) \lesssim r^n \alpha_d(T_I)^\gamma |\det(T_I)|$$

for all $n \in \mathbb{N}$ and $I \in \Sigma_n$.

- By Inequalities (I) and (II), for $N > t + d$,

$$\begin{aligned}
&\int_{\mathbb{R}^{md}} \int_{\mathbb{R}^d} \psi(\mathbf{a}) |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 |\xi|^\gamma \, d\xi d\mathbf{a} \\
&\lesssim \int_\Sigma \int_\Sigma \int_{\mathbb{R}^d} (1 + |T^*_{x \wedge y}\xi|)^{-N} |\xi|^\gamma d\xi d\mu(x) d\mu(y) \\
&\lesssim \int_\Sigma \int_\Sigma \frac{1}{\alpha_d(T_{x\wedge y})^\gamma |\det(T_{x \wedge y})|} d\mu(x) d\mu(y) < \infty.
\end{aligned}$$

An open question

Our results might not be sharp. Recall that JPS07 proved that $\mathcal{L}^d(K) > 0$ for a.e. $(a_1, \dots, a_m)$ under the conditions that $\|T_i\| < 1/2$ and $\sum_{i=1}^m |\det(T_i)| > 1$.

Open Question: Does K have non-empty interior a.e. under the above condition?

Thank you for listening !!!