Non-stationary version of Furstenberg Theorem on random matrix products

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Thermodynamic Formalism: Non-additive Aspects and Related Topics

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The results are joint
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Powers of a matrix and their rate of growth

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Suppose $A \in SL(d, \mathbb{R})$. What can be said about the rate of growth of $\|A^n\|$ as $n \to \infty$?

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n\| = \lambda,$$

where λ is a logarithm of a maximal eigenvalue of A (i.e. logarithm of the spectral radius).

Furstenberg-Kesten Theorem

Let μ be a probability distribution on $SL(d, \mathbb{R})$. Choose matrices $A_1, A_2, \ldots$ randomly and independently with respect to the distribution μ. Denote $A^n = A_n \ldots A_2 A_1$. What can one say about the rate of growth of $\|A^n\|$ as $n \to \infty$?

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Theorem (Furstenberg-Kesten, 1960)

Suppose that

$$\int \log \|A\| d\mu(A) < \infty.$$

Then almost surely there exists a limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n\| = \lambda,$$

where $\lambda = \lambda(\mu)$ is a non-random constant (Lyapunov exponent).

Example 1

Set $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, $B = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$. As A_i let us choose A or B (with probablity $1/2$), $A^n = A_n \dots A_2 A_1$.

Then

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n\| = 0.$$

Example 2

Set $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ (i.e. $B = A^{-1}$). As A_i let us choose A or B (with probability $1/2$), $A^n = A_n \dots A_2 A_1$.

Then almost surely

$$\lim_{n \to \infty} \frac{1}{n} \log \|A^n\| = 0.$$

Example 3

Set $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As A_i let us choose A or B (with probability $1/2$), $A^n = A_n \ldots A_2 A_1$.

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$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n\| = 0.$$

Furstenberg Theorem

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Theorem (Furstenberg Theorem, 1963)

Suppose that

- $\int \log \|A\| d\mu(A) < \infty$;
- *the smallest closed subgroup of* $SL(d, \mathbb{R})$ *that contains* $\operatorname{supp} \mu$ *is not compact, and*
- *there is no finite collection of proper subspaces on* $\mathbb{R}^d$ *invariant under action of each of the linear maps from* $\operatorname{supp} \mu$.

Then almost surely the Lyapunov exponent

$$\lim_{n \to \infty} \frac{1}{n} \log \|A^n\| = \lambda$$

is strictly positive, $\lambda > 0$.

Furstenberg Theorem

Remark 1:

If $d = 2$, then the conditions

• *the smallest closed subgroup of* $SL(d, \mathbb{R})$ *that contains* $\operatorname{supp} \mu$ *is not compact, and*

• *there is no finite collection of proper subspaces on* $\mathbb{R}^d$ *invariant under action of each of the linear maps from* $\operatorname{supp} \mu$

can be replaced by

• *There exist no measure* ν *on* $\mathbb{RP}^1$ *invariant under all* f_A, $A \in \operatorname{supp} \mu$.

Furstenberg Theorem

Remark 2:

To see why the condition

$$\int \log \|A\| d\mu(A) < \infty$$

is relevant, consider the case $d = 1$. If $\{a_n\}$, $a_n \in \mathbb{R}$, $a_n > 0$, are iid random variable, then to study $a_1 \cdot a_2 \cdot \ldots \cdot a_n$, consider $\log a_1 + \log a_2 + \ldots + \log a_n$. We have

$$\frac{\log a_1 + \log a_2 + \ldots + \log a_n}{n} \to \mathbb{E} \log a_1,$$

if $\mathbb{E} \log a_1 < \infty$.

Example 4: Schrödinger operators.

Cosider discrete Schrödinger operator on $l^2(\mathbb{Z})$:

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n),$$

where $\{V(n)\}$ are iid random variables.

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If $E \in \mathbb{R}$ is an eigenvalue, then $Hu = Eu$, that is,

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

Example 4: Schrödinger operators.

If

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n),$$

then

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}.$$

Denote $\Pi_{E,n} = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}$.

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Denote $\Pi_{E,n} = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}$.

So if $\{V(n)\}$ are iid random variables, by Furstenberg Theorem, with probability 1 for a fixed value of energy E

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{E,n}\Pi_{E,n-1}\ldots\Pi_{E,1}\| = \lambda > 0$$

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What if

$$V(n) = V_{background}(n) + V_{random}(n) \quad ?$$

Question: Suppose $A_i \in SL(d, \mathbb{R})$ is distributed w.r.t. μ_i (independent, but not identically distributed)? What one can say about the growth rate of $\|T_n\|$, where $T_n = A_n \cdot \ldots \cdot A_1$?

Non-stationary Furstenberg Theorem

Theorem (G, Kleptsyn, 2022)

Assume K is a compact set in the space of probability distributions on $SL(2,\mathbb{R})$ such that

- *For some $\gamma, C > 0$ and any $\mu \in K$ we have $\int \|A\|^\gamma < C$;*
- *("measures condition") For any $\mu \in K$, there are no probability measures ν_1, ν_2 on $\mathbb{RP}^1$ such that $(f_A)_*\nu_1 = \nu_2$ for all $A \in supp\,\mu$.*

Choose any sequence $\{\mu_i\} \subset K^{\mathbb{N}}$, and let A_i be chosen w.r.t. μ_i, independently. Then

1) For some $\lambda > 0$ we have $\mathbb{E}\log\|T_n\| = L_n > \lambda n$ for all $n \in \mathbb{N}$;

2) Almost surely, $\lim_{n\to\infty} \frac{1}{n}(\log\|T_n\| - L_n) = 0$.

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Addendum:

Almost surely, there exists a unit vector $\bar{v} \in \mathbb{R}^2$ such that $T_n \bar{v} \to 0$, and

$$\lim_{n\to\infty} \frac{1}{n}(\log | T_n \bar{v} | + L_n) = 0.$$

Non-stationary Anderson Model

Consider a discrete Schrödiger operator H on $\ell^2(\mathbb{Z})$

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n).$$

Suppose that $\{V(n)\}$ are independent (but not necessarily identically distributed) random variables, distributed with respect non-degenerate probability measures $\{\mu_n\}$ with compact support. In particular, non-stationary Anderson-Bernoulli Model (measure μ_n is supported on two points, and depends on n) satisfies our conditions. Denote

$$\mathbb{P} = \prod_{n=-\infty}^{\infty} \mu_n$$

Non-stationary Anderson Model

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n).$$

Theorem (Spectral Anderson Localization) *Suppose that a random potential* $\{V(n)\}$ *of operator* H *is defined by independent probability distributions* $\{\nu_n\}$*, such that*

1) $\operatorname{supp}\mu_n \subseteq [-K, K]$*;*

2) $Var(\mu_n) > \varepsilon$*,*

where $\varepsilon > 0, K < \infty$ *are some uniform constant. Then* $\mathbb{P}$*-almost surely operator* H *has p.p. spectrum, with exponentially decaying eigenfunctions.*

Non-stationary Furstenberg Theorem

Theorem (the case $d \geq 2$)

Assume additionally that

• *for any* $\mu \in K$*, there are no union of proper subspaces* U, U' *in* $\mathbb{R}^d$ *such that* $A(U) = U'$ *for all* $A \in \mathrm{supp}\,\mu$.

Then the same conclusion holds. That is, if we choose any sequence $\{\mu_i\} \subset K^{\mathbb{N}}$*, and set* A_i *to be chosen w.r.t.* μ_i*, independently, then*

1) For some $\lambda > 0$ *we have* $\mathbb{E} \log \|T_n\| = L_n > \lambda n$ *for all* $n \in \mathbb{N}$*;*

2) Almost surely, $\lim_{n\to\infty} \frac{1}{n}(\log \|T_n\| - L_n) = 0$.

Remark:

The statement that $\liminf_{n\to\infty} \frac{1}{n} L_n > 0$, was proven by I.Goldsheid (independently and by different methods).

Non-stationary Furstenberg Theorem

Theorem (Large Deviation Estimates)

Under the assumptions above, for any ε *there exists* $\delta > 0$ *such that for all large enough* $n \in \mathbb{N}$ *we have*

$$\mathbb{P}\left\{ \left| \log \| T_n \| - L_n \right| > \varepsilon n \right\} < e^{-\delta n}$$

Exponential growth of the norms

Consider any closed Riemannian manifold M and the set of its C^1-diffeomorphisms $\mathrm{Diff}^1(M)$. Set

$$\mathrm{Jac}(f)|_x = |\det df|_x| = \left.\frac{d\mathrm{Leb}_M}{df_*\mathrm{Leb}_M}\right|_x .$$

We will measure the maximum volume contraction rate of a diffeomorphism by the following quantity:

$$\mathcal{N}(f) := \max_{x\in M} \mathrm{Jac}(f)|_x^{-1} = \max_{x\in M} \left.\frac{df_*\mathrm{Leb}_M}{d\mathrm{Leb}_M}\right|_x .$$

Exponential growth of the norms

Let K_M be a compact subset of the space of probability measures on $\mathrm{Diff}^1(M)$ (equipped with the weak-$*$ convergence topology). Let M, K_M satisfy the following assumptions:

- **(log-moment condition)** For any $\mu \in \mathsf{K}_M$ one has

$$\int_{\mathrm{Diff}^1(M)} \log \mathcal{N}(f) d\mu(f) < \infty$$

- **(measures condition)** For any $\mu \in \mathsf{K}_M$ there are no Borel probability measures ν_1, ν_2 on M such that $f_* \nu_1 = \nu_2$ for μ-almost every $f \in \mathrm{Diff}^1(M)$.

Exponential growth of the norms

Theorem
If M*,* K_M *satisfy the assumptions above, then there exists* $\mathsf{h} > 0$ *such that for any* n *and any* $\mu_1, \dots, \mu_n \in \mathsf{K}_M$ *we have*

$$\mathbb{E} \log \mathcal{N}(F_n) \geq n\mathsf{h},$$

where $F_n = f_n \circ \cdots \circ f_1$*, and every* f_i *is chosen independently with respect to the corresponding measure* μ_i*, so that the expectation is taken over the distribution* $\mu_1 \times \mu_2 \times \ldots \times \mu_n$*.*

Exponential growth of the norms

Theorem

If M, K_M satisfy the assumptions above, then for any given sequence $(\mu_i)_{i\in\mathbb{N}}$, $\mu_i \in \mathsf{K}_M$ of measures on $\mathrm{Diff}^1(M)$ we have

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}\,\log\mathcal{N}(F_n) \geq \mathsf{h} > 0,$$

where $F_n = f_n \circ \cdots \circ f_1$, and the expectation is taken with respect to the infinite product measure $\prod_i \mu_i$.

Remark:

If $A \in SL(d,\mathbb{R})$, then $\mathcal{N}(f_A) = \|A\|^d$.

Probability of cancellations

For a given (large but fixed) k we decompose the product of matrices of length $n = km$,

$$T_n = A_n \dots A_1,$$

into m groups of products of length k:

$$T_n = (A_n \dots A_{k(m-1)+1}) \dots (A_k \dots A_1) = B_m \dots B_1,$$

where

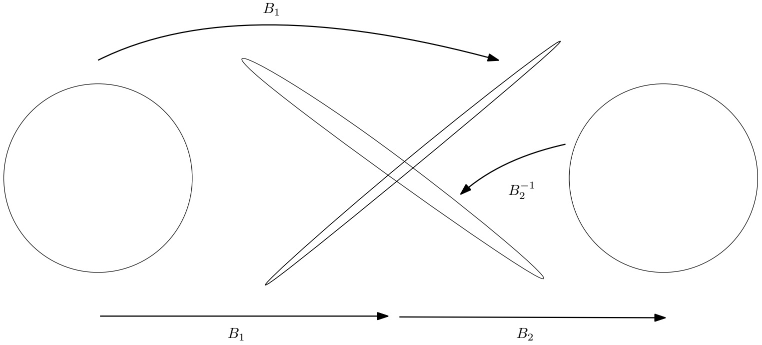
$$B_j := (A_{kj} \dots A_{k(j-1)+1}).$$

One has

$$\log \|T_n\| = \log \|B_m \dots B_1\| \leq \sum_{j=1}^{m} \log \|B_j\|.$$

Probability of cancellations

$$T_n = (A_n \dots A_{k(m-1)+1}) \dots (A_k \dots A_1) = B_m \dots B_1,$$
$$B_j = (A_{kj} \dots A_{k(j-1)+1}).$$



Stationary case and stationary measures

In the stationary case:

Definition:
Let μ be a probability distribution on $SL(d, \mathbb{R}^d)$. A probability measure ν on $\mathbb{RP}^{d-1}$ is stationary if

$$\mu * \nu = \mathbb{E}_\mu((f_A)_* \nu) = \int (f_A)_* \nu d\mu(A) = \nu$$

Claim: Under natural conditions on μ (that hold generically), a stationary measure is unique and has no atoms.

Atom Dissolving

Let X be a metric compact. For a measure μ on the space of homeomorphisms $\mathrm{Homeo}(X)$, we say that there is

- *no finite set with a deterministic image*, if there are no two finite sets $F, F' \subset X$ such that $f(F) = F'$ for μ-a.e. $f \in \mathrm{Homeo}(X)$;
- *no measure with a deterministic image*, if there are no two probability measures ν, ν' on X such that $f_* \nu = \nu'$ for μ-a.e. $f \in \mathrm{Homeo}(X)$.

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Denote by $\mathfrak{Max}(\nu)$ the weight of a maximal atom of a probability measure ν. In particular, if ν has no atoms, then $\mathfrak{Max}(\nu) = 0$.

Atom Dissolving

Theorem (Atoms Dissolving)

Let K_X *be a compact set of probability measures on* $\mathrm{Homeo}(X)$.

- *Assume that for any* $\mu \in \mathsf{K}_X$ *there is no finite set with a deterministic image. Then for any* $\varepsilon > 0$ *there exists* n *such that for any probability measure* ν *on* X *and any sequence* $\mu_1, \ldots, \mu_n \in \mathsf{K}_X$ *we have*

$$\mathfrak{Max}\,(\mu_n * \cdots * \mu_1 * \nu) < \varepsilon.$$

In particular, for any probability measure ν *on* X *and any sequence* $\mu_1, \mu_2, \ldots \in \mathsf{K}_X$ *we have*

$$\lim_{n\to\infty} \mathfrak{Max}\,(\mu_n * \cdots * \mu_1 * \nu) = 0.$$

Atom Dissolving

Theorem (Atoms Dissolving)

Let K_X be a compact set of probability measures on $\mathrm{Homeo}(X)$.

- *If for any $\mu \in \mathsf{K}_X$ there is no measure with a deterministic image, then the convergence is exponential and uniform over all sequences $\mu_1, \mu_2, \ldots$ from $K^{\mathbb{N}}$ and all probability measures ν. That is, there exists $\lambda < 1$ such that for any n, any ν and any $\mu_1, \mu_2, \dots \in \mathsf{K}_X$*

$$\mathfrak{Max}\,(\mu_n * \cdots * \mu_1 * \nu) < \lambda^n.$$

Non-stationary Ergodic Theorem

Let X be a compact metric space, and μ be a probability measure on the space of continuous maps $C(X, X)$. The iterations of the corresponding random dynamical system are the sequences of compositions

$$f_1, \ f_2 \circ f_1, \ \ldots, \ f_n \circ \cdots \circ f_1, \ \ldots,$$

where $f_i : X \to X$ are chosen randomly and independently w.r.t. the measure μ.

Non-stationary Ergodic Theorem

The following theorem corresponds to the Birkhoff Ergodic Theorem for the stationary random dynamics:

Theorem (Random Birkhoff Ergodic Theorem)

For any ergodic stationary measure ν on X, for any $\varphi \in C(X, \mathbb{R})$, for ν-a.e. $x \in X$, $\mu^{\mathbb{N}}$-almost surely one has

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi(f_k \circ \ldots \circ f_1(x)) \to \int_X \varphi(x) d\nu(x),$$

where $f_1, f_2, \ldots$ are chosen randomly and independently, with respect to the distribution μ.

Non-stationary Ergodic Theorem

Assume that a "test function" $\varphi \in C(X)$ is given. Also, for every n we assume that a measure μ_n on $C(X, X)$ is given (if the dynamics is assumed to be invertible, one can ask instead for the measure on the set of homeomorphisms of X). Denote

$$\mathbb{P} := \prod_{n=1}^{\infty} \mu_n, \tag{1}$$

and our goal is to describe time averages along $\mathbb{P}$-almost every sequence of iterations.

Let us denote by $\mathcal{M} = \mathcal{M}(X)$ the space of Borel probability measures on the compact metric space X. We consider it to be equipped with the Wasserstein distance (that is one of the ways to metrize the weak-* topology in the space of probability measures).

Non-stationary Ergodic Theorem

Standing Assumption: *We will say that a sequence of distributions* $\mu_1, \mu_2, \mu_3, \ldots$ *on* $C(X, X)$ *satisfies the Standing Assumption if for any* $\delta > 0$ *there exists* $m \in \mathbb{N}$ *such that the images of any two initial measures after averaging over m random steps after any initial moment n become* δ*-close to each other:*

$$\forall \nu, \nu' \in \mathcal{M}, \quad \forall n \in \mathbb{N},$$
$$\mathrm{dist}_{\mathcal{M}}(\mu_{n+m} * \ldots * \mu_{n+1} * \nu, \mu_{n+m} * \ldots * \mu_{n+1} * \nu') < \delta.$$

Non-stationary Ergodic Theorem

Theorem (G, Kleptsyn, 2023)

Suppose the sequence of distributions $\mu_1, \mu_2, \mu_3, \dots$ *satisfies the Standing Assumption above. Given any Borel probability measure* ν_0 *on* X*, define*

$$\nu_n := \mu_n * \nu_{n-1}, \quad n = 1, 2, \dots .$$

Then for any $\varphi \in C(X, \mathbb{R})$ *and any* $x \in X$*, almost surely*

$$\frac{1}{n} \left| \sum_{k=1}^{n} \varphi(f_k \circ \dots \circ f_1(x)) - \sum_{k=1}^{n} \int_X \varphi \, d\nu_k \right| \to 0 \text{ as } \quad n \to \infty,$$

where $\nu_n := \mu_n * \nu_{n-1}, \quad n = 1, 2, \dots,$ *and* ν_0 *is arbitrary.*

Non-stationary Ergodic Theorem

Moreover, an analogue of the Large Deviations Theorem holds. Namely, for any $\varepsilon > 0$ there exist $C, \delta > 0$ such that for any $x \in X$

$$\forall n \in \mathbb{N},$$

$$\mathrm{P}\left(\frac{1}{n} \left| \sum_{k=1}^{n} \varphi(f_k \circ \ldots \circ f_1(x)) - \sum_{k=1}^{n} \int_X \varphi \, d\nu_k \right| > \varepsilon \right) < \\ < C \exp(-\delta n).$$

Random Iterated Function Systems

Proposition:
Let X be a compact metric space, and $\lambda \in (0, 1)$ be a constant. Suppose $\{\mu_i\}_{i \in \mathbb{N}}$ be a sequence of probability distributions in the space of contractions $X \to X$ with Lipschitz constant at most λ. Then Standing Assumption holds, i.e. for any Borel probability measures ν, ν' on X we have

$$\mathrm{dist}_{\mathcal{M}}(\mu_{n+m} * \ldots * \mu_{n+1} * \nu, \mu_{n+m} * \ldots * \mu_{n+1} * \nu') \to 0 \text{ as } \quad m \to \infty,$$

uniformly in (ν, ν') and $n \in \mathbb{N}$.

Random Matrix Products

Suppose K is a compact set in the space of probability distributions on $SL(2,\mathbb{R})$ such that for any $\mu \in$ K *the measures condition* is satisfied, i.e. there are no probability measures ν_1, ν_2 on $\mathbb{RP}^1$ such that $(f_A)_*(\nu_1) = \nu_2$ for all $A \in \operatorname{supp}\mu$. Slightly abusing the notation, we will treat μ also as a measure on the space of projective maps $f_A : \mathbb{RP}^1 \to \mathbb{RP}^1$.

Proposition:
Suppose the measure condition holds for each $\mu \in$ K. Then for any sequence $\{\mu_i\} \in \mathrm{K}^{\mathbb{N}}$ and any probability measures $\nu, \nu' \in \mathcal{M}$ we have:

$$\mathrm{dist}_{\mathcal{M}}(\mu_{n+m} * \ldots * \mu_{n+1} * \nu, \mu_{n+m} * \ldots * \mu_{n+1} * \nu') \to 0 \ \text{ as } \ m \to \infty,$$

uniformly in (ν, ν') and $n \in \mathbb{N}$.

Central Limit Theorem

Theorem (CLT, La Page, 1982)

Suppose $\{A_i\}$ *are random iid* $SL(d, \mathbb{R})$ *matrices, distributed w.r.t.* μ*, which is strongly irreducible and contracting, and*

$$\int \|A\|^{\gamma} d\mu < \infty$$

for some $\gamma > 0$*. Then there exists* $a > 0$ *such that*

$$\frac{\log \|T_n\| - n\lambda_F}{\sqrt{n}} \xrightarrow{dist} \mathcal{N}(0, a^2).$$

Central Limit Theorem

Conjecture:
Under suitable (non-stationary) assumptions, if $\{A_i\}$ *are random independent (but not neccessarily identically distributed)* $SL(d, \mathbb{R})$ *matrices, distributed w.r.t.* μ_i, $Var(\log \|T_n\|)$ *growth linearly, and*

$$\frac{\log \|T_n\| - L_n}{\sqrt{Var(\log \|T_n\|)}} \xrightarrow{dist} \mathcal{N}(0,1).$$

Work in progress, joint with Victor Kleptsyn and Grigorii Monakov.

Thank you!