Holder regularity of stationary measures

Joint work with A. Gorodetski and G. Monakov

Victor Kleptsyn

CNRS, Institute of Mathematical Research of Rennes, University of Rennes 1

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Not guaranteed to be regular even for minimal analytic diffeomorphisms: Sullivan's example for circle diffeomorphisms (perturbations near periodic points).

Stationary measures

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A *random dynamical system* *on X: a measure μ on $\mathrm{Homeo}(X)$.*

a measure ν on X such that

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In the example above:

$$\nu = \sum_i p_i (f_i)_* \nu.$$

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Can we claim Hölder regularity for stationary measures?

It turns out, that in absence of invariant measures the answer is "Yes".

Main results-1

Definition

$\mathcal{L}(f) := \max(\mathrm{Lip}(f), \mathrm{Lip}(f^{-1}))$.

- *For some $\gamma > 0$, one has $\mathbb{E}_\mu \mathcal{L}(f)^\gamma < \infty$.*
- *There is no probability measure m on M such that $f_* m = m$ for μ-almost all f.*

Then there exist $\alpha > 0$, C such that any μ-stationary measure ν is (C, α)-Hölder.

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Then there exist $\alpha > 0$, C, $\kappa < 1$ such that for any initial measure ν one has

$$\forall n \quad \forall r > \kappa^n \quad \forall x \quad (\mu^{*n} * \nu)(B_r(x)) < Cr^\alpha.$$

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What if we are doing nonstationary iterations? That is: we have a compact $\mathbf{K}$ in the set of measures on $\mathrm{Diff}^1(M)$.

Theorem (A. Gorodetski, V.K., G. Monakov, 2022)

Assume that

- **(finite positive moment)** *For some* $\gamma > 0$, C_0, *one has*

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Assumptions on measures

Proposition

Assume that there is no measure m such that $f_* m = m$ for μ-a.e. f.

Take

$$\nu_{n,0} := m, \quad \nu_{n,j} := f_* \nu_{n,j-1} \quad \text{for } \mu\text{-a.e. } f, \quad j = 1, 2, \dots, n,$$

$$\overline{\nu}_n := \frac{1}{n} \sum_{j=0}^{n-1} \nu_{n,j}.$$

Then any weak limit of $\overline{\nu}_n$ is a common invariant measure. □

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Remark

In other words: Frostman dimension is at least half of the correlation one.

First steps

Theorem

Under our assumptions, there exists $\alpha > 0$, $\lambda < 1$ and C such that

$$\mathcal{E}_\alpha(\mu * \nu) < \lambda \mathcal{E}_\alpha(\nu) + C.$$

Their energies do not tend to infinity, hence the same holds for their Cesaro averages. Extract a convergent subsequence. □

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Start with any initial measure ν_0; consider averaged images $\mu^{*n} * \nu_0$.

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For $\alpha > 0$ and a measure ν, define

$$\rho_\alpha[\nu](y) := \int_M \varphi_\alpha(d(x,y))\, d\nu(x),$$

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□

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Corollary

For $\mathbb{R}^k$, one has an exact equality

$$\widetilde{\mathcal{E}}_\alpha[\nu] = c_\alpha \mathcal{E}_\alpha[\nu]$$

Proof of the theorem

- As α gets smaller, $\mathcal{E}_\alpha$ gets more and more f-invariant:

$$\mathcal{L}(f)^{-\alpha}\mathcal{E}_\alpha(\nu) \leq \mathcal{E}_\alpha(f_*\nu) \leq \mathcal{L}(f)^{\alpha}\mathcal{E}_\alpha(\nu).$$

Proof of the theorem

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- In the former case, we will find a measure with a deterministic image, and the latter is the conclusion of the theorem.

Constructing measures

Definition

Define a non-probability measure:

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be its normalization.

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Proposition

If the conclusion of the theorem does not hold, one has

$$f_*\theta_\alpha[\nu] \approx \theta_\alpha[\mu * \nu]$$

for μ-most f.

Passing to the limit provides $f_*m = m'$ for μ-a.e. f.

General theorem: cut-off

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Define $\rho_{\alpha,\varepsilon}[\nu]$ and $\widetilde{\mathcal{E}}_{\alpha,\varepsilon}[\nu]$ accordingly:

Definition

$$\rho_{\alpha,\varepsilon}[\nu](y) := \int_M \varphi_{\alpha,\varepsilon}(d(x,y))\, d\nu(x),$$

$$\widetilde{\mathcal{E}}_{\alpha,\varepsilon}[\nu] := \int_M \rho_{\alpha,\varepsilon}[\nu]^2(y)\, d\mathrm{Leb}(y).$$

Function $\varphi_{\alpha,\varepsilon}$ belongs to $L_2(M)$, thus the energy $\widetilde{\mathcal{E}}_{\alpha,\varepsilon}$ is always finite (and bounded uniformly from above).

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- For n-fold convolution, we can choose ε so that $\lambda^n \widetilde{\mathcal{E}}_{\alpha,\varepsilon}(\nu) \sim \mathrm{const}$, that is, $\varepsilon = (\lambda^{1/\alpha})^n$.
- Markov inequality: this provides an estimate of measures of $B_r(x)$ for $r > \varepsilon$.

Thank you for your attention!