Thermodynamic formalism for $\mathscr{B}$-free systems

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$\mathscr{B}$-free systems

Class of systems of number theoretic origin that seem interesting both from the number theory and dynamical systems viewpoint.

For each $\mathscr{B} \subseteq \mathbb{N}$ define $\mathcal{M}_{\mathscr{B}} := \bigcup_{b\in\mathscr{B}} b\mathbb{Z}$ (**set of multiples**) and $\mathcal{F}_{\mathscr{B}} := \mathbb{Z} \setminus \mathcal{M}_{\mathscr{B}}$ ($\mathscr{B}$**-free set**)

- each $M \subseteq \mathbb{Z}$ is closed under taking multiples is a set of multiples: $M = \mathcal{M}_M$
- we usually assume that $\mathscr{B}$ is **primitive** ($b|b'$ for $b, b' \in \mathscr{B} \implies b = b'$)
- for any sets of multiples M, there exists a primitive set $\mathscr{B}$ such that $M = \mathcal{M}_{\mathscr{B}}$

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While it's clear that these objects can be of an interest in number theory (introduced and studied first already in the 1930's!), they attracted recently (since 2010) some attention from the dynamical systems point of view. My talk will be devoted mostly to the dynamical aspects, but I will give some examples and applications interesting from number theory or combinatorics viewpoint.

Number-thoretic roots

Square-free integers $\mathscr{B} = \mathbb{P}^2 =$ **squares of primes**.

Then $\mathcal{F}_{\mathscr{B}} =$ **square-free numbers**; $\mathbf{1}_{\mathcal{F}_{\mathscr{B}}} = \mu^2$

- $\mu \colon \mathbb{N} \to \{-1, 0, 1\}$ is the **Möbius function** ($\mu(0) = 0$, $\mu(p_1 \cdot \ldots \cdot p_k) = (-1)^k$, $\mu(p^2 n) = 0$).

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Abundant numbers

Three classes: **abundant numbers A** (sum of divisors $\sigma(n) > 2n$), **perfect numbers P** ($\sigma(n) = 2n$) and **deficient numbers D** ($\sigma(n) < 2n$).

- $\mathbf{A}$ is closed under taking multiples, so $\mathbf{A} = \mathcal{M}_{\mathscr{B}_A}$ for some primitive set $\mathscr{B}_A$
- same for $\mathbf{A} \cup \mathbf{P}$: we have $\mathcal{M}_{\mathscr{B}_A \cup \mathbf{P}}$
- Bessel-Hagen 1929: Does $d(\mathbf{A}) = \lim_{n \to \infty} \frac{1}{n} |\mathbf{A} \cap [1, n]|$ exist?
  Davenport-Erdös-Chowla (independently) 1930: yes

Density questions

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  $\delta(\mathcal{M}_{\mathscr{B}}) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k\leq n, k\in\mathcal{M}_{\mathscr{B}}} \frac{1}{k} = \underline{d}(\mathcal{M}_{\mathscr{B}}) = \lim_{K\to\infty} d(\mathcal{M}_{\mathscr{B}_K})$, where
  $\mathscr{B}_K = \{b \in \mathscr{B} : b < K\}$

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  $\mathscr{B}_K = \{b \in \mathscr{B} : b < K\}$
- Erdős 1948: $d(\mathcal{M}_{\mathscr{B}})$ exists iff

$$\lim_{\varepsilon\to 0} \limsup_{x\to\infty} x^{-1} \sum_{x^{1-\varepsilon} < a \leq x, a\in\mathscr{B}} M(x, a, \mathscr{B}) = 0,$$

where $M(x, a, \mathscr{B}) = |\{1 \leq n \leq x : a|n \text{ and } n \in \mathcal{F}_{\mathscr{B}_a}\}|$.

Some important classes of $\mathscr{B}$'s

- **Besicovitch**: whenever $d(\mathcal{M}_{\mathscr{B}})$ exists
- **Erdős condition**: $\mathscr{B}$ infinite, pairwise coprime, $\sum_{b\in\mathscr{B}} 1/b < \infty$ (then $\mathscr{B}$ is Besicovitch), e.g. $\mathscr{B} = \mathbb{P}^2$, $d(\mathcal{M}_{\mathscr{B}}) = \frac{6}{\pi^2}$
- **Behrend**: if $d(\mathcal{M}_{\mathscr{B}}) = 1$ (e.g. $\mathscr{B} = \mathbb{P}'$, where $\mathscr{P}' \subseteq \mathbb{P}$ satisfies $\sum_{p\in\mathbb{P}'} \frac{1}{p} = \infty$)
- **taut**: for any $b \in \mathscr{B}$, we have $\delta(\mathcal{M}_{\mathscr{B}\setminus\{b\}}) < \delta(\mathcal{M}_{\mathscr{B}})$
  - $\mathscr{B}$ is taut iff $d\mathcal{A} \not\subseteq \mathscr{B}$ for $\mathscr{A}$ Behrend

Language of dynamics

Let $\eta := \mathbf{1}_{\mathcal{F}_{\mathscr{B}}} \in \{0,1\}^{\mathbb{Z}}$. Let $\sigma\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ stand for the left shift.

Three subshifts: $X_\eta \subseteq \widetilde{X}_\eta \subseteq X_{\mathscr{B}}$

- $X_\eta := \overline{\{\sigma^n \eta : n \in \mathbb{Z}\}}$ ($\mathscr{B}$-**free subshift**)
- $\widetilde{X}_\eta := M(X_\eta \times \{0,1\}^{\mathbb{Z}})$, where $M\colon (\{0,1\}^{\mathbb{Z}})^2 \to \{0,1\}^{\mathbb{Z}}$ is the coordinatewise multiplication (**hereditary closure** of X_η)
- $X_{\mathscr{B}} := \{x \in \{0,1\}^{\mathbb{Z}} : |\mathit{supp}\ x \bmod b| < b - 1 \text{ for each } b \in \mathscr{B}\}$ (**admissible subshift**) $\rightsquigarrow$ admissible subsets of integers

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Dynamical properties of these subshifts $\leftrightarrow$ number theoretic properties of $\mathcal{M}_{\mathscr{B}}$ and $\mathcal{F}_{\mathscr{B}}$:

- patterns,
- recurrence,
- combinatorics.

Consequences of dynamical results – samples: patterns

(A. Dymek, S. Kasjan, JKP, Mariusz Lemańczyk 2018)

If $\mathscr{B}$ is taut, $F \subseteq \mathcal{F}_{\mathscr{B}}$, $M \subseteq \mathcal{M}_{\mathscr{B}}$ are finite then
$\overline{d}(\{n \in \mathbb{N} : F + n \subseteq \mathcal{F}_{\mathscr{B}}, M + n \subseteq \mathcal{M}_{\mathscr{B}}\}) > 0$.

If $\mathscr{B}$ is taut and contains an infinite pairwise coprime subset then
$\limsup_{j\to\infty} \inf_{0\leq k\leq K}(n_{j+k+1} - n_{j+k}) = \infty$, where (n_j) is the sequence of consecutive $\mathscr{B}$-free numbers.

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Primitive abundant numbers $\mathscr{B}_A$ are taut. In fact $\sum_{b\in\mathscr{B}_A} 1/b < \infty$, so, in particular, $\mathscr{B}_A$ is Besicovitch. E.g., we get

- $d\{n \in \mathbb{N} : n+1, \dots, n+5 \in \mathbf{D}\} > 0$
- $\limsup_{j\to\infty} \inf_{0\leq k\leq K}(n_{j+k+1} - n_{j+k}) = \infty$, $(n_j) =$ consecutive deficient numbers

Consequences of dynamical results – samples: recurrence

(V. Bergelson, JKP, Mariusz Lemańczyk, F.K. Richter 2018)

$R \subseteq \mathbb{N}$ is a **set of recurrence** if for each measure-preserving system (X, μ, T) and each A with $\mu(A) > 0$, there exists $n \in R$ such that $\mu(A \cap T^{-n}A) > 0$.

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$R \subseteq \mathbb{N}$ is an **averaging set of polynomial multiple recurrence** if for each (X, μ, T) and each A with $\mu(A) > 0$, $\ell \in \mathbb{N}$ and any polynomials $p_i \in \mathbb{Q}[t]$ with $p_i(\mathbb{Z}) \subseteq \mathbb{Z}$ and $p_i(0) = 0$ $(1 \leq i \leq \ell)$,

$$\lim_{N\to\infty} \frac{1}{|R \cap [1, N]|} \sum_{n=1}^{N} \mathbf{1}_R(n)\mu(A \cap T^{-p_1(n)}A \cap \ldots \cap T^{-p_\ell(n)}(A)) > 0.$$

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Theorem: Suppose that $\mathscr{B}$ is Besicovitch, not Behrend. Then there exists $D \subseteq \mathcal{F}_{\mathscr{B}}$ such that $d(\mathcal{F}_{\mathscr{B}} \setminus D) = 0$ such that $r \in D \iff \mathcal{F}_{\mathscr{B}} - r$ is an averaging set of polynomial multiple recurrence.

Dynamics for Behrend sets

If $\mathscr{B}$ is Behrend – unclear how "rich dynamically" is the corresponding subshift X_η (the only invariant measure is $\delta_{\ldots 000 \ldots}$).

Examples:

- we have $\mathcal{F}_{\mathbb{P}} = \{\pm 1\}$.
- $\mathbb{P}_2 = \{pq : p, q \in \mathbb{P}\} \implies \mathcal{F}_{\mathbb{P}_2} = \mathbb{P} \cup (-\mathbb{P}) \cup \{\pm 1\}$. $\implies$ studying $\mathcal{F}_{\mathscr{B}}$ in whole generality must be hard! $X_{\mathbf{1}_{\mathcal{F}_{\mathbb{P}_2}}}$ = subshift of prime numbers
- $\mathbb{P}_3 = \{pqr : p, q, r \in \mathbb{P}\} \implies \mathcal{F}_{\mathbb{P}_3} = (\mathbb{P} \cup (-\mathbb{P})) \cup (\mathbb{P}_2 \cup (-\mathbb{P}_2)) \cup \{\pm 1\}$

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Problem: what is the cardinality of $X_{\mathbf{1}_{\mathcal{F}_{\mathbb{P}_2}}}$?

Both, **prime k-tuples conjecture** and **Dickson's conjecture** imply $X_{\mathbb{P}} \subseteq X_{\mathbf{1}_{\mathcal{F}_{\mathbb{P}_2}}}$. In particular, the subshift of primes is uncountable ($\mathfrak{c}$) since $X_{\mathbb{P}}$ is uncountable.

Unconditional proof was recently (2023) found by T. Tao and T. Ziegler.

Dynamics for non-Behrend sets

P. Sarnak (2010): study X_η for $\mathscr{B} = \mathbb{P}^2$ (i.e. the **square-free system**). It might give us knowledge about μ!

- X_{μ^2} is a **topological factor** of $X_\mu = \overline{\{\sigma^n \mu : n \in \mathbb{Z}\}}$ via $\pi \colon X_\mu \to X_{\mu^2}$ given by $\pi(x) = |x|$ (clearly, $\sigma \circ \pi = \pi \circ \sigma$)

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Results for μ^2:

- μ^2 is a generic point: $\frac{1}{N}\sum_{n\leq N} \delta_{\sigma^n \mu^2} \nu_{\mu^2}$ (**Mirsky measure** – *ergodic, zero entropy, discrete spectrum*)
- $X_{\mu^2} = X_{\mathbb{P}^2}$ and $X_{top}(X_{\mu^2}) = \frac{6}{\pi^2}$
- X_{μ^2} is proximal: for any $x, y \in X_{\mu^2}$, there exists (n_k) with $d(\sigma^{n_k} x, \sigma^{n_k} y) \to 0$

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El Abdalaoui, Mariusz Lemańczyk, de la Rue: Sarnak's program in the Erdős case (2013/2015). Then M. Boshernitzan during a conference in Toruń (2014) asked what happens for general $\mathscr{B}$'s ... $\rightsquigarrow$ rich intersting class, with new phenomena...

Topological dynamics phenomena: proximality vs minimality

$T \colon X \to X$ is **proximal** if for all (x, y), we have $\liminf_{n\to\infty} d(\sigma^n x, \sigma^n y) = 0$.

Theorem (A. Dymek, S. Kasjan, JKP, Mariusz Lemańczyk 2018): X_η is proximal $\iff$ $\ldots 000 \ldots \in X_\eta \iff \mathscr{B} \supseteq \mathcal{A}$ (infinite pairwise coprime).

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Proximality is an "opposite" notion to minimality. $T\colon X \to X$ is **minimal** if there is no closed invariant $\emptyset \neq A \subsetneq X$.

- the only system that is both minimal and proximal is the 1-pt dynamical system: if x and Tx are proximal, we get a fixed point and two distinct points are never proximal.

Topological dynamics phenomena: proximality vs minimality

(A. Dymek, S. Kasjan, G. Keller, JKP, Mariusz Lemańczyk 2017-2022)

Theorem: X_η is minimal $\iff$ η is z Toeplitz sequence (for all $n \in \mathcal{F}_{\mathscr{B}}$ there exists $s \geq 1$ such that $n + s\mathbb{Z} \subseteq \mathcal{F}_{\mathscr{B}}$ $\iff$ $c\mathcal{A} \not\subseteq \mathscr{B}$ for infinite pairwise coprime $\mathcal{A}$).

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$T: X \to X$ is **essentially minimal** if there is a unique minimal subset $A \subseteq X$.

<u>Theorem</u>: Each X_η is essentially minimal. There exists $\mathscr{B}^*$ such that

- $X_{\eta^*} \subseteq X_\eta$ is the unique minimal subset of X_η
- $\mathscr{B}^* = (\mathscr{B} \cup C)^{prim}$, where $C = \{c \in \mathbb{N} : c\mathcal{A} \subseteq \mathscr{B}$ for an inf. pairwise coprime $\mathcal{A}\}$

E.g. for any Behrend set $\mathscr{B}$, we have $\mathscr{B}^* = \{1\}$.

Toeplitz sequence x if **regular** if $d(\{n : x(n)$ is periodic with a period $\leq K\}) \to 1$.
$\mathscr{B}$-free Toeplitz sequences can be both regular and irregular.

Measure theoretic phenomena: invariant measures

Davenport-Erdős theorem:

$\delta(\mathcal{M}_{\mathscr{B}}) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k\le n, k\in\mathcal{M}_{\mathscr{B}}} \frac{1}{k} = \underline{d}(\mathcal{M}_{\mathscr{B}}) = \lim_{K\to\infty} d(\mathcal{M}_{\mathscr{B}_K})$.

Theorem (A. Dymek, S. Kasjan, JKP, Mariusz Lemańczyk 2018): For (N_k) realizing $\underline{d}(\mathcal{M}_{\mathscr{B}})$, we have $\frac{1}{N_k}\sum_{n\le N_k} \delta_{\sigma^n\eta} \to \nu_\eta$ (frequency of 0-1 blocks in $[1, N_k]$).

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This question appears first in the Erdős case, where $\widetilde{X}_\eta = X_\eta$, so we need to look at $[M_k + 1, M_k + N_k]$ with M_k arbitrary and $N_k \to \infty$.

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Let M be the coordinatewise multiplication of 0-1-sequences. We have $M\colon X_\eta \times \{0,1\}^{\mathbb{Z}} \to \widetilde{X}_\eta$, so $M_*(\nu_\eta \vee \kappa) \in \mathcal{M}(\widetilde{X}_\eta)$.

Theorem (Erdös case: JKP, Mariusz Lemańczyk, B. Weiss 2015, general case: AD, SK, JKP, ML 2018): $\mathcal{M}(\widetilde{X}_\eta) = \{M_*(\nu_\eta\vee\kappa) : \kappa\in\mathcal{M}(\{0,1\}^{\mathbb{Z}})\}$.

Measure theoretic phenomena: tautness

$\mathscr{B}$ is taut if $\delta(\mathcal{M}_{\mathscr{B}\setminus\{b\}}) < \delta(\mathcal{M}_{\mathscr{B}})$ for each $b \in \mathscr{B}$.

Theorem (A. Dymek, S. Kasjan, JKP, Mariusz, Lemańczyk 2018 and 2022): For all $\mathscr{B}$ there exists a (unique!) taut set $\mathscr{B}'$ such that $\nu_\eta = \nu_{\eta'}$. Additionally, we have:

- $\eta' \leq \eta$
- $\mathcal{M}(X_\eta) = \mathcal{M}(X_{\eta'})$
- $\mathscr{B}' = (\mathscr{B} \cup D)^{prim}$, where $D = \{d \in \mathbb{N} : d\mathcal{A} \subseteq \mathscr{B}$ for some Behrend set $\mathcal{A}\}$

(this is simliar to the construction of $\mathscr{B}^*$ yielding the unique minimal susbet of X_η – instead of scales of infinite pairwise coprime subset, we search fo scales of Behrend sets!)

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Theorem (AD, SK, JKP, ML 2018): If $\mathscr{B}_1, \mathscr{B}_2$ are taut then $X_{\mathscr{B}_1} = X_{\mathscr{B}_2} \iff \mathscr{B}_1 = \mathscr{B}_2 \iff \nu_{\eta_1} = \nu_{\eta_2}$.

The first proof (in the Erdős case!) used the intrinsic ergodicity of $X_\eta = \widetilde{X}_\eta$.

Measure theoretic phenomena: entropy

Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be a subshift.

- **Topological entropy** $h_{top}(X)$ = complexity of blocks = $\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}^{(n)}|$, where $\mathcal{L}^{(n)}$ = all 0-1 blocks of length n appearing in X.
- **Variational principle**: $h_{top}(X) = \sup\{h(X,\mu) : \mu \in \mathcal{M}(X)\}$.
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Theorem (A. Dymek, JKP, S. Kasjan, Mariusz Lemańczyk, R.Peckner, B. Weiss 2015 and 2018):

- $h_{top}(\widetilde{X}_\eta) = \overline{d}(\mathcal{F}_{\mathscr{B}})$
- $\widetilde{X}_\eta$ is intrinsically ergodic, the unique measure of maximal entropy: $M_*(\nu_\eta \otimes B_{1/2})$

Measure theoretic phenomena: entropy

Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be a subshift.

- **Topological entropy** $h_{top}(X)$ = complexity of blocks = $\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}^{(n)}|$, where $\mathcal{L}^{(n)}$ = all 0-1 blocks of length n appearing in X.
- **Variational principle**: $h_{top}(X) = \sup\{h(X,\mu) : \mu \in \mathcal{M}(X)\}$.
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Main tool: finite approximations $\mathscr{B}_K = \{b \in \mathscr{B} : b < K\}$ of $\mathscr{B} \rightsquigarrow$ periodic approximation $\eta_K \searrow \eta \rightsquigarrow$ periodic approximation ν_{η_K} of ν_η (weak topology).

Measure theoretic phenomena: convolution systems

$X \subseteq \{0,1\}^{\mathbb{Z}}$ is a **convolution system** if $\mathcal{M}(X) = \{M_*(\nu \vee \kappa) : \kappa \in \mathcal{M}(\{0,1\}^{\mathbb{Z}})\}$.

Proposition (JKP, ML, MR 2022/23): The measure ν is unique and ergodic. It is the unique measure of maximal density. (We call ν the **base measure**.)

$\widetilde{X}_\eta$ is a convolution system with base measure ν_η (earlier slide).

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Theorem (JKP, ML, MR 2023/23): Let X be a convolution system with zero entropy base measure ν. Then X is intrinsically ergodic, with $h_{top}(X) = \nu(1)$ and the unique measure of maximal entropy equals $M_*(\nu \otimes B_{1/2,1/2})$.

Measure theoretic phenomena: entropy

What about X_η?

- $X_{\eta^*} \subseteq X_\eta$, so we need to "control" $\mathcal{M}(X_{\eta^*}) \implies$ assumption: η^* is a regular Toeplitz sequence $\implies$ periodic approximations $\underline{\eta}_k^* \nearrow \eta^*$ such that $\nu_{\underline{\eta}_k^*} \to \nu_{\eta^*}$ (recall that we always have $\eta_k \searrow \eta$)
- G. Keller (2021) proved that X_η is **in a sense** hereditary: if $\mathscr{B}$ is taut then $X_\eta = \overline{\{\sigma^n x : \eta^* \leq x \leq \eta\}}$ (implicit)

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<u>Theorem</u> (A. Dymek, JKP, D. Sell 2023, motivated by a conjecture of Keller 2021): If η^* is a regular Toeplitz sequence then $\mathcal{M}(X_\eta) = \{N_*(\nu_{\eta^*} \vee \nu_\eta \vee \kappa) : \kappa \in \mathcal{M}(\{0,1\}^{\mathbb{Z}})\}$.

- $\nu_{\eta^*} \vee \nu_\eta$ is the weak limit of $\frac{1}{N_k}\sum_{n \leq N_k} \delta_{(\sigma^n \eta^*, \sigma^n \eta)}$
- $N\colon (\{0,1\}^{\mathbb{Z}})^3 \to \{0,1\}^{\mathbb{Z}}$, $N(w,x,y) = yw + (1-y)x$

Moreover, X_η is intrinsically ergodic, with $h_{top}(X_\eta) = \overline{d}(\mathcal{F}_{\mathscr{B}}) - d(\mathcal{F}_{\mathscr{B}^*})$.

Measure-theoretic phenomena: bi-convolution systems

$X \subseteq \{0,1\}^{\mathbb{Z}}$ is a bi-convolution system if $\mathcal{M}(X) = \{N_*(\rho \vee \kappa) : \kappa \in \mathcal{M}(\{0,1\}^{\mathbb{Z}})\}$, where $N(w,x,y) = yw + (1-y)x$.

- If X is a convolution system then there exists ρ satisfying the above such that $\rho(\{(w,x) : w \leq x\}) = 1$. We call each such ρ a **base measure**.
- We don't know if ρ is unique (its marginals are unique and ergodic).

If η^* is a regular Toeplitz sequence then X_η is a bi-convolution system (previous slide).

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If η^* is a regular Toeplitz sequence then X_η is a bi-convolution system (previous slide).

<u>Theorem</u> (JKP, ML, MR 2022/23): Let X be a bi-convolution system with a zero entropy base measure ρ. Then X is intrinsically ergodic, with $h_{top}(X) = \rho(* \times 1) - \rho(1 \times *)$ and the unique measure of maximal entropy equals $N_*(\rho \otimes B_{1/2,1/2})$.

Thermodynamic formalism: absence of Gibbs property

Measure $\mu \in \mathcal{M}(X)$ has **Gibbs property** if there exists $a > 0$ such that $\mu(C) \geq a \cdot 2^{-|C|h_{top}(X)}$ for all blocks C with $\mu(C) > 0$.

- Gibbs property $\implies$ μ is a **Gibbs measure**: additionally, for some $b > 0$, we have $\mu(C) \leq b \cdot 2^{-|C|h_{top}(X)}$ (B. Weiss, M. Hochman).

Motivation:

- for sofic systems, systems with specification property/ies there is a unique measure of maximal entropy and it has (some) Gibbs property.
- B. Weiss: if μ has Gibbs property and is a measure of maximal entropy then X is intrinsically ergodic.

Theorem (R. Pecker 2015: square-free case, JKP, ML 2021): For any $\mathscr{B}$ such that ν_η is non-atomic, the measure of maximal entropy on $\widetilde{X}_\eta$ does not have the Gibbs property.

Thermodynamic formalism: topological pressure $\varphi(x) = \varphi(x_0)$

Let X be a subshift and $\varphi\colon X \to \mathbb{R}$ (**potential**).

Topological pressure $\mathcal{P}_{X,\varphi} = \lim_{n\to\infty} \log \sum_{A\in\mathcal{L}^{(n)}} 2^{\sup_{x\in A} \varphi^{(n)}(x)}$

- **variational principle**: $\mathcal{P}_{X,\varphi} = \sup\{h(X,\mu) + \int_X \varphi \, d\mu : \mu \in \mathcal{M}(X)\}$
- μ is a **Gibbs measure** corresponding to φ if there exists $c > 0$ and P such that $c^{-1} \leq \frac{\mu(x[0,n-1])}{2^{\varphi^{(n)}(x)-nP}} \leq c$ for any $C \in \mathcal{L}^{(n)}$ with $\mu(C) > 0$ and $x \in C$. Then $P = \mathcal{P}_{X,\varphi}$.

Theorem (JKP, ML, MR 2022/23): For any $\mathscr{B}$ and $\varphi(x) = \varphi(x_0)$, we have $\mathcal{P}_{\widetilde{X}_\eta,\varphi} = (1-d)\varphi(0) + d\log(2^{\varphi(0)} + 2^{\varphi(1)})$, where $d = \nu_\eta(1)$. Moreover, there is a unique equilibrium measure.

- This result is more general. We don't need the special structure of $\widetilde{X}_\eta$ for this $\rightsquigarrow$ convolution / bi-convolution systems.

Thermodynamic formalism: topological pressure $\varphi(x) = \varphi(x_0, x_1)$

Let now $\varphi(x) = \varphi(x_0, x_1)$ and let $M := \begin{pmatrix} 2^{\varphi(0,0)} & 2^{\varphi(0,1)} \\ 2^{\varphi(1,0)} & 2^{\varphi(1,1)} \end{pmatrix}$

$\det M = 0 \implies$ (for **any subshift**!) we are back to $\psi(x) = \psi(x_0)$ since $\int \varphi \, d\mu = \int \psi \, d\mu$, where $\psi(i) = \varphi(i, i)$.

- Use $\varphi(0,0) + \varphi(1,1) = \varphi(1,0) + \varphi(0,1)$ and $\mu(01) = \mu(10)$.

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$\det M \neq 0 \implies \mathcal{P}_{\{0,1\}^{\mathbb{Z}}} = \log \lambda^+$, where $|\lambda^-| < \lambda^+$ are the eigenvalues of M.

- Walters' method for finite type shifts.
- $\widetilde{X}_{\eta_k}$ is a factor of a finite type subshift (with finite fibers!). Each 0-1 block w yields the following edge Markov shift: if $w_j = 0$ then the corresponding edge is single (labeled with a_j), if $w_j = 1$ then the edge is double (labeled with a_j and c_j). The factor (corresp. to $\widetilde{X}_{\eta_k}$ when $\eta_k = w^\infty$) arises by taking $a_j \mapsto 0$, $c_j \mapsto 1$.
- The arising matrices are huge... $\rightsquigarrow$ relations between the eigenvalues = ???

Thermodynamic formalism: topological pressure $\varphi(x) = \varphi(x_0, x_1)$

$\varphi(x) = \varphi(x_0, x_1)$, $\det M \neq 0$. Second attempt.

$\mathcal{P}_{\{0,1\}^{\mathbb{Z}},\varphi} = \lim_{n\to\infty} \log \sum_{A\in\{0,1\}^{n-1}} 2^{\varphi^{n)}(x)} = \lim_{n\to\infty} \frac{1}{n} \sum_{A\in\{0,1\}^{n-1}} 2^{\varphi^{(n)}(0A0)}$.

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Let $Z_n^0 = Z_n^{0,0} := \sum_{A\in\{0,1\}^{n-1}} 2^{\varphi^{(n)}(0A0)}$, $Z_n^1 = Z_n^{1,0} := \sum_{A\in\{0,1\}^{n-1}} 2^{\varphi^{(n)}(1A0)}$. We have the following recurrence relations ($Z_0^0 = 1, Z_0^1 = 0$):

- $\begin{pmatrix} Z_n^0 \\ Z_n^1 \end{pmatrix} = M \begin{pmatrix} Z_{n-1}^0 \\ Z_{n-1}^1 \end{pmatrix} = \cdots = M^n \begin{pmatrix} Z_0^0 \\ Z_0^1 \end{pmatrix}$.

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Since $M = I \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} I^{-1}$, we get $Z_n^0 = c_1(\lambda^+)^n \left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^n\right)$ for some $c_1, c_2 > 0$ that depend only on φ. Hence $\mathcal{P}_{\{0,1\}^{\mathbb{Z}},\varphi} = \lim_{n\to\infty} \frac{1}{n} \log Z_n^0 = \log \lambda^+$.

Thermodynamic formalism: topological pressure $\varphi(x) = \varphi(x_0, x_1)$

Theorem (JKP, ML, MR 2022/23):

$$\mathcal{P}_{\widetilde{X}_{\eta},\varphi} = \sum_{\ell=1}^{b_1} \nu(B_\ell 0) \log Z_\ell^0 = \log \lambda^+ + (1-d) \log c_1 + \sum_{\ell=1}^{b_1} \nu_\eta(B_\ell 0) \log \left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right),$$

where $B_\ell = 01\ldots1 \in \{0,1\}^\ell$ and $Z_\ell^0 = \sum_{A\in\{0,1\}^{\ell-1}} 2^{\varphi^{(\ell)}(0A0)}$. Constants $c_1, c_2 > 0$ depend only on φ.

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Step 1. Reduction to finite $\mathscr{B}$.

Theorem: We have $\widetilde{X}_\eta \subseteq \cdots \subseteq \widetilde{X}_{\eta_{k+1}} \subseteq \widetilde{X}_{\eta_k}$. Moreover, $\mathcal{M}(\widetilde{X}_\eta) = \mathcal{M}(\bigcap_{k\geq 1} \widetilde{X}_{\eta_k})$.

Corollary: $\mathcal{P}_{\widetilde{X}_\eta,\varphi} = \lim_{k\to\infty} \mathcal{P}_{\widetilde{X}_{\eta_k},\varphi}$ (upper semicontinuity of entropy).

Thermodynamic formalism: topological pressure $\varphi(x) = \varphi(x_0, x_1)$

Step 2. Proof for finite $\mathscr{B}$.

Let s :=period of η. Then

$$\mathcal{P}_{\widetilde{X}_{\eta},\varphi} = \lim_{n\to\infty} \frac{1}{sn} \log \sum_{A \leq \eta[0,sn-1]} 2^{\varphi^{(sn)}(A0)}$$

$$= \lim_{n\to\infty} \frac{1}{sn} \log \left(\sum_{A \leq \eta[0,s-1]} \right)^n = \frac{1}{s} \log \sum_{A \leq \eta[0,s-1]} 2^{\varphi^{(s)}(A0)}.$$

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We have $\eta[0, s-1] = B_{\ell_1} \dots B_{\ell_{(1-d)s}}$, where $B_\ell = 01\dots1 \in \{0,1\}^\ell$.

Each $A \le \eta[0, s-1]$ has at least $(1-d)s$ zeros (with $d = \nu_\eta(1)$), whose positions are the same as the positions of zeros on $\eta[0, s-1]$. The remaining positions can be filled either with 0's or 1's in an arbitrary way.

Thermodynamic formalism: topological pressure $\varphi(x) = \varphi(x_0, x_1)$

Recall: $\mathcal{P}_{\widetilde{X}_{\eta,\varphi}} = \frac{1}{s} \log \sum_{A \leq \eta[0,s-1]} 2^{\varphi^{(s)}(A0)}$ and $\eta[0, s-1] = B_{\ell_1} \dots B_{\ell_{(1-d)s}}$.

$Z_\ell^0 = \sum_{A \in \{0,1\}^{\ell-1}} 2^{\varphi^{(\ell)}(0A0)} = \sum_{A \leq B_\ell} 2^{\varphi^{(\ell)}(A0)} \implies \mathcal{P}_{\widetilde{X}_{\eta,\varphi}} = \frac{1}{s} \log \prod_{i=1}^{(1-d)s} Z_{\ell_i}^0$.

First formula

$$\mathcal{P}_{\widetilde{X}_{\eta,\varphi}} = \frac{1}{s} \log \prod_{i=1}^{(1-d)s} Z_{\ell_i}^0 = \frac{1}{s} \sum_{i=1}^{(1-d)s} \log (Z_\ell^0)^{\#\{1 \leq i \leq (1-d)s : \ell_i = \ell\}}$$

$$= \sum_{\ell=1}^{b_1} \frac{\#\{1 \leq i \leq (1-d)s : \ell_i = \ell\}}{s} \log Z_\ell^0 = \sum_{\ell=1}^{b_1} \nu_\eta(B_\ell 0) \log Z_\ell^0.$$

Second formula

Follows immediately by $Z_n^0 = c_1 (\lambda^+)^n \left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^m\right)$.

Thermodynamic formalism: "approximation" of $\{0,1\}^{\mathbb{Z}}$ with $\widetilde{X}_{\eta_k}$

Recall: $\mathcal{P}_{\widetilde{X}_{\eta},\varphi} = \log\lambda^+ + (1-d)\log c_1 + \sum_{\ell=1}^{b_1} \nu_\eta(B_\ell 0)\log\left(1 + c_2\left(\frac{\lambda^-}{\lambda^+}\right)^{\ell}\right)$,

Question: what happens to the third term when $\widetilde{X}_\eta$ more and more resembles $\{0,1\}^{\mathbb{Z}}$?

By "resembling" we mean that $\overline{d}(\mathcal{F}_{\mathscr{B}})$ (it is not sufficient to look at $\mathcal{L}^{(n)}(\widetilde{X}_\eta)$ as it equals $\{0,1\}^n$, whenver $\min\mathscr{B}$ is large enough!

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Theorem (JKP, ML, MR 2022/23): For any $\varepsilon \in (0,2)$, within the family of Erdős sets $\mathscr{B}$, as $d \to 1$ we have

$$\sum_{\ell=1}^{b_1} \nu_\eta(B_\ell 0)\log\left(1 + c_2\left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right) \ll (1-d)^\varepsilon.$$

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Tools:

- $d = \nu_\eta(1) = \prod_{i=1}^\infty \left(1 - \frac{1}{b_i}\right)$, $\nu_\eta(B_\ell 0) \leq \sum_{i,j\geq 1} \frac{1}{b_i b_j} = S^2$, where $S = \sum_{i\geq 1} \frac{1}{b_i}$.
- for $d_0 \in (0,1)$, there exists $C > 0$ such that for any $\mathscr{B}$ that is Erdős, with $d \in [d_0, 1]$, we have $1 - d \leq S \leq C(1-d)$

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For $\varepsilon \in (0,2)$, we split our sum at $\ell_0 \simeq (1-d)^{-2+\varepsilon}$.

Part 1. $\left|\sum_{\ell=1}^{\ell_0-1} \nu_\eta(B_\ell 0) \log\left(1 + c_2\left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right)\right| \leq \sum_{\ell=1}^{\ell_0-1} \nu_\eta(B_\ell 0) \left|\log\left(1 + c_2\left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right)\right| \leq \ell_0 \cdot S^2 \cdot \text{const} = \text{const} \cdot (1-d)^\varepsilon$.

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Part 2.

$$\left|\sum_{\ell=\ell_0}^{b_1} \nu_\eta(B_\ell 0) \log\left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right)\right| \leq \sum_{\ell=\ell_0}^{b_1} \nu_\eta(B_\ell 0) \max_{\ell_0 \leq \ell \leq b_1} \left|\log\left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right)\right|$$

$\max_{\ell_0 \leq \ell \leq b_1} \left|\log\left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right)\right| = \left|\log\left(1 + c_2 \left(\frac{\lambda^-}{\lambda^+}\right)^{\ell_0}\right)\right|$ (ℓ_0 large $\iff$ $1-d$ small).

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$\max_{\ell_0 \leq \ell \leq b_1} \left|\log\left(1 + c_2\left(\frac{\lambda^-}{\lambda^+}\right)^\ell\right)\right| = \left|\log\left(1 + c_2\left(\frac{\lambda^-}{\lambda^+}\right)^{\ell_0}\right)\right|$ (ℓ_0 large $\iff$ $1-d$ small).

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Question: What happend for non-Erdős $\mathscr{B}$?

Thank you!