Restricted variational principle of Lyapunov exponents for typical cocycles

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Papers

- R. Mohammadpour, Entropy spectrum of Lyapunov exponents for typical cocycles, ArXiv:2210.11574.
- R. Mohammadpour, Restricted variational principle of Lyapunov exponents for typical cocycles, ArXiv:2301.01721.

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- (Σ, T) is either a topologically mixing subshift of finite type or a full shift.
- $\mathcal{M}(\Sigma, T)$= the space of all T-invariant Borel probability measures on Σ.This space is a nonempty convex set and is compact with respect to the weak-$*$ topology.

Variational principle

Let $f : \Sigma \to \mathbb{R}$ be a continuous function over (Σ, T). The pressure $P : C(\Sigma) \to \mathbb{R}$ defined by

$$P(f) := \sup_{\mu \in \mathcal{M}(\Sigma, T)} \left\{ h_\mu(T) + \int f d\mu \right\}. \tag{1.1}$$

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If the supremum is attained, then such measures will be called *equilibrium state*. When $f \equiv 0$, the pressure $P(0)$ is equal to the topological entropy $h_{\text{top}}(\Sigma, T)$, which measures the complexity of the system (Σ, T). By (1.1),

$$h_{\text{top}}(T) := h_{\text{top}}(\Sigma, T) = \sup_{\mu \in \mathcal{M}(\Sigma, T)} h_\mu(T). \tag{1.2}$$

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If μ is an ergodic invariant probability measure, then the Birkhoff average converges to $\int f d\mu$ for μ-almost all points, but there are plenty of ergodic invariant measures, for which the limit exists but converges to a different quantity. Furthermore, there are plenty of points which are not generic points for any ergodic measure or even for which the Birkhoff average does not exist.

Level set

Therefore, one may ask about the size of the set of points

$$E_f(\alpha) = \left\{ x \in \Sigma : \frac{1}{n} S_n f(x) \to \alpha \ \text{as} \ n \to \infty \right\},$$

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$$L = \left\{ \alpha \in \mathbb{R} : \exists x \in \Sigma \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n f(x) = \alpha \right\},$$

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This type of question was considered by Barreira and Saussol ('01). There is actually quite a large literature on multifractal analysis (or multifractal formalism) which addresses various questions related to this one. Pesin, Weiss, Olsen, Barreira, Saussol, Feng, Fan, Schmeling, Climenhaga, Kucherenko, Wolf and ...

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Matrix(linear) cocycles

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$$(x, v) \mapsto (T^n(x), \mathcal{A}^n(x)v).$$

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We denote by $\mathcal{L}$, and $\mathcal{L}_n$ the set of words, and the set of words with the length n, respectively. Let $(\mathcal{A}, T)$ be a one-step cocycle. For any $n \in \mathbb{N}$ and $I = i_0 i_1 \ldots i_{n-1} \in \mathcal{L}_n$, we define

$$\mathcal{A}_I = A_{i_{n-1}} \ldots A_{i_0}.$$

Lyapunov exponents

Let $\mathcal{A} : X \to GL(d, \mathbb{R})$ be a matrix cocycle over (Σ, T). By Kingman's subadditive ergodic theorem, for any $\mu \in \mathcal{M}(\Sigma, T)$ and μ almost every $x \in X$ such that $\log^+ \|\mathcal{A}\| \in L^1(\mu)$, the following limit, called the *top Lyapunov exponent* at x, exists:

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Let us denote $\chi(\mu, \mathcal{A}) = \int \chi(., \mathcal{A}) d\mu$. If the measure μ is ergodic then $\chi(x, \mathcal{A}) = \chi(\mu, \mathcal{A})$ for μ-almost every $x \in \Sigma$.

Level sets of the top Lyapunov exponent

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$h_{\text{top}}(E(\alpha)) := h_{\text{top}}(T_{|E(\alpha)})$.

All Lyapunov exponents

Let $(\mathcal{A}, T)$ be matrix cocycle. Let μ be an T-invariant measure. By Oseledets' theorem, there might exist several Lyapunov exponents. We denote by $\chi_1(x, \mathcal{A}) \geqslant \chi_2(x, \mathcal{A}) \geqslant \ldots \geqslant \chi_d(x, \mathcal{A})$ the Lyapunov exponents, counted with multiplicity, of the cocycle $(\mathcal{A}, T)$. Also, we denote $\chi_i(\mu, \mathcal{A}) := \int \chi_i(x, \mathcal{A}) d\mu$. Therefore, one may ask the size of the $\vec{\alpha}$-level set.

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$$E(\vec{\alpha}) = \left\{ x \in \Sigma : \frac{1}{n} \log \sigma_i(\mathcal{A}^n(x)) \to \alpha_i \text{ as } n \to \infty \right\},$$

$$\vec{L} = \left\{ \vec{\alpha} \in \mathbb{R}^d : \exists x \in \Sigma \text{ such that } \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(\mathcal{A}^n(x)) = \alpha_i \right\}.$$

$$\vec{\Omega} := \{(\chi_1(\mu, \mathcal{A}), \chi_2(\mu, \mathcal{A}), ..., \chi_d(\mu, \mathcal{A})) : \mu \in \mathcal{M}(\Sigma, T)\}.$$

Dominated cocycles-Bochi and Gourmelon('09)

We say that a matrix cocycle $\mathcal{A} : X \to GL(d, \mathbb{R})$ over a homeomorphism map (X, T) is *dominated* with index i if there exist constants $C > 1$, $0 < \tau < 1$ such that

$$\frac{\sigma_{i+1}(\mathcal{A}^n(x))}{\sigma_i(\mathcal{A}^n(x))} \leqslant C\tau^n, \ \ \forall n \in \mathbb{N}, x \in X.$$

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Let $\mathbf{A}$ be a compact set in $GL(d, \mathbb{R})$. We say that $\mathbf{A}$ is *dominated* of index i iff there exist $C > 0$ and $0 < \tau < 1$ such that for any finite sequence $A_1, \dots, A_N$ in $\mathbf{A}$ we have

$$\frac{\sigma_{i+1}\left(A_1 \cdots A_N\right)}{\sigma_i\left(A_1 \cdots A_N\right)} < C\tau^N.$$

We say that $\mathbf{A}$ is dominated iff it is dominated of index i for each i.

A one step cocycle $\mathcal{A}$ generated by $\mathbf{A}$ is dominated if $\mathbf{A}$ is dominated.

Theorem (Barreira and Gelfert, CMP('06))

Let Λ be a repeller of a $C^{1+\alpha}$ map $f:\mathbb{R}^2\to\mathbb{R}^2$ such that:

- *d_xf has bounded distortion on Λ;*
- *d_xf is dominated,*

Then for each $q\in\mathbb{R}^2$ and each $\vec{\alpha}\in\nabla P(\langle q,(\log\sigma_1(d_xf),\log\sigma_2(d_xf)\rangle)$,

$$h_{top}(E(\vec{\alpha}))=\inf_{q\in\mathbb{R}^2}\{P(\langle q,(\log\sigma_1(d_xf),\log\sigma_2(d_xf))\rangle)-\langle q,\vec{\alpha}\rangle\}.$$

Legendre transform

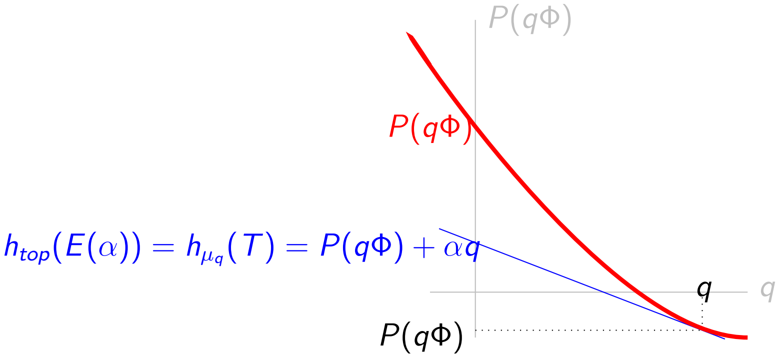

Figure: $P(q\Phi)$ is a convex function for $q \in \mathbb{R}$. The blue line is tangent to $P(q\Phi)$ at q with slope $-\alpha = P'(q\Phi)$.

A crucial technique in their work

Barreira and Gelfert considered a C^1 local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, and a compact f-invariant set $\Lambda \subset \mathbb{R}^2$. They showed that if there is a dominated splitting $T_\Lambda \mathbb{R}^2 := E(x) \oplus F(x)$, then there exists $C \geqslant 1$ such that for every $x \in \Lambda$ and $n, m \in \mathbb{N}$ we have

$$C^{-1} \sigma_i (d_x f^n) \sigma_i (d_{f^n x} f^m) \leqslant \sigma_i \left(d_x f^{n+m} \right) \leqslant \sigma_i (d_x f^n) \sigma_i (d_{f^n x} f^m) .$$

Almost additivity of all singular values

Theorem (M('22), JSP)

Let X be a compact metric space, and let $\mathcal{A} : X \to GL(d, \mathbb{R})$ be a matrix cocycle over a homeomorphism (X, T). Assume that the cocycle $\mathcal{A}$ is dominated with index 1. Then, there exists $\kappa > 0$ such that for every $m, n > 0$ and for every $x \in X$ we have

$$||\mathcal{A}^{m+n}(x)|| \geqslant \kappa ||\mathcal{A}^m(x)|| \cdot ||\mathcal{A}^n(T^m(x))||.$$

Entropy spectrum of all almost additive potentials

Theorem (Feng and Huang('10), CMP)

Let (Σ, T) *be a topologically mixing SFT, and* $\Phi_i := \{\log \phi_n^i\}_{n\in\mathbb{N}}$ $(i = 1, \ldots, d)$ *be an almost additive potentials on* Σ. *Then*

$$h_{top}(E(\vec{\alpha})) = \inf_{q\in\mathbb{R}^d} \left\{ P(\sum_{i=1}^{d} q_i \log \Phi_i) - \langle q, \vec{\alpha}\rangle \right\}$$
$$= \sup\{h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), (\chi(\mu, \Phi_1), \ldots, \chi(\mu, \Phi_d)) = \vec{\alpha}\}$$

for any $\vec{\alpha} \in \vec{\Omega}$.

Generic matrix cocycles

Theorem (Feng ('09), Isr. J. Math.)

Assume that $(A_1, \ldots, A_k) \in GL(d, \mathbb{R})^k$ *generates a one-step cocycle* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$. *Suppose that* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *is irreducible. Then,*

$$h_{top}(E(\alpha)) = \inf_{q \in \mathbb{R}} \{P(q \log \|\mathcal{A}\|) - \alpha q\},$$

for $\alpha \in L$.

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for $\alpha \in L$.

Theorem (M('22), JSP)

Assume that $T : \Sigma \to \Sigma$ *is a topologically mixing subshift of finite type. Suppose that* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *is a typical cocycle. Then,*

$$\begin{aligned} h_{top}(E(\alpha)) &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi(\mu, \mathcal{A}) = \alpha\} \\ &= \inf_{q \in \mathbb{R}} \{P(q \log \|\mathcal{A}\|) - \alpha . q\} \;\; \forall \alpha \in \mathring{\Omega}. \end{aligned}$$

Typical cocycles-Bonatti and Viana ('04), Avila and Viana('07)

We say that a one-step cocycle $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$

- is pinching if there is $I \in \mathcal{L}$ such that the matrix $\mathcal{A}_I$ is simple, where the logarithms $\theta_i = \log \sigma_i (\mathcal{A}_I)$ of the singular values $\sigma_i(\mathcal{A}_I)$ of $\mathcal{A}_I$ satisfy the following inequality

$$\theta_i > \theta_{i+1}.$$

- is twisting if for any $1 \leqslant k \leqslant d-1$, any $F \in Gr(k)$, and any finite $G_1, \ldots, G_n \in Gr(d-k)$, there exists $J \in \mathcal{L}$ such that $\mathcal{A}_J(F) \cap G_i = \{0\}$.

We say that the cocycle $\mathcal{A}$ is *typical* if it is pinching and twisting.

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We say that the cocycle $\mathcal{A}$ is *typical* if it is pinching and twisting. Bonatti–Viana and Avila–Viana showed that the set of typical cocycles is open and dense.

Falconer's singular value function

We define *Falconer's singular value function* $\varphi^s(\mathcal{A})$ as follows. Let $k \in \{0, \dots, d-1\}$ and $k \leqslant s < k+1$. Then,

$$\varphi^s(\mathcal{A}) = \sigma_1(\mathcal{A}) \cdots \sigma_k(\mathcal{A}) \sigma_{k+1}(\mathcal{A})^{s-k},$$

and if $s \geqslant d$, then $\varphi^s(\mathcal{A}) = (\det(\mathcal{A}))^{\frac{s}{d}}$.

For $s := (s_1, \cdots, s_d) \in \mathbb{R}^d$, we define the *generalized singular value function* $\psi^{s_1,\dots,s_d}(\mathcal{A}) : \mathbb{R}^{d\times d} \to [0, \infty)$ as

$$\psi^{s_1,\dots,s_d}(\mathcal{A}) := \sigma_1(\mathcal{A})^{s_1} \cdots \sigma_d(\mathcal{A})^{s_d} = \left(\prod_{m=1}^{d-1} \|\mathcal{A}^{\wedge m}\|^{s_m - s_{m+1}} \right) \left\| \mathcal{A}^{\wedge d} \right\|^{s_d}.$$

When $s \in [0, d]$, the singular value function $\varphi^s(\mathcal{A}(\cdot))$ coincides with the generalized singular value function $\psi^{s_1,\cdots,s_d}(\mathcal{A}(\cdot))$ where

$$(s_1, \dots, s_d) = (\underbrace{1, \dots, 1}_{m \text{ times}}, s-m, 0, \dots, 0),$$

with $m = \lfloor s \rfloor$. We denote $\psi^s(\mathcal{A}) := \psi^{s_1,\cdots,s_d}(\mathcal{A})$.

Main result1

Theorem (M('22))

Assume that $(A_1, \ldots, A_k) \in GL(d, \mathbb{R})^k$ *generates a one-step cocycle* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$. *Let* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *be a typical cocycle. Then*

$$h_{\mathrm{top}}(E(\vec{\alpha})) = \inf_{q \in \mathbb{R}^d} \{P(\log \psi^q(\mathcal{A})) - \langle q, \vec{\alpha} \rangle\}$$

for all $\vec{\alpha} \in \mathring{\vec{L}}$.

Main result2

Theorem (M('23))

Assume that $(A_1, \dots, A_k) \in GL(d, \mathbb{R})^k$ *generates a one-step cocycle* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$. *Let* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *be a typical cocycle. Suppose that*

$$\sup \left\{ h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi_i(\mu, \mathcal{A}) = \alpha_i \right\} = \inf_{q \in \mathbb{R}^d} \left\{ P(\log \psi^q(\mathcal{A})) - \langle q, \vec{\alpha} \rangle \right\},$$

for $\vec{\alpha} \in ri(\vec{\Omega})$, *where* $ri(\vec{\Omega})$ *denotes the relative interior of* $\vec{\Omega}$.

Main result2

Theorem (M('23))

Assume that $(A_1, \ldots, A_k) \in GL(d, \mathbb{R})^k$ *generates a one-step cocycle* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$. *Let* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *be a typical cocycle. Suppose that*

$$\sup \left\{ h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi_i(\mu, \mathcal{A}) = \alpha_i \right\} =$$

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$$\Rightarrow h_{\text{top}}(E(\vec{\alpha})) = \sup \left\{ h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi_i(\mu, \mathcal{A}) = \alpha_i \right\}.$$

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Affirmative answer to Breuillard and Sert's question.

QM

We say that a one-step cocycle $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ is *simultaneously quasi-multiplicative* if there exist $C > 0$ and $k \in \mathbb{N}$ such that for all $I, J \in \mathcal{L}$, there is $K = K(I, J) \in \mathcal{L}_k$ such that $IKJ \in \mathcal{L}$ and for each $i \in \{1, \ldots, d-1\}$, we have

$$\|\mathcal{A}_{IKJ}^{\wedge i}\| \geqslant C \|\mathcal{A}_I^{\wedge i}\| \|\mathcal{A}_J^{\wedge i}\|.$$

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Theorem (Park('20), CMP)

Typical cocycles are simultaneously quasi-multiplicative.

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More information about QM,
R. Mohammadpour and K. Park, Uniform quasi-multiplicativity of locally constant cocycles and applications. ArXiv:2209.08999.

Topological pressure

For any $q \in \mathbb{R}^d$, note that $\psi^q(\mathcal{A})$ is neither submultiplicative nor supermultiplicative. For one-step cocycles, the limsup topological pressure of $\log \psi^q(\mathcal{A})$ can be defined by

$$P^*(\log \psi^q(\mathcal{A})) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(q), \quad \forall q \in \mathbb{R}^d,$$

where $s_n(q) := \sum_{I\in\mathcal{L}_n} \psi^q(\mathcal{A}_I)$. When the limit exists, we denote the topological pressure by $P(\log \psi^q(\mathcal{A}))$.

Topological pressure

For any $q = (q_1, \ldots, q_d) \in \mathbb{R}^d$, we can write

$$\psi^q(\mathcal{A}_{IKJ}) = \underbrace{\prod_{i=1}^{d} \|\mathcal{A}_{IKJ}^{\wedge i}\|^{t_i}}_{(1)},$$

where $t_i = q_i - q_{i+1}$, and $q_{d+1} = 0$ for $1 \leqslant i \leqslant d$.
If $t_i < 0$, then by the sub-multiplicativity property, there is $C_0 > 0$ such that

$$\|\mathcal{A}_{IKJ}^{\wedge i}\|^{t_i} \geqslant C_0^{t_i} \|\mathcal{A}_I^{\wedge i}\|^{t_i} \|\mathcal{A}_J^{\wedge i}\|^{t_i}. \tag{1.3}$$

If $t_i \geqslant 0$, then by the simultaneous quasi-multiplicativity of $\mathcal{A}$, we have

$$\|\mathcal{A}_{IKJ}^{\wedge i}\|^{t_i} \geqslant C^{t_i} \|\mathcal{A}_I^{\wedge i}\|^{t_i} \|\mathcal{A}_J^{\wedge i}\|^{t_i}. \tag{1.4}$$

Topological pressure

By (1.3) and (1.4),

$$(1) \geqslant C_1 \prod_{i=1}^{d} \|\mathcal{A}_I^{\wedge i}\|^{t_i} \prod_{i=1}^{d} \|\mathcal{A}_J^{\wedge i}\|^{t_i},$$

where $C_1 := C_1(C_0^{t_i}, C^{t_i})$. Therefore,

$$\psi^q(\mathcal{A}_{IKJ}) \geqslant C_1 \psi^q(\mathcal{A}_I)\psi^q(\mathcal{A}_J).$$

Then,

$$s_{n+k+m}(q) \geqslant C_1 s_n(q) s_m(q).$$

Upper bound

Theorem (M('22))

Assume that a one-step cocycle $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *is simultaneously quasi-multiplicative. Then,*

$$h_{\mathrm{top}}\left(E(\vec{\alpha})\right) \leqslant \inf_{t \in \mathbb{R}^d} \left\{P\left(\log \psi^t(\mathcal{A})\right) - \langle t, \vec{\alpha} \rangle\right\}$$

for all $\alpha \in \overset{\circ}{\vec{L}}$*.*

Dominated subsystem

Theorem (M('22))

Assume that $(A_1, \ldots, A_k) \in GL(d, \mathbb{R})^k$ *generates a one-step cocycle* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$. *Assume that* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *is a typical cocycle. Then, there exists* $K_0 \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and* $I \in \mathcal{L}_n$ *there exist* $J_2 = J_2(I)$ *and* $J_1 = J_1(I)$ *with* $|J_i| \leqslant K_0$ *for* $i = 1, 2$ *such that the tuple*

$$(\mathcal{A}_{\mathrm{k}})_{\mathrm{k} \in \mathcal{L}^{\mathcal{D}}_{\ell(I)}}, \quad \textit{where } \mathcal{L}^{\mathcal{D}}_{\ell(I)} := \{J_1 I J_2 : I \in \mathcal{L}_n\},$$

is dominated.

For simplicity, we denote by $\ell := \ell(I)$ the length of each $I \in \mathcal{L}^{\mathcal{D}}_{\ell(I)}$, where $\ell \in [n, n + 2K_0]$.

Topological pressure for dominated subsystems

We define the one-step cocycle $\mathcal{B} : (\mathcal{L}_\ell^{\mathcal{D}})^{\mathbb{Z}} \to GL(d, \mathbb{R})$ over a full shift $((\mathcal{L}_\ell^{\mathcal{D}})^{\mathbb{Z}}, f)$ defined by $\mathcal{B}(\omega) := \mathcal{A}_{J_1(I)IJ_2(I)}$, where $\mathcal{B}$ depends only on the zero-th symbol $J_1(I)IJ_2(I)$ of $\omega \in (\mathcal{L}_\ell^{\mathcal{D}})^{\mathbb{Z}}$, is dominated. It is easy to see that $(\mathcal{L}_\ell^{\mathcal{D}})^{\mathbb{Z}} \subset \Sigma$.

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We define a pressure on the dominated subsystem $\mathcal{L}_\ell^{\mathcal{D}}$ by setting

$$P_{\ell,\mathcal{D}}(\log\varphi) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{I_1,\dots,I_k \in \mathcal{L}_\ell^{\mathcal{D}}} \varphi(I_1 \dots I_k),$$

where $\varphi : \mathcal{L} \to \mathbb{R}_{\geqslant 0}$ is submultiplicative, i.e.,

$$\varphi(\mathrm{I})\varphi(\mathrm{J}) \geqslant \varphi(\mathrm{IJ}).$$

for all $I, J \in \mathcal{L}$ with $IJ \in \mathcal{L}$.

Relation between the dominated subsystems and the original system

$$\lim_{\ell \to \infty} \frac{1}{\ell} P_{\ell,\mathcal{D}}(\psi^q(\mathcal{B})) = P(\log \psi^q(\mathcal{A})),$$

$$\lim_{\ell\to\infty}\frac{1}{\ell}P_{\ell,\mathcal{D}}(\psi^q(\mathcal{B}))=P(\log\psi^q(\mathcal{A})),$$

For any $\mu'\in\mathcal{M}((\mathcal{L}^{\mathcal{D}}_{\ell(I)})^{\mathbb{Z}},f)$, there is $\mu\in\mathcal{M}(\Sigma,T)$ such that

$$h_{\mu'}(f)\leqslant(n+2K_0)h_\mu(T)+\frac{n+2K_0}{n}\log(2K_0+1),$$

and

$$\lim_{k\to\infty}\frac{1}{k}\int\log\psi^q(\mathcal{B}^k(x))d\mu'(x)\leqslant(n+2K_0)\lim_{k\to\infty}\frac{1}{k}\int\log\psi^q(\mathcal{A}^k(x))d\mu(x).$$

Proof

$$s_0(\vec{\alpha}) := \inf_{q \in \mathbb{R}^d} \{ P(\log \psi^q(\mathcal{A})) - \langle q, \vec{\alpha} \rangle \},$$
$$s_\ell(\vec{\alpha}) := \inf_{q \in \mathbb{R}^d} \{ P_{\ell,\mathcal{D}}(\psi^q(\mathcal{B}))\rangle) - \langle q, \ell\vec{\alpha} \rangle \},$$

Proof

$$s_0(\vec{\alpha}) := \inf_{q \in \mathbb{R}^d} \{P(\log \psi^q(\mathcal{A})) - \langle q, \vec{\alpha} \rangle\},$$
$$s_\ell(\vec{\alpha}) := \inf_{q \in \mathbb{R}^d} \{P_{\ell,\mathcal{D}}(\psi^q(\mathcal{B}))\rangle) - \langle q, \ell\vec{\alpha} \rangle\},$$

$$\begin{aligned} \frac{1}{\ell} s_\ell(\vec{\alpha}) &= \frac{n+2K_0}{n+2K_0} \frac{1}{\ell} h_{\mathrm{top}}(E^{\ell,\mathcal{D}}(\vec{\alpha})) \\ &\leqslant \frac{n+2K_0}{\ell} h_{\mathrm{top}}(E(\vec{\alpha})) \\ &\leqslant \frac{n+2K_0}{\ell} s_0(\vec{\alpha}). \end{aligned}$$

Therefore,

$$h_{\mathrm{top}}(E(\vec{\alpha})) = s_0(\vec{\alpha}),$$

when $\ell \to \infty$.

Variational principle

Theorem (M('23))

Assume that $(A_1, \ldots, A_k) \in GL(d, \mathbb{R})^k$ *generates a one-step cocycle* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$. *Let* $\mathcal{A} : \Sigma \to GL(d, \mathbb{R})$ *be a typical cocycle. Then,*

$$P(\log \psi^q(\mathcal{A})) = \sup_{\mu \in \mathcal{M}(\Sigma, T)} \left\{ h_\mu(T) + \lim_{n \to \infty} \frac{1}{n} \int \log \psi^q(\mathcal{A}^n(x)) d\mu(x) \right\}$$

for any $q \in \mathbb{R}^d$.

LT

As an application of Legendre transform, we have

Theorem

Assume that S *is a non-empty, convex set in* $\mathbb{R}^d$ *and let* $g : S \to \mathbb{R}$ *be a concave function. Set*

$$W(x) = \sup\{g(a) + \langle a, x\rangle : a \in S\}, \quad x \in \mathbb{R}^d$$

and

$$G(a) = \inf\{W(x) - \langle a, x\rangle : x \in \mathbb{R}^d\}, \quad a \in S.$$

Then $G(a) = g(a)$ *for* $a \in ri(S)$.

Thanks

Thanks for your attention!