# A variational principle relating self-affine measures and self-affine sets

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# Iterated function systems and their attractors

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- By default I will write the transformation $T_i$ as $T_i x = A_i x + v_i$.
- General problem: find the Hausdorff dimensions of the associated *self-affine set* $X = \bigcup_{i\in\mathcal{I}} T_i X$ and *self-affine measures* $m = \sum_{i\in\mathcal{I}} p_i (T_i)_* m$, where $(p_i)_{i\in\mathcal{I}}$ is an arbitrary non-degenerate probability vector.

# The classical self-similar case

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and if $m = \sum_{i \in \mathcal{I}} p_i (T_i)_* m$ then

$$\dim_{\mathrm{H}} m = \sum_{i \in \mathcal{I}} \frac{\log p_i}{\log r_i}.$$

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- Define a continuous function $\pi \colon \Sigma_{\mathcal{I}} \to \mathbb{R}^d$ by

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- The measure $m := \pi_* \nu$ has Hausdorff dimension $\sum_{i\in I} \log p_i / \log r_i$ and support $X$.

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- Covering directly by sets of the form $T_{i_1} \cdots T_{i_n} U$ is no longer useful: these sets are (in general) long and narrow, but the definition of Hausdorff dimension rewards covers which use "round" sets.
- Computing the dimensions of self-affine measures is also much harder and remains a wide open problem in dimension $d \geq 4$.
- If the linear parts $A_i$ are algebraically degenerate (e.g. if they are all diagonal matrices) then various exceptional examples occur (e.g. the "carpet" fractals of Bedford and McMullen and their extensions by Das-Simmons, Feng-Wang, Fraser &c.).

- In 1988, Falconer obtained an upper bound for the dimension of a self-affine set essentially by "chopping" the sets $T_{i_1} \cdots T_{i_n} X$ into round pieces to create a more efficient cover. This gives a bound called the *affinity dimension* of $(T_i)_{i\in\mathcal{I}}$, written $\dim_{\mathrm{aff}}(T_i)_{i\in\mathcal{I}}$.

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- Feng more recently showed that the set of good translation vectors is also residual.
- *Explicit* examples of self-affine sets with known Hausdorff dimension remained rare until the late 2010s (e.g. Hueter-Lalley '95).

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- Given $B \in GL_d(\mathbb{R})$, let $\sigma_1(B) \geq \sigma_2(B) \cdots \geq \sigma_d(B)$ denote the singular values. For each $s \in [0, d]$ define

$$\varphi^s(B) := \sigma_1(B)\sigma_2(B) \cdots \sigma_{\lfloor s \rfloor}(B)\sigma_{\lceil s \rceil}(B)^{s-\lfloor s \rfloor}.$$

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- The affinity dimension $s$ of $(T_i)_{i\in\mathcal{I}}$ is defined to be the unique $s\geq 0$ such that the quantity

$$P((T_i)_{i\in\mathcal{I}};s):=\lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1,\ldots,i_n\in\mathcal{I}}\varphi^s(A_{i_1}\cdots A_{i_n})$$

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$$h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_{\mathcal{I}}} \varphi^s(A_{i_1} \cdots A_{i_n}) d\mu \left[(i_k)_{k=1}^{\infty}\right] = 0.$$

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- He showed that $\dim_{\mathsf{H}} \pi_* \mu \leq \dim_{\mathsf{Lyap}} \mu$ and that there always exists an ergodic *equilibrium state* $\mu$ such that $\dim_{\mathsf{Lyap}} \mu = \dim_{\mathsf{aff}}(T_i)_{i\in\mathcal{I}}$.

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- A possible strategy for the general problem: understand enough about the equilibrium states $\mu$, and the dimensions of measures of the form $\pi_*\mu$, to find a measure on the attractor $X$ with Hausdorff dimension equal to $\dim_{\mathrm{aff}}(T_i)_{i\in\mathcal{I}}$.

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- If we knew that the equilibrium states were Bernoulli measures then we could make the lower and upper bounds meet.
- However, this is *never* the case except when the maps $T_i$ are similitudes (M.-Sert '19).

# Approximating equilibrium states using Bernoulli measures

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- Clearly, $(T_{\mathtt{i}})_{\mathtt{i} \in \mathcal{I}^n}$ is an IFS with the same attractor as $(T_i)_{i \in \mathcal{I}}$. It also has the same affinity dimension.
- By taking $n$ sufficiently large, can we find Bernoulli measures on $\Sigma_{\mathcal{I}^n}$ with Lyapunov dimension close to $\dim_{\mathrm{aff}}(T_i)_{i \in \mathcal{I}}$?

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- This smaller IFS may fail to inherit key algebraic properties from $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{I}^n}$ such as strong irreducibility, which are necessary for theorems on self-affine measures to work.
- We need a theorem showing that the desired *analytic* properties described above can be obtained in a way which ensures that $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{J}}$ has the same *algebraic* features as $(T_i)_{i\in\mathcal{I}}$.

# The variational principle for planar affine IFS

**Theorem (M. - Shmerkin '16)**

*If $(T_i)_{i\in\mathcal{I}}$ is an irreducible affine IFS acting on $\mathbb{R}^2$, then for every $\varepsilon > 0$ we may find $n \geq 1$ and $\mathcal{J} \subset \mathcal{I}^n$ such that:*

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4. *If $(T_i)_{i\in\mathcal{I}}$ satisfies the SOSC then $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{J}}$ satisfies the SSC.*

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This allowed deep results of Bárány, Hochman and Rapaport on planar self-affine *measures* to translate directly into results on planar self-affine *sets*.

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- Using a pigeonhole argument and non-atomicity of the distribution of the Oseledec subspaces, we can do this in a way which ensures that $\{A_{\mathtt{i}} \colon \mathtt{i} \in \mathcal{J}\}$ is an irreducible and dominated semigroup. Strong irreducibility follows.

# Proof idea in the proximal & strongly irreducible case

- Choose an equilibrium state $\mu$. It can be shown to have distinct Lyapunov exponents.
- Using the Subadditive Ergodic Theorem and Shannon-McMillan-Breiman, find a collection $\mathcal{J}$ of at least $e^{n(h(\mu)-\varepsilon)}$ words $\mathtt{i} \in \mathcal{I}^n$ such that $A_{\mathtt{i}}$ has norm $n\varepsilon$-close to the top Lyapunov exponent of $(A_i)_{i\in\mathcal{I}}$ with respect to $\mu$.
- Using a pigeonhole argument and non-atomicity of the distribution of the Oseledec subspaces, we can do this in a way which ensures that $\{A_{\mathtt{i}} \colon \mathtt{i} \in \mathcal{J}\}$ is an irreducible and dominated semigroup. Strong irreducibility follows.
- Control on cardinality of $\mathcal{J}$ and on Lyapunov exponents implies control of the Lyapunov dimension of the measure of maximal entropy on $\Sigma_{\mathcal{J}}$.

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- We need to consider irreducibility and proximality across multiple representations (e.g. different exterior powers).
- There are very few subgroups of $\mathrm{GL}_2(\mathbb{R})$, resulting in what could be seen as a case-by-case argument depending on which linear algebraic group $(A_i)_{i\in\mathcal{I}}$ generates. In general dimensions no analogous case-by-case argument is possible.

# ...and now the result:

## Theorem (M. - Sert '23)

*If* $(T_i)_{i\in\mathcal{I}}$ *is a completely reducible affine IFS acting on* $\mathbb{R}^d$*, then for every* $\varepsilon > 0$ *we may find* $n \geq 1$ *and* $\mathcal{J} \subset \mathcal{I}^n$ *such that:*

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2. *If $G \leq GL_d(\mathbb{R})$ denotes the Zariski closure of the semigroup generated by $\{A_i \colon i \in \mathcal{I}\}$, then the semigroup generated by $\{A_{\mathtt{i}} \colon \mathtt{i} \in \mathcal{J}\}$ is Zariski dense in the identity component of $G$.*

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3. *If $(T_i)_{i\in\mathcal{I}}$ is k-proximal and k-strongly irreducible then $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{J}}$ is k-dominated and k-strongly irreducible.*

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3. *If $(T_i)_{i\in\mathcal{I}}$ is $k$-proximal and $k$-strongly irreducible then $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{J}}$ is $k$-dominated and $k$-strongly irreducible.*
4. *If $(T_i)_{i\in\mathcal{I}}$ satisfies the SOSC then $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{J}}$ satisfies the SSC.*

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**Corollary**

*Let $(T_i)_{i\in\mathcal{I}}$ be a strongly irreducible affine iterated function system acting on $\mathbb{R}^3$ and satisfying the strong open set condition. Then the Hausdorff dimension of the attractor is equal to the affinity dimension of $(T_i)_{i\in\mathcal{I}}$.*

# An overview of the proof (with simplifications)

1. Choose an equilibrium state $\mu$ and use SAET and SMBT as before to find a set $\mathcal{J}_0 \subset \mathcal{I}^n$ of at least $e^{n(h(\mu)-\varepsilon)}$ words $\mathtt{i}$ such that the singular values of every $A_{\mathtt{i}}$ are $n\varepsilon$-close to the respective Lyapunov exponents.

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3. Extending those words by an *a priori* bounded amount, pass to a new set $\mathcal{J}_2 \subset \mathcal{I}^{n+k}$ of at least $e^{n(h(\mu)-3\varepsilon)}$ words which generate a narrow Schottky subsemigroup of the identity component and where the singular values are still $2n\varepsilon$-close to the respective Lyapunov exponents.

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4. Select some additional words $\mathtt{k}_1, \ldots, \mathtt{k}_t$ which, when appended to $\mathcal{J}_3$, ensure that a Zariski-dense subsemigroup of the identity component is generated. (Moreover, do this in such a way that substituting any power of $\mathtt{k}_i$ for the relevant word $\mathtt{k}_i$ has the same effect.)

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5. The set $\mathcal{J}_3 \cup \{\mathtt{k}_1, \ldots, \mathtt{k}_t\}$ no longer consists of words of a consistent length, so choose integers $m$, $r_1, \ldots, r_t$ such that

$$\mathcal{J}_4 = \{\mathtt{i}_1 \cdots \mathtt{i}_m \colon \mathtt{i}_j \in \mathcal{J}_3\} \cup \{\mathtt{k}_1^{r_1}, \ldots, \mathtt{k}_t^{r_t}\}$$

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# An overview of the proof (with simplifications)

6. Some of those words do not have *a priori* control on their singular values, so instead consider

$$\mathcal{J}_5 = \{\mathtt{i}_1 \cdots \mathtt{i}_{m+p} \colon \mathtt{i}_j \in \mathcal{J}_3\} \cup \{\mathtt{i}^p \mathtt{k}_1^{r_1}, \ldots, \mathtt{i}^p \mathtt{k}_t^{r_t}\}$$

where $\mathtt{i} \in \mathcal{J}_3$ is arbitrary, and $p$ is large enough that $A_{\mathtt{i}}^p$ generates a Zariski-connected semigroup, and also large enough that the singular values are $3n\varepsilon$-controlled.

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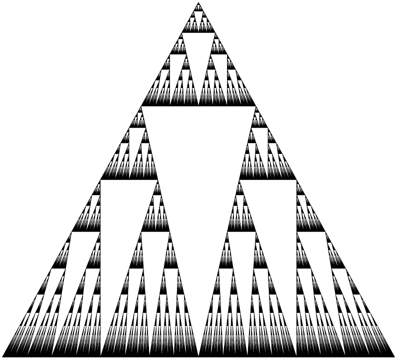
7. The number of elements, their length & singular values are now controlled, and they generate a semigroup which is dominated and has the correct Zariski closure.
8. Control on the number of elements and their Lyapunov exponents implies control on the Lyapunov dimension.

## A list of ingredients (not exhaustive):

- Bochi-Gourmelon: characterisations of domination
- Benoist: finding Zariski dense, narrow Schottky subsemigroups of semigroups of linear maps
- Tits: finding small generating sets for Zariski dense subsemigroups
- Abels-Margulis-Soifer: finding large proximal subsets of semigroups of linear maps
- Guivarc'h-Raugi: separating Lyapunov exponents for Bernoulli measures
- . . . and numerous others for bringing the theory of affine IFS to its present state.



Thanks for listening!