Subshifts with slow forbidden word growth

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Thermodynamic Formalism: Non-additive Aspects and Related Topics
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  - Examples 1,2 are SFTs, Example 3 is not

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  - Such μ called **measure of maximal entropy/MME**

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- We'll take a different approach here

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- It can't approach 0, but ‘slow growth’ with n may be enough

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- Idea was to use such hypothesis to prove unique MME

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- Shows that unique MME not always fully supported ($\mu([0^m]) = 0$)

Application: β-shifts

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- Then sum is $\displaystyle\sum_{n=M}^{\infty} n^2 k (3/k)^{n/3}$, small enough when M large
- In fact, we can also apply our results to some of the much more complicated α-β shifts (coding $x \mapsto \alpha + \beta x \pmod 1$)

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    - Often unique MME is limit of some sort of average over periodic points; this shift has none!

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  - Proof works even if G only has specification, i.e. can have gap of length R indep. of v, w

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- X bounded supermultiplicative: technical argument based on Miller's

Thanks for listening!