Multifractal Analysis of Random Substitutions

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joint w/ Andrew Mitchell

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Multifractal Analysis of Measures

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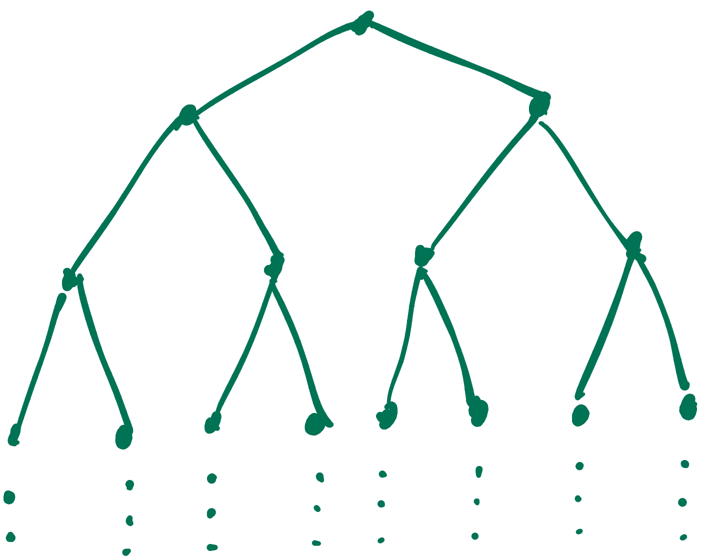
Metric tree Υ



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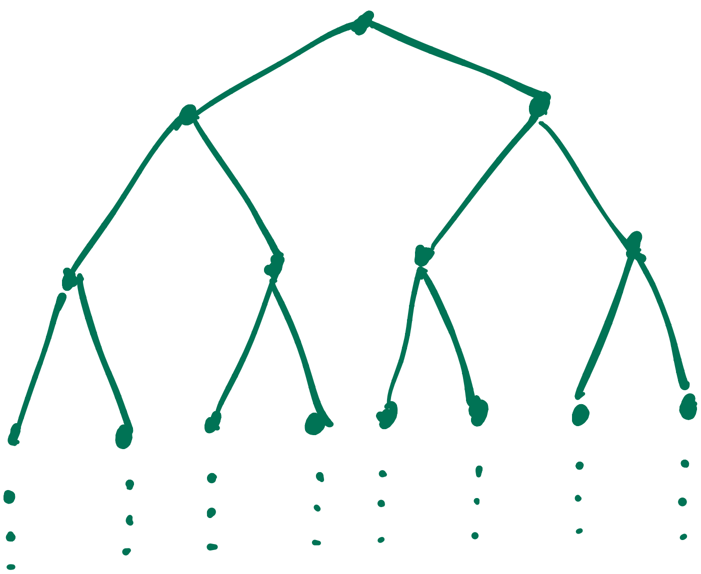
Metric tree Υ



- $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$
- dyadic cubes in $\mathbb{R}^d$
- Markov partition of manifold

Main object of interest:

(Borel) probability measure ν with $\operatorname{supp} \nu \subset \Upsilon$.

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Local dimension of ν at $(i_n)_{n=1}^{\infty} = x$

$$\dim_{loc}(\nu, x) = \lim_{n \to \infty} \frac{\log \nu([i_1 \dots i_n])}{-n}$$

(when the limit exists)

If ν is "well-behaved" (e.g. if there are some dynamics) then

$$\dim_{loc}(\nu, x) = \alpha \quad \text{for } \nu\text{-a.e. } x$$

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What about non-typical points?

Multifractal spectrum

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$$\boxed{\begin{array}{l}\text{Reasonable guess: } f(\alpha) = \tau^*(\alpha) \\ \tau = L^q\text{-spectrum of } \nu.\end{array}}$$

L^q-spectrum

$$\tau(q) = \liminf_{n\to\infty} \frac{-\log \sum_{i_1 \cdots i_n} \nu([i_1 \cdots i_n])^q}{n}$$

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Always holds: $f(\alpha) \leq \tau^*(\alpha)$

(inequality strict in general)

Canonical Example: Bernoulli measure on full shift

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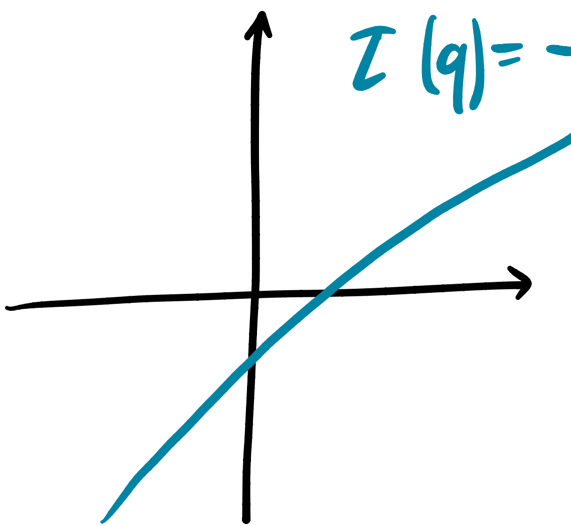
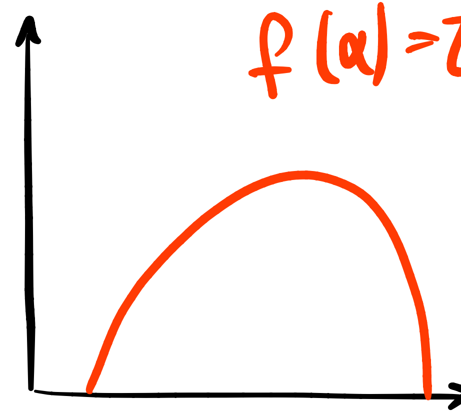
$$\Sigma = \{1, \dots, m\}^{\mathbb{N}}$$

$$\mu = (p_1, \dots, p_m)^{\otimes \mathbb{N}}$$

$$\tau(q) = -\log \sum p_i^q$$



$$f(\alpha) = \tau^*(\alpha)$$



General Questions:

Given a measure ν:

① Can we determine the L^q-spectrum of ν?
$\tau(q)$

② Is there a meaningful relationship between L^q-spectrum and multifractal
$\tau(q)$ $f(\alpha)$ spectrum?

Random substitutions

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$$\begin{cases} 0 \longmapsto 01 \\ 1 \longmapsto 0 \end{cases}$$

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$$\begin{cases} 0 \longmapsto \begin{cases} 01 & p \\ 10 & 1-p \end{cases} \\ 1 \longmapsto 0 \end{cases}$$

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$$\begin{tikzcd}[column sep=small]
 & 01^{p} \arrow[r] \arrow[dr] & 010^{p^2} \\
0 \arrow[ur] \arrow[dr] & & 100^{p(1-p)} \\
 & 10^{1-p} \arrow[r] \arrow[dr] & 001^{(1-p)p} \\
 & & 010^{(1-p)^2}
\end{tikzcd}$$

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$$0 \to 01 \to 010 \to 01001$$

$$\begin{cases} 0 \longmapsto \begin{cases} 01 & p \\ 10 & 1-p \end{cases} \\ 1 \longmapsto 0 \end{cases}$$

$$100 \to \begin{cases} 0\,01\,01 & p^2 \\ 0\,01\,10 & p(1-p) \\ 0\,10\,10 & (1-p)p \\ 0\,10\,10 & (1-p)^2 \end{cases}$$

Frequency measures

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$$\mathrm{freq}(v) = \lim_{k\to\infty} \frac{\mathbb{E}[\#\text{ occurences of } v \text{ in } \vartheta^k(a)]}{\mathbb{E}[\text{length of } \vartheta^k(a)]}$$

Frequency measures

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Extends to ergodic shift-invariant measure on $\mathcal{A}^{\mathbb{Z}}$ satisfying

$$\mu([v]) = \text{freq}(v)$$

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$$0 \longrightarrow \left\{ \begin{matrix} 011 \\ 101 \end{matrix} \right\} \text{\# 0s, \#1s is same.}$$

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$\Longrightarrow$ frequency of letters is uniform

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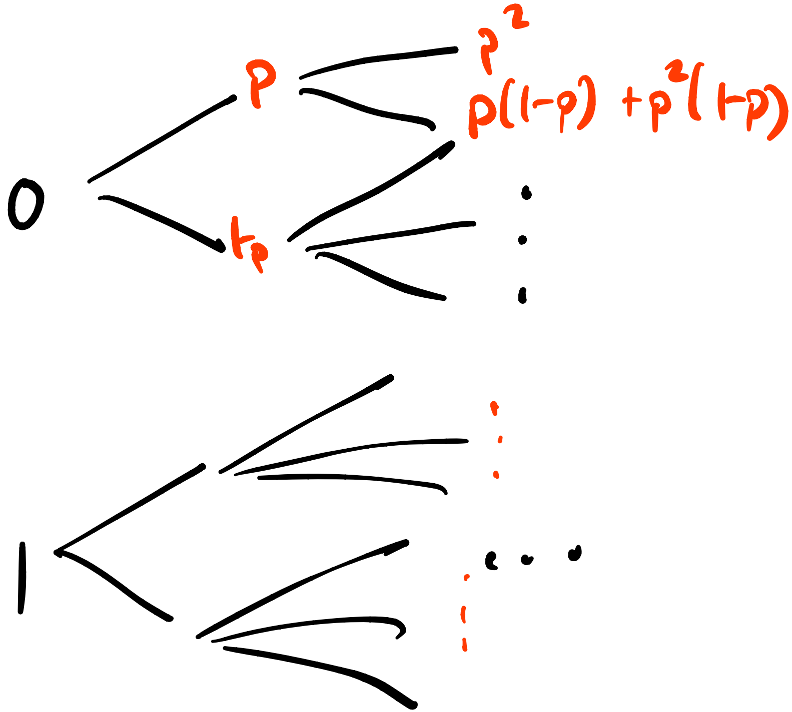
Natural guess: $I(q)$ "fully determined by substitution tree"


0
p
p^2
$p(1-p)+p^2(1-p)$
$1-p$
1

$$T(q) = \liminf_{k \to \infty} \frac{\sum_{a \in A} R_a \log\left(\sum_s \mathbb{P}[\nu^k(a) = s]^q\right)}{\lambda^k}$$

$$T(q) = \liminf_{k\to\infty} \frac{\sum_{a\in A} R_a \log\left(\sum_s \mathbb{P}[v^k(a)=s]^q\right)}{\lambda^k}$$

Challenge: overlaps

Words v can occur as:

(1) images under substitution (OK)

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Challenge: overlaps

Words v can occur as:

(1) images under substitution (OK)

(2) boundary between two words (BAD)

$\ldots 0 \mid 1 \ldots$

$11\underbrace{0 \mid 01}1$ $\rightsquigarrow$ 001 appears on boundary

$\cdots 0 \mid 1 \cdots$

$11\underbrace{0 \mid 01}1$ $\rightsquigarrow$ 001 appears on boundary

<u>Multiple ways</u> for a given word to appear.

Theorem (A. Mitchell + AR)

Assume compatible. Then

$$\tau(q) = T(q)$$

for $q \geq 0$. Moreover, $\tau'(1)$ exists.

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Example: NOT TRUE for $q < 0$.

Recognizable: no boundary effects

e.g.

$$\begin{cases} 0 \longmapsto \begin{cases} 011 \\ 101 \end{cases} \\ 1 \longmapsto 00 \end{cases}$$

Theorem (A. Mitchell + AR)

Assume compatible + recognizable.

Then $T(q) = \tau(q)$ for all $q \in \mathbb{R}$.

Moreover $f(\alpha) = \tau^*(\alpha)$; $T(q)$ analytic.

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Moreover $f(\alpha) = \tau^*(\alpha)$; $T(q)$ analytic.

[moreover $\forall \alpha$ $\exists$ frequency measure ν s.t.

$\dim_{loc}(\nu, x) = \alpha$ for ν-a.e. x and $h(\nu) = f(\alpha)$]

Example: There exist compatible + recognizable systems with (unique) measures of maximal entropy satisfying

$$|v|^{-1}\exp\left(-|v|\cdot h_{top}(\text{subshift})\right) \approx \mathcal{N}([v])$$

for infinitely many v. In particular system can fail specification.