Continuous piecewise linear iterated function systems on the line

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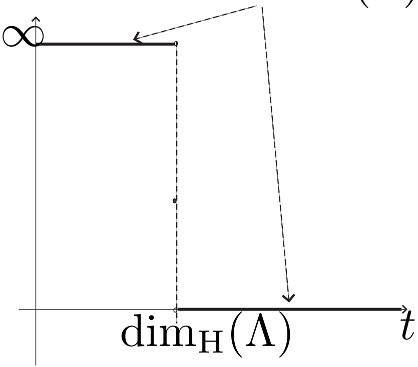
$$(1) \qquad \mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \left\{ \underbrace{\inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : \Lambda \subset \bigcup_{i=1}^{\infty} A_i ; |A_i| < \delta \right\}}_{\mathcal{H}^s_\delta(A)} \right\}.$$

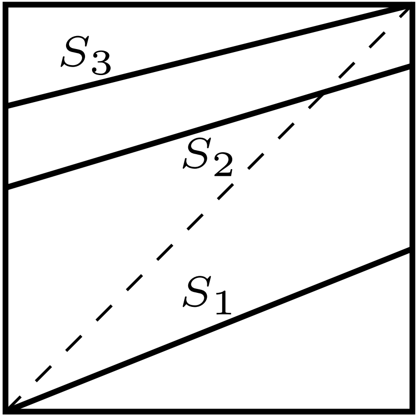
$t \to \mathcal{H}^t(\Lambda)$



Above: $\Lambda \subset \mathbb{R}^d$, $t \geqslant 0$,

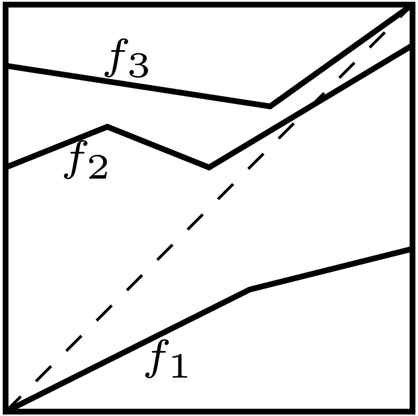
The Hausdorff dimension of Λ

$$\dim_H(\Lambda) = \inf \{t : \mathcal{H}^t(\Lambda) = 0\} = \sup \{t : \mathcal{H}^t(\Lambda) = \infty\}.$$



Self-similar IFS

$\mathcal{S} = \{S_1, S_2, S_3\}$



CPLIFS

$\mathcal{F} = \{f_1, f_2, f_3\}$

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a finite list of strict contractions on $\mathbb{R}$. We call it Iterated Function system (IFS). The attractor $\Lambda^{\mathcal{F}}$ of the IFS $\mathcal{F}$ is the unique non-empty compact set

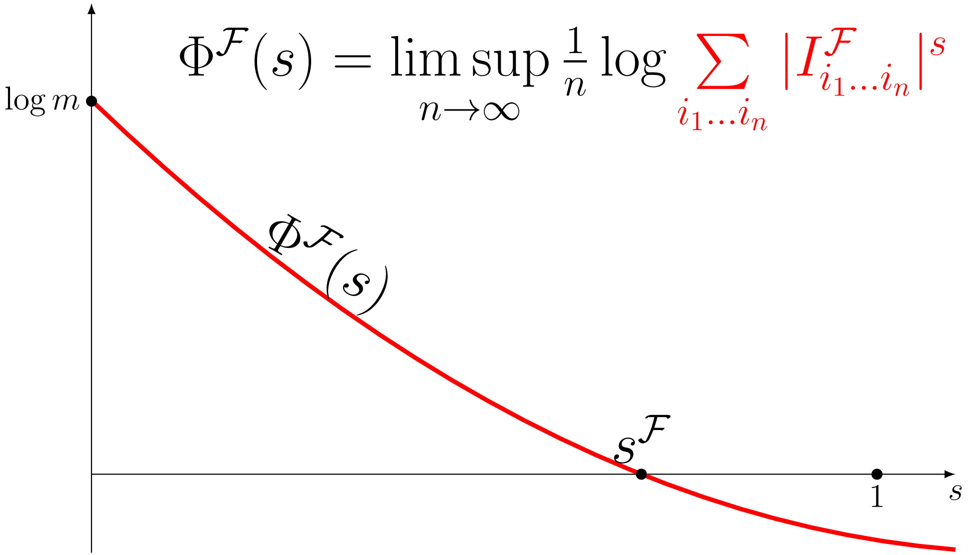
$$\Lambda^{\mathcal{F}} = \bigcup_{k=1}^{m} f_k(\Lambda^{\mathcal{F}}). \tag{2}$$

Let $I^{\mathcal{F}}$ be the smallest non-empty compact interval such that $f_i(I^{\mathcal{F}}) \subset I^{\mathcal{F}}$ for all $i \in [m] := \{1, \ldots, m\}$.

$$\Lambda^{\mathcal{F}} = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\ldots,i_n)\in[m]^n} I^{\mathcal{F}}_{i_1\ldots i_n}, \tag{3}$$

where $I^{\mathcal{F}}_{i_1\ldots i_n} := f_{i_1\ldots i_n}(I^{\mathcal{F}})$ are the `cylinder intervals`, and we use the common shorthand notation $f_{i_1\ldots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$.

$$\Phi^{\mathcal{F}}(s) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\ldots i_n} \left| I^{\mathcal{F}}_{i_1\ldots i_n} \right|^s$$



(4)
$$\Phi^{\mathcal{F}}(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} |I^{\mathcal{F}}_{i_1\dots i_n}|^s.$$

It is easy to see that we can obtain $\Phi^{\mathcal{F}}(s)$ above as a special case of the non-additive upper capacity topological pressure introduced by Barreira[1] in $s \mapsto \Phi^{\mathcal{F}}(s)$ is strictly decreasing, continuous, $\Phi^{\mathcal{F}}(0) = \log m$ and $\Phi^{\mathcal{F}}(s)$ tends to $-\infty$ as $s \to \infty$. So, the zero of $\Phi^{\mathcal{F}}(s)$ is well defined

(5)
$$s_{\mathcal{F}} := (\Phi^{\mathcal{F}})^{-1}(0).$$

(6)
$$\overline{\dim}_{\mathrm{B}}\Lambda^{\mathcal{F}} \leqslant \min\{1, s_{\mathcal{F}}\}.$$

[1] Luis M Barreira. A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. *Ergodic Theory and Dynamical Systems*, 16(5):871–928, 1996

The natural projection

The points of the attractor Λ are coded by the elements of the symbolic space Σ

$$\Sigma := \{\mathbf{i} = (i_1, i_2, \dots) : i_k \in [m]\},$$

where as we mentioned above we write $[m] := \{1, \dots, m\}$ by the natural coding (or natural projection) $\Pi : \Sigma \to \Lambda$

$$\Pi^{\mathcal{F}}(\mathbf{i}) := \lim_{n\to\infty} f_{i_1\dots i_n}(x) = \bigcap_{n=1}^{\infty} I^{\mathcal{F}}_{i_1\dots i_n}, \quad (7)$$

where $x \in \Lambda$ is arbitrary. Clearly, $\Pi^{\mathcal{F}}(\Sigma) = \Lambda$.

Let ρ_k be the maximum (in absolute value) of the slopes of f_k and ρ be the maximum of ρ_k (the greatest slope in the system).

We say that $\mathcal{F}$ is small if both (a) and (b) below hold

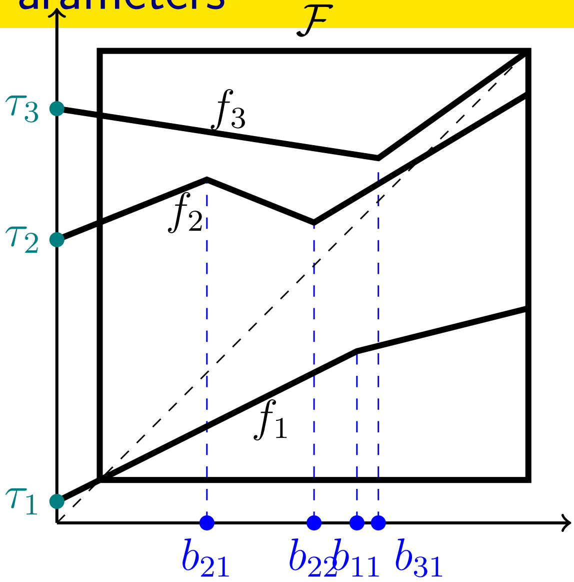
(a) $\sum_{k=1}^{m} \rho_k < 1$, and

(b)
- if all functions of $\mathcal{F}$ are injective then $\rho < \frac{1}{2}$,
- otherwise, $\rho < \frac{1}{3}$.

The main result: for packing dimension typical small CPLIFS $\mathcal{F}$

$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} = s_{\mathcal{F}}. \tag{8}$$

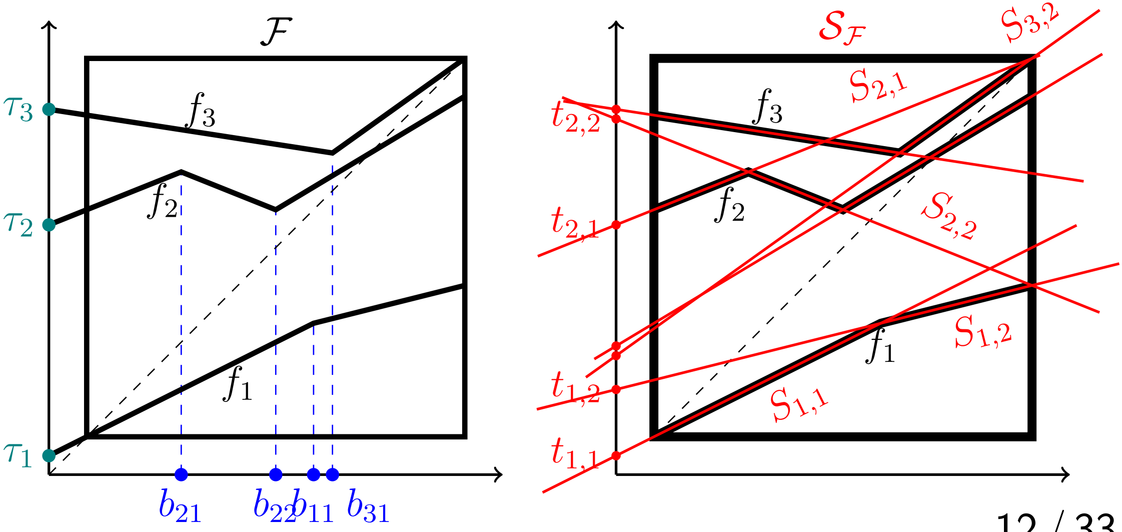
Parameters



The meaning of "packing dimension typical":

We fix all slopes! The parematers are the vertical translations and the breaking points. "Packing dimension typical" means: the packing dimension of the parameters of the exceptional CPLIFS is less than the dimension of the parameter space.

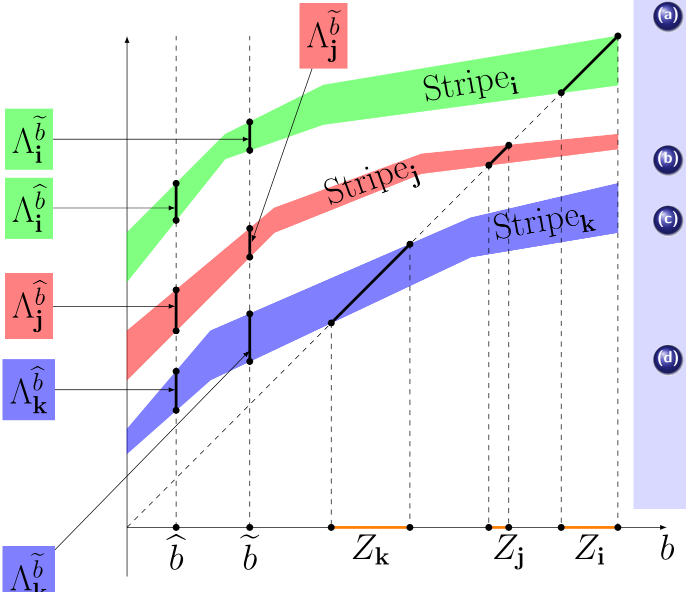
The generated self-similar IFS by a CPLIFS



(a) We verify: If the contraction ratios are small then the attrcator does not contain any breaking points, at least typically.
(b) In this case, there is an N such that there are no breaking points in any level N cylinder intervals.
(c) We consider the N-th iterate of the generated self-similar IFS $S_{\mathcal{F}}$:

$$\mathcal{S}_{\mathcal{F}}^{N} := \{S_{k_1,i_1} \circ \cdots \circ S_{k_N,i_N}\}. \tag{9}$$

(d) We take an appropriate subsystem of $\mathcal{S}_{\mathcal{F}}^{N}$.
(e) We create a graph directed self-similar IFS from the functions of this subsystem of $\mathcal{S}_{\mathcal{F}}^{N}$.
(f) The dimension of an appropriate ergodic measure for this graph directed IFS is the dimension of the attractor of our original CPLIFS.



(a) In this simplified example the total number of breaking points is 1.

(b) $\mathbf{i}, \mathbf{j}, \mathbf{k} \in [m]^n$.

(c) $\Lambda^{\widetilde{b}} \subset \left\{b = \widetilde{b}\right\} \bigcap \bigcup_{\mathbf{i} \in [m]^n} \mathrm{Stripe}_{\mathbf{i}}$

(d) $\widetilde{b} \in \Lambda^{\widetilde{b}} \Longrightarrow \widetilde{b} \in \bigcup_{\mathbf{i} \in [m]^n} Z_{\mathbf{i}}$.

Small CPLIFS

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS on the line. Let ρ_k be the maximal slope of f_k and ρ be the maximum of ρ_k:

(10) $$\rho_k := \max\left\{|f_k'(x)| : x \in I^{\mathcal{F}}\right\}, \quad \rho := \max\{\rho_k\}_{k=1}^m .$$

We say that $\mathcal{F}$ is small if both

(a) $\sum_{k=1}^m \rho_k < 1$, and

(b)
- if all functions of $\mathcal{F}$ are injective then $\rho < \frac{1}{2}$,
- otherwise, $\rho < \frac{1}{3}$.

From now on we always assume that $\mathcal{F}$ is small and there are no breaking points on the attractor.

Exponential Separation Condition (ESC) I

The distance of two similarity mappings $g_1(x) = r_1 x + \tau_1$ and $g_2(x) = r_2 x + \tau_2$, $r_1, r_2 \in (-1,1) \setminus \{0\}$, on $\mathbb{R}$.

(11) $$\operatorname{dist}(g_1, g_2) := \begin{cases} |\tau_1 - \tau_2|, & \text{if } r_1 = r_2; \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 3.1

Given a self-similar IFS $\mathcal{S} = \{S_k(x)\}_{k=1}^{M}$ on $\mathbb{R}$. We say that $\mathcal{F}$ satisfies the Exponential Separation Condition (ESC) if there exists a $c > 0$ and a strictly increasing sequence of natural numbers $\{n_\ell\}_{\ell=1}^{\infty}$ such that

(12) $\operatorname{dist}(S_{\mathbf{i}}, S_{\mathbf{j}}) \geqslant c^{n_\ell}$ for all ℓ and for all $\mathbf{i}, \mathbf{j} \in \{1, \ldots, M\}^{n_\ell}$, $\mathbf{i} \neq \mathbf{j}$.

Exponential Separation Condition (ESC) II

Consider a family of self-similar IFSs on the line with fixed contraction ratios $r_1, \ldots, r_M$ and the parameters are the vertical translations $\mathbf{t} = (t_1, \ldots, t_M)$.

$$\mathcal{S}^{\mathbf{t}} := \{S_k(x) = r_k x + t_k\}_{k=1}^{M}.$$

(a) $\dim_{\mathrm{P}} \{\boldsymbol{t} \in \mathbb{R}^M : \mathcal{S}^{\boldsymbol{t}}$ does not satisfy the ESC $\} \leqslant M - 1$.

(b) Assume that $\mathcal{S}^{\mathbf{t}}$ satisfies the ESC. Then

  (i) For all n, the n-th iterate $(\mathcal{S}^{\mathbf{t}})^n$ also satisfies the ESC.

  (ii) All subsystem of $(\mathcal{S}^{\mathbf{t}})^n$ also satisfies ESC.

Recall that the N-th iterate is:

$$\mathcal{S}_{\mathcal{F}}^{N} := \{S_{i_1} \circ \cdots \circ S_{i_N}\}_{(i_1,\ldots,i_N)\in[M]^N}.$$

Definition 4.1

We say that a CPLIFS $\mathcal{F}$ is regular if

(a) $\mathcal{F}$ is small and

(b) the attractor $\Lambda^{\mathcal{F}}$ does not contain any breaking points.

We have indicated that packing-dimension typical small CPLIFS is regular and from now on we always assume that the CPLIFS $\mathcal{F} = \{f_i\}_{i=1}^{m}$ under consideration is regular. Fix N so big that

$$\bigcup_{\mathbf{u}\in[m]^n} I_{\mathbf{u}}^{\mathcal{F}} \text{ contains no breaking points,}$$

where $I_{\mathbf{u}}^{\mathcal{F}} = f_{u_1} \circ \cdots \circ f_{u_N}(I^{\mathcal{F}})$ for $\mathbf{u} = (u_1, \ldots, u_N) \in [m]^N$.

$$\Lambda^{\mathcal{F}} = \bigcup_{\mathbf{v}\in[m]^N} \Lambda^{\mathcal{F}}_{\mathbf{v}} = \bigcup_{\mathbf{v}\in[m]^N} \bigcup_{\mathbf{u}\in[m]^N} f_{\mathbf{v}}(\Lambda^{\mathcal{F}}_{\mathbf{u}}),$$

where $\Lambda^{\mathcal{F}}_{\mathbf{u}} = f_{\mathbf{u}}(\Lambda^{\mathcal{F}})$. Clearly, $\Lambda^{\mathcal{F}}_{\mathbf{u}} \subset I^{\mathcal{F}}_{\mathbf{u}}$. Recall that

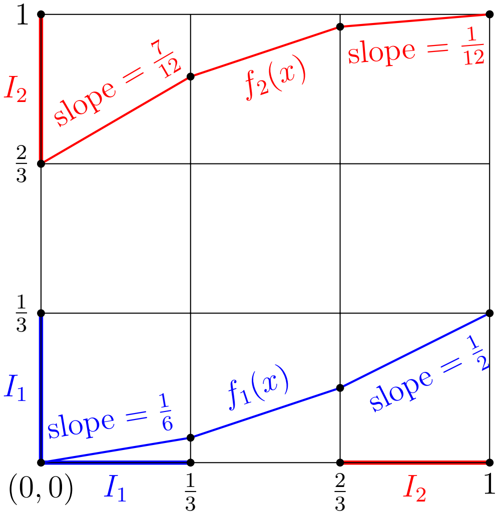
$$\bigcup_{\mathbf{u}\in[m]^n} I^{\mathcal{F}}_{\mathbf{u}} \text{ contains no breaking points.}$$

Hence for every pair $(\mathbf{v},\mathbf{u})$ we can find an $\mathbf{a} = (a_1,\dots,a_N) = \psi(\mathbf{v},\mathbf{u})$ such that $S_{a_i} \in \mathcal{S}^{\mathcal{F}}$ and for the similarity mapping $S_{\mathbf{a}} = S_{a_1} \circ \cdots \circ S_{a_n}$

$$f_{\mathbf{v}}|_{I_{\mathbf{u}}} = S_{\mathbf{a}}|_{I_{\mathbf{u}}}.$$

(13) $$\Lambda^{\mathcal{F}} = \bigcup_{\mathbf{v}\in[m]^n} \Lambda^{\mathcal{F}}_{\mathbf{v}} \quad \text{and} \quad \Lambda^{\mathcal{F}}_{\mathbf{v}} = \bigcup_{\mathbf{u}} S_{\psi(\mathbf{v},\mathbf{u})}(\Lambda^{\mathcal{F}}_{\mathbf{u}}).$$

A trivial motivational example



$\Lambda \subset (I_1 \bigcup I_2)$.

$\Lambda_j := f_j(\Lambda) = \Lambda \bigcap I_j$.

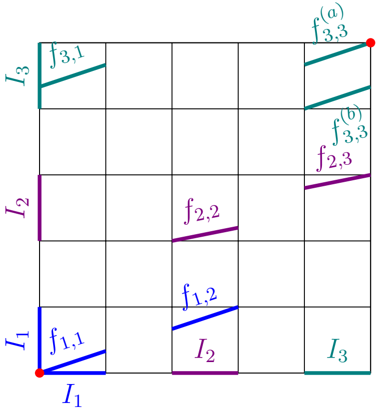
$\Lambda = \Lambda_1 \bigcup \Lambda_2$. Let $f_{i,j} := f_i|_{I_j}$.

Then $\Lambda_1 = f_{1,1}(\Lambda_1) \cup f_{1,2}(\Lambda_2)$, $\Lambda_2 = f_{2,1}(\Lambda_1) \cup f_{2,2}(\Lambda_2)$.

$$A^{(s)} := \begin{pmatrix} \left(\frac{1}{6}\right)^s & \left(\frac{1}{2}\right)^s \\ \left(\frac{7}{12}\right)^s & \left(\frac{1}{12}\right)^s \end{pmatrix}.$$

$\rho(A^{(\alpha)}) = 1$ for $\alpha = 0.577295\ldots$,

so, $\dim_{\mathrm{H}} \Lambda = 0.577295\ldots$

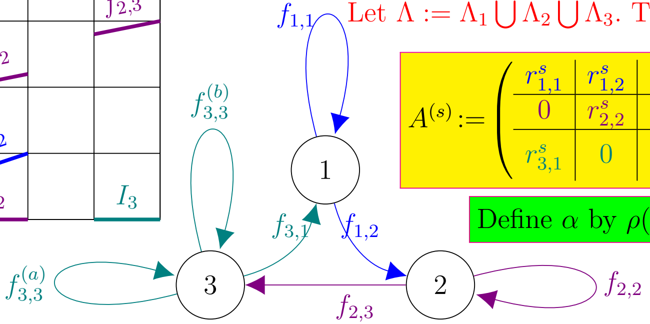


$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a strongly connected directed graph . $\forall e \in \mathcal{E}$ given a similarity $f_e : \mathbb{R} \to \mathbb{R}$ with ratio $r_e \in (0,1)$.

Then $\exists_3 \, \{\Lambda_i\}_{i \in \mathcal{V}}$ unique non empty compact sets with

$\Lambda_i = \bigcup_{j=1}^{3} \bigcup_{e \in \mathcal{E}_{i,j}} f_e(\Lambda_j)$, where $\mathcal{E}_{i,j}$ is the set of edges from j to i.

Let $\Lambda := \Lambda_1 \bigcup \Lambda_2 \bigcup \Lambda_3$. Then Λ is the attractor.



$$A^{(s)} := \begin{pmatrix} r^s_{1,1} & r^s_{1,2} & 0 \\ 0 & r^s_{2,2} & r^s_{2,3} \\ r^s_{3,1} & 0 & \left(r^{(a)}_{3,3}\right)^s + \left(r^{(b)}_{3,3}\right)^s \end{pmatrix}$$

Define α by $\rho(A^{(\alpha)}) = 1$.

Always: $\dim_{\mathrm{H}} \Lambda \leq \alpha$. If the interior of $\{f_{i,j}(I_j)\}_{(i,j)\in\mathcal{E}}$ are pairwise disjoint then $\dim_{\mathrm{H}} \Lambda = \alpha$.

2. [2]Kenneth J Falconer and KJ Falconer. *Techniques in fractal geometry*, volume 3. Wiley Chichester, 1997

The Lyapunov exponents for ergodic measures

Recall: $\mathcal{S} := \left\{ S_i(x) = r_i x + t_i \right\}_{i=1}^{m}$ Let μ be an invariant ergodic probability measure on the symbolic space Σ. The Lyapunov exponent of μ is

(14) $$\chi(\mu) := -\sum_{k=1}^{m} \mu([k]) \log r_k,$$

where the cylinder is defined by $[i_1, \dots, i_n] := \{\mathbf{j} = (j_1, j_2, \dots) \in \Sigma : j_1 = i_1, \dots, j_n = i_n\}$. If $\mu = \mathbf{p}^{\mathbb{N}}$ for a probability vector $\mathbf{p} = (p_1, \dots, p_m)$ then the push forward measure $\Pi_* \mu$ is called self-similar measure. In this case $\chi(\mu) = -\sum_{k=1}^{m} p_k \log r_k$.

Theorem 5.1 (Jordan and Rapaport [a] (2020))

[a] Thomas Jordan and Ariel Rapaport. Dimension of ergodic measures projected onto self-similar sets with overlaps.
Proceedings of the London Mathematical Society, 2020

Let $\mathcal{S}$ be a self-similar IFS on the line as above. We assume that $\mathcal{S}$ satisfies the so called Exponential Separation Condition (ESC). If μ is an ergodic invariant probability measure then

$$\dim_{\mathrm{H}} \Pi_* \mu = \min\left\{1, \frac{h_\mu}{\chi(\mu)}\right\}. \tag{15}$$

This theorem extends M. Hochman's celebrated result from self-similar measures to ergodic measures.

Using Jordan-Rapaport Theorem we obtain the following assertion which was proved in the appendix of the paper[3], and it plays a crusial role in our proofs.

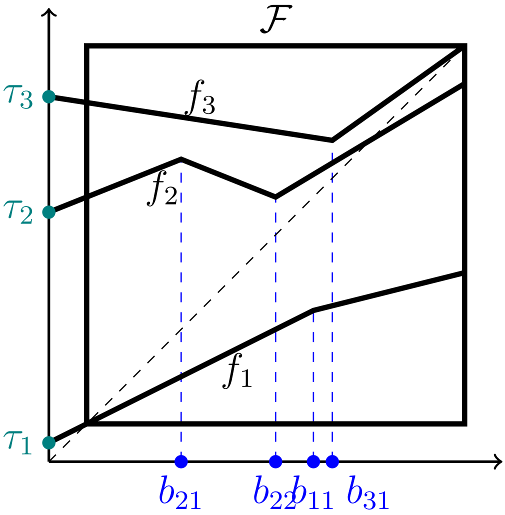
Theorem 5.2 (Prokaj, S.)

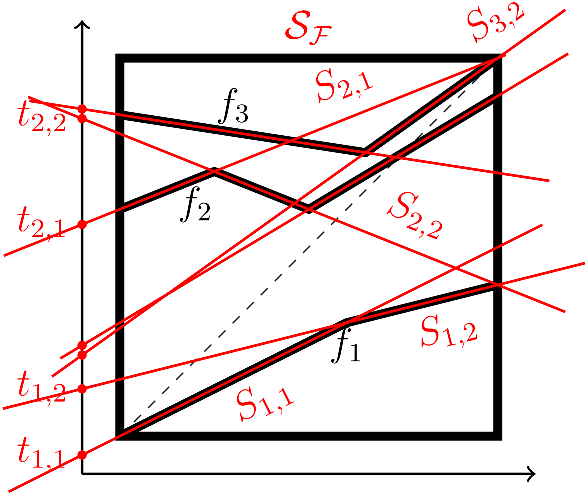
Let Λ be the attractor of the self-similar graph directed IFS $\mathcal{F}$. Let $\mathcal{S}_{\mathcal{F}}$ be the generated self-similar IFS.

$$\mathcal{S}_{\mathcal{F}} \text{ satisfies the ESC} \Longrightarrow \dim_{\mathrm{H}} \Lambda = \{1, s^{\mathcal{F}}\}. \tag{16}$$

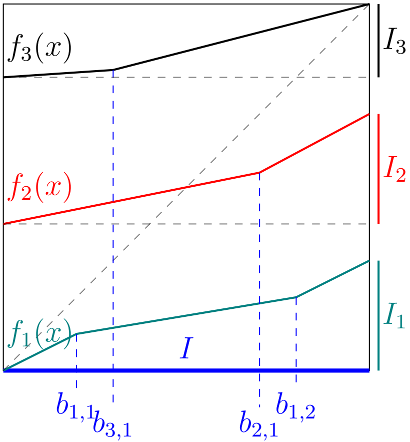
[3] R Dániel Prokaj and Károly Simon. Piecewise linear iterated function systems on the line of overlapping construction.
Nonlinearity, 35(1):245, 2021

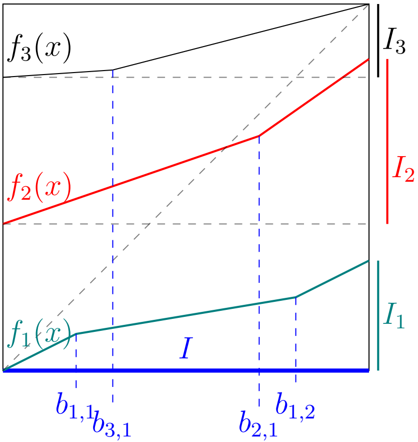
The generated self-similar IFS by a CPLIFS



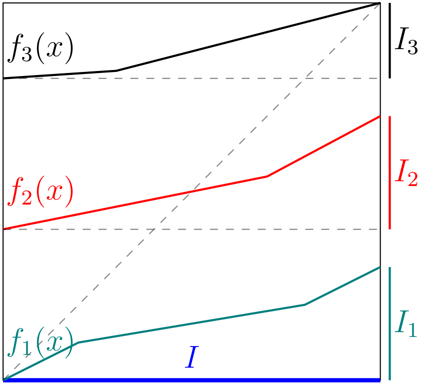


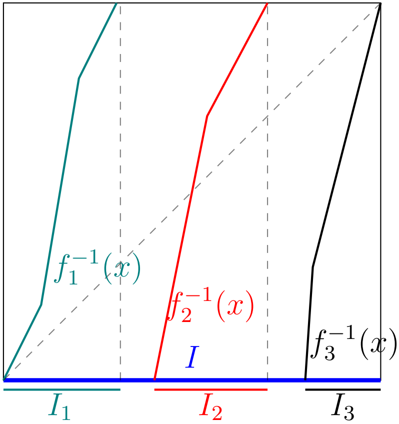
Injective, non-overlapping and overlapping CPLIFS





Hofbauer and Raith theory in the injective and non-overlapping case





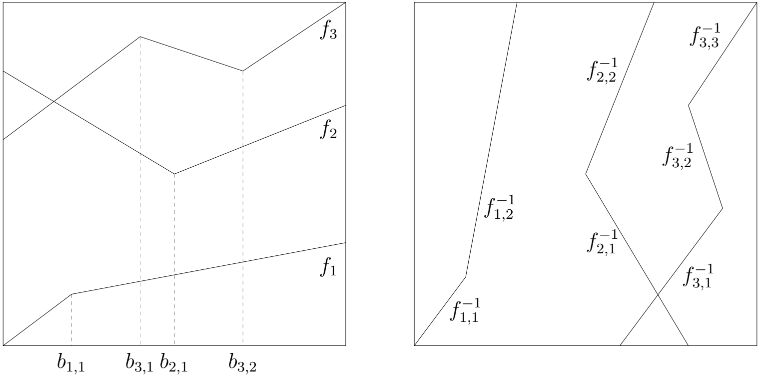


Figure: A CPLIFS $\mathcal{F} = \{f_k\}_{k=1}^{m}$ is on the left with its associated expansive multi-valued mapping T on the right.

Theorem 6.1 (Prokaj,Raith, S.)

Let $\mathcal{F}$ be a CPLIFS with generated self-similar system $\mathcal{S}$ and attractor Λ. If $\mathcal{S}$ satisfies the ESC, then

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\{1, s_{\mathcal{F}}\}. \tag{17}$$

Theorem 6.2 (Prokaj, Raith, S.)

Let $\mathcal{F}$ be a $\dim_{\mathrm{P}}$-typical CPLIFS with attractor Λ. Then

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\{1, s_{\mathcal{F}}\}. \tag{18}$$

Dániel Prokaj will talk more about this and another related theorem after the break.