Absolute continuity of self-similar measures on the plane

Adam Śpiewak

Institute of Mathematics, Polish Academy of Sciences

joint work with Boris Solomyak

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Thermodynamic Formalism: Non-additive Aspects and Related Topics

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and let $\nu = \nu^p_{\lambda, t}$ be the corresponding **self-similar measure**, i.e. the unique Borel prob. measure on $\mathbb{C}$ satisfying

$$\nu = \sum_{i=1}^{k} p_i g_{\lambda_i, t_i} \nu.$$

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Fact/Definition

Self-similar measure ν is exact dimensional, i.e. there exists $\alpha \in [0, 2]$ such that

$$\lim_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} = \alpha \quad \text{for } \nu\text{-a.e.} x \in \mathbb{C}.$$

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Similiarity dimension of $\nu^p_{\lambda,t}$

$$s(\lambda, p) = \frac{-\sum_{i=1}^{k} p_i \log p_i}{-\sum_{i=1}^{k} p_i \log |\lambda_i|}$$

Fact: $\dim \nu^p_{\lambda,t} \leq \min\{2, s(\lambda, p)\}$

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The proof extends the strategy of Saglietti-Shmerkin-Solomyak from $\mathbb{R}$ to $\mathbb{C}$.

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1. Transversality

- Solomyak '95: $\{\lambda x, \lambda x + 1\}$ on $\mathbb{R}$, almost sure a.c. for $\lambda \in (1/2, 1)$
- Solomyak-Xu '03: $\{\lambda x, \lambda x + 1\}$ on $\mathbb{C}$, almost sure a.c. for $2^{-1/2} \leq |\lambda| \leq 2 \times 5^{-5/8}$
- Neunhäuserer '01, Ngai-Wang '05: $\{\lambda_1 x, \lambda_2 x + 1\}$ on $\mathbb{R}$, almost sure a.c. on a subset of the super-critical region

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2. **Hochman's theory**
   - Hochman '14: $\dim(\nu^p_{\lambda,t}) = \min\{1, s(\lambda, p)\}$ on $\mathbb{R}$ for every $\lambda \notin E$ with $\dim_P E \leq k - 1$
   - Hochman '15: $\dim(\nu^p_{\lambda,t}) = \min\{2, s(\lambda, p)\}$ on $\mathbb{C}$ for every $\lambda \notin E$ with $\dim_P E \leq 2k - 1$

- Shmerkin '14: $\{\lambda x + t_1, \ldots, \lambda x + t_k\}$ homogeneous case on $\mathbb{R}$: a.c. for $\lambda \in (0,1) \setminus E$ with $s(\lambda, p) > 1$, $\dim_H E = 0$

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Natural projection map: $\Pi_\lambda : \{1, \ldots, k\}^{\mathbb{N}} \to \mathbb{R}$

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Lemma (Shmerkin)

Let η_1, η_2 be Borel probability measures on $\mathbb{R}^d$. Assume that $\dim \eta_1 = d$ and η_2 has power Fourier decay. Then $\eta_1 * \eta_2$ is absolutely continuous.

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Extend dimension and power Fourier decay properties to $\eta_\lambda^{(\omega)}$ for $\mathbb{P}$-almost every ω.

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Theorem (Solomyak-Ś)

Assume $\mathbb{P}(\#\Phi^{(\omega_1)} > 1) > 0$ and $\mathbb{P}(\lambda_{\omega_1} \notin \mathbb{R}) > 0$.

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Assume $\mathbb{P}(\#\Phi^{(\omega_1)} > 1) > 0$ and $\mathbb{P}(\lambda_{\omega_1} \notin \mathbb{R}) > 0$. If the model satisfies the exponential separation condition, then

$$\dim \eta^{(\omega)} = \min\{2, \mathrm{sdim}(\Sigma)\} \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Theorem (Solomyak-Ś)

Fix $\beta_i \in \mathbb{C} \setminus \{0\},\ i \in I$. If $\mathbb{P}$ is Bernoulli and $\Sigma_\lambda = \left((\Phi_\lambda^{(i)})_{i\in I}, (p^{(i)})_{i\in I}, \mathbb{P}\right)$ are models as above, such that $\Phi_\lambda^{(i)}$ has linear part λ^{β_i}, then for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\eta_\lambda^{(\omega)}$ has power Fourier decay for $\lambda \in \mathbb{D}_* \setminus E(\omega)$ with $\dim_H(E(\omega)) \le 1$.

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- Similar result on the line with translation parameter was proved by Käenmäki and Orponen

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Actually, we have to prove this for finite scale entropy and for a positive proportion of small-scale components of $\eta^{(\omega)}$, uniformly in V.

This is a preparation to apply Hochman's inverse theorem for entropy to measures $\eta^{(\omega)}$.

Thank you for your attention!