Invariant measures for Iterated Function Systems with inverses

Yuki Takahashi

Department of Mathematics, Saitama University

May 18, 2023

Outline of the talk

- Part I: Iterated Function Systems (IFS)
- Part II: Invariant measures for IFS
- Part III: Transversality argument
- Part IV: Iterated Function Systems with inverses
- Part V: Furstenberg measure

Part I: Iterated Function Systems (IFS)

Iterated Function Systems (IFS)

- Let $\mathcal{F} = \{f_i\}_{i \in \Lambda}$ be a finite collection of contractive self-maps of $\mathbb{R}$.
- There exists a unique nonempty compact set K such that

$$K = \bigcup_{i \in \Lambda} f_i(K).$$

- K is called the *attractor* of $\mathcal{F}$.

Example

Let

$$f_0(x) = \frac{1}{3}x, \;\; f_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then the attractor is the middle-$1/3$ Cantor set.

Cantor set

Example (Middle-$1/3$ Cantor set)

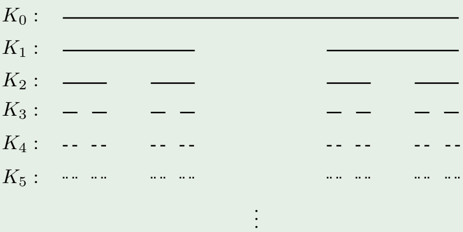


Figure: $K := \bigcap_{n=0}^{\infty} K_n$ is the *middle*-$1/3$ *Cantor set*.

- We have $K = f_0(K) \cup f_1(K)$, where $f_0(x) = \frac{1}{3}x$ and $f_1(x) = \frac{1}{3}x + \frac{2}{3}$.

Open set condition

- We say that an IFS $\mathcal{F} = \{f_i\}_{i\in\Lambda}$ satisfies the *open set condition* if there exists a nonempty open set $O \subset \mathbb{R}$ such that

$$(i) : \bigcup_{i\in\Lambda} f_i(O) \subset O;$$

$$(ii) : \{f_i(O)\}_{i\in\Lambda} \text{ are pairwise disjoint.}$$

Example

Let

$$f_0(x) = \frac{1}{3}x, \;\; f_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then the IFS $\mathcal{F} = \{f_0, f_1\}$ satisfies the open set condition.

$\because$) For $O = (0,1)$, we have $f_1(O) = (0, 1/3)$ and $f_2(O) = (2/3, 1)$.

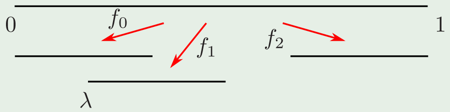
Open set condition

Example

Let $0 < \lambda < 1/3$, and let

$$f_0(x) = \frac{1}{3}x,\ f_1(x) = \frac{1}{3}x + \frac{2}{3} \text{ and } f_2(x) = \frac{1}{3}x + \lambda.$$

Then the IFS $\mathcal{F} = \{f_0, f_1, f_2\}$ may *not* satisfy the open set condition.



Coding

Example (Coding)

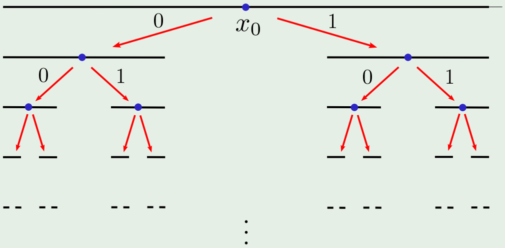
- Let $\mathcal{F} = \{f_0, f_1\}$ be an IFS, where

$$f_0(x) = \frac{1}{3}x,\ \ f_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

- Fix $x_0 \in \mathbb{R}$. The IFS $\mathcal{F}$ induces a natural projection map $\Pi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ called *coding* by

$$\Pi(\omega) = \lim_{n\to\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(x_0).$$

Example (Continued)



- We have, for example,

$$\Pi(0000\cdots) = 0,\ \Pi(10000\cdots) = 2/3,\ \Pi(010101\cdots) = 1/4, \quad \text{etc.}$$

Part II: Invariant measures for IFS

Invariant measures

- Let $\mathcal{F} = \{f_i\}_{i\in\Lambda}$ be an IFS and $p = (p_i)_{i\in\Lambda}$ be a probability vector. Then there exists a unique Borel probability measure ν such that

$$\nu = \sum_{i\in\Lambda} p_i f_i \nu.$$

- The measure ν is called the *invariant measure* associated to the IFS $\mathcal{F}$ and the weight p.

Example (Self-similar measure)

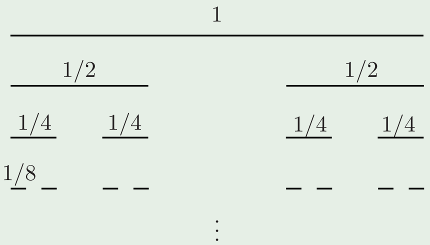
- Let

$$f_0(x) = \frac{1}{3}x,\ \ f_1(x) = \frac{1}{3}x + \frac{2}{3} \ \text{ and } \ p = (\frac{1}{2}, \frac{1}{2}).$$

- Then the associated self-similar measure ν satisfies

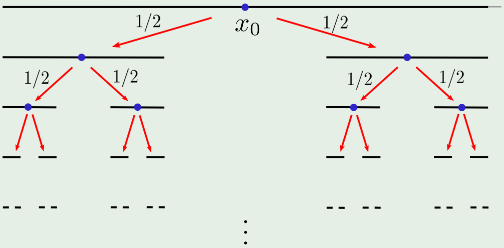
$$\nu = \frac{1}{2} f_1 \nu + \frac{1}{2} f_2 \nu.$$

- The support of ν is the middle-$1/3$ Cantor set.



Example (Continued)

- The measure ν agrees with the probability distribution of the following random walk:



- Let $\Pi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be the coding map, let μ be the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ associated with the probability vector $p = (1/2, 1/2)$. Then we have

$$\nu = \Pi\mu.$$

Entropy and the Lyapunov exponent

- Let $\mathcal{F} = \{f_0, f_1\}$ be an affine IFS and $p = (p_0, p_1)$ be a probability vector.
- $$h(p) = -(p_0 \log p_0 + p_1 \log p_1)$$
  is called the *entropy*, and
  $$\chi(\mathcal{F}, p) = -(p_0 \log r_0 + p_1 \log r_1)$$
  is called the *Lyapunov exponent*, where r_i is the contracting ratio of f_i.

Example

Let
$$f_0(x) = \frac{1}{3}x,\ \ f_1(x) = \frac{1}{3}x + \frac{2}{3} \ \text{ and } \ p = (\frac{1}{2}, \frac{1}{2}).$$
Then $h(p) = \log 2$ and $\chi = \log 3$.

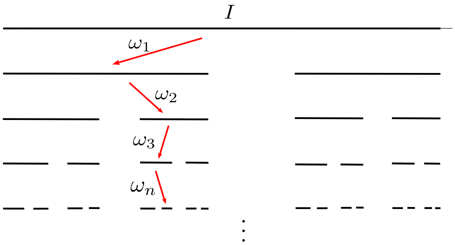
Dimension of self-similar measures (heuristic argument)

- Assume that there is no "overlap".
- For "typical" $\omega \in \{0,1\}^{\mathbb{N}}$, we have

$$\log\left(f_{\omega_1} \circ f_{\omega_2} \cdots \circ f_{\omega_n}(I)\right) \approx \log\left((r_0^{p_0} r_1^{p_1})^n |I|\right) \approx -n\chi$$

and

$$\log\left(p_{\omega_1} p_{\omega_2} \cdots p_{\omega_n}\right) \approx \log\left((p_0^{p_0} p_1^{p_1})^n\right) = -nh(p).$$



Dimension of self-similar measures (continued)

- Therefore, the dimension $\dim \nu$ should satisfy

$$(-n\chi)^{\dim \nu} = -nh(p),$$

which implies

$$\dim \nu = \frac{-nh(p)}{-n\chi} = \frac{h(p)}{\chi}.$$

Dimension of self-similar measures

- Let $\mathcal{F} = \{f_i\}_{i\in\Lambda}$ be an IFS and let $p = (p_i)_{i\in\Lambda}$ be a probability vector. Let ν be the associated self-similar measure.
- We have
$$\dim \nu \leq \min\left\{1, \frac{h(p)}{\chi}\right\}.$$
- If $\mathcal{F}$ satisfies the open set condtion, then
$$\dim \nu = \frac{h(p)}{\chi}.$$

Part III: Bernoulli convolution and the transversality argument

Bernoulli convolution

Definition (Bernoulli convolution)

Let $1/2 < \lambda < 1$, $p = (1/2, 1/2)$ and

$$f_{-1}^{(\lambda)}(x) = \lambda x - 1, \quad f_{1}^{(\lambda)}(x) = \lambda x + 1.$$

The associated self-similar measure ν_λ is called the *Bernoulli convolution*.

- We are interested in a one-parameter family of self-similar measure (e.g. Bernoulli convolution).

Bernoulli convolution

Theorem ('95, Solomyak)

For a.e. $\lambda \in (1/2, 1)$, ν_λ is absolutely continuous.

Theorem ('14, Shmerkin)

There exists a Hausdorff dimension zero set $E \subset [1/2, 1]$ such that for all $\lambda \in [1/2, 1] \setminus E$, the measure ν_λ is absolutely continuous.

Theorem ('19, Varju)

If $\lambda = 1 - 10^{-50}$ then ν_λ is absolutely continuous.

Bernoulli convolution

Definition (Pisot number)

A *Pisot number* is a real algebraic integer greater than 1 such that all of its Galois conjugates are less than 1 in absolute value.

Example (Pisot number)

(i) $\frac{1+\sqrt{5}}{2}$; minimal polynomial: $x^2 - x - 1$.

(ii) $2 + \sqrt{5}$; minimal polynomial: $x^2 - 4x - 1$.

(iii) $\alpha > 1$ such that $\alpha^3 - \alpha - 1 = 0$; minimal polynomial: $x^3 - x - 1$.

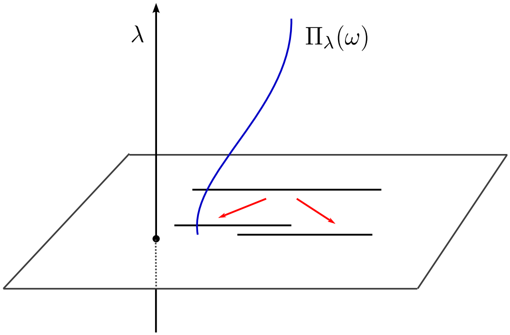
Theorem (Erdös, 1939)

Bernoulli convolution ν_λ is singular if λ is the inverse of a Pisot number.

Transversality condition

- Let $\Pi_\lambda : \{-1,1\}^{\mathbb{N}} \to \mathbb{R}$ be the natural projection map.
- We have

$$\Pi_\lambda(\omega) = \sum_{n=0}^{\infty} \omega_n \lambda^n.$$



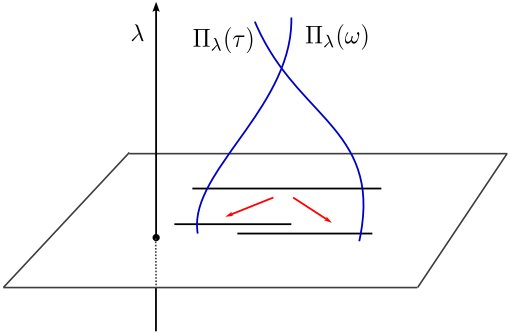
Transversality condition

Definition (Transversality condition)

We say that ν_λ satisfies the *transversality condition* on $I \subset [1/2, 1)$ if for all ω and τ in $\{0,1\}^{\mathbb{N}}$ with $\omega_1 \neq \tau_1$, the two curves

$$\{\Pi_\lambda(\omega) \mid \lambda \in I\} \text{ and } \{\Pi_\lambda(\tau) \mid \lambda \in I\}$$

are transversal.



Absolute continuity of measures

- Let ν be a measure on $\mathbb{R}$.
- Define the *lower density* of the measure ν by

$$\underline{D}(\nu, x) = \liminf_{r \downarrow 0} \frac{\nu[x-r, x+r]}{2r}.$$

- It is known that

$$\underline{D}(\nu, x) < \infty \text{ for } \nu\text{-a.e.} \implies \nu \ll \mathcal{L}.$$

- Therefore,

$$\int \underline{D}(\nu, x)\, d\nu < \infty \implies \nu \ll \mathcal{L}.$$

Bernoulli convolution: absolute continuity

- Let ν_λ be the Bernoulli convolution and let $I \subset [1/2, 1)$.
- We have

$$\mathcal{I} := \int_I \int \underline{D}(\nu_\lambda, x)\, d\nu_\lambda d\lambda < \infty \implies \nu_\lambda \ll \mathcal{L} \text{ for a.e. } \lambda \in I.$$

- By Fatou's lemma, we obtain

$$\mathcal{I} \leq \liminf_{r \downarrow 0} \int_I \int_{\mathbb{R}} \frac{\nu_\lambda[x-r, x+r]}{2r}\, d\nu_\lambda d\lambda.$$

Bernoulli convolution: absolute continuity

- By changing the variable and exchanging the order of integration, we have

$$\mathcal{I} \le \liminf_{r \downarrow 0} (2r)^{-1} \iint_{\Omega^2} \mathcal{L}\left\{\lambda \in I : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \le r\right\} d\mu(\omega) d\mu(\tau)$$

- By the transversality condition, we conclude $\mathcal{I} < \infty$.

General case

- $\mathcal{F}_t = \{f_i^t\}_{i\in\Lambda}$: one-parameter family of IFS.
- $p = (p_i)_{i\in\Lambda}$: probability vector.

Theorem ('01, Simon & Solomyak & Urbanski)

Assume that the transversality condition is satisfied. Then

(i) *For a.e. t,*

$$\dim \nu_t = \min\left\{1, \frac{h}{\chi_t}\right\},$$

where $h = h(p)$ is the entropy and χ_t is the Lyapunov exponent.

(ii) *The measure ν_t is absolutely continuous for a.e. t in*

$$\left\{t : \frac{h}{\chi_t} > 1\right\}.$$

Part IV: Iterated Function Systems with inverses

Self-similar measures (review)

- Let $\mathcal{F} = \{f_i\}_{i\in\Lambda}$ be an affine IFS and $p = (p_i)_{i\in\Lambda}$ be a probability vector. Then there exists a unique Borel probability measure ν such that

$$\nu = \sum_{i\in\Lambda} p_i f_i \nu.$$

- The measure ν is called the *self-similar measure* associated to the IFS $\mathcal{F}$ and the weight p.

Example

Let

$$f_0(x) = \frac{1}{3}x,\ \ f_1(x) = \frac{1}{3}x + \frac{2}{3} \ \text{ and } \ p = (\frac{1}{2}, \frac{1}{2}).$$

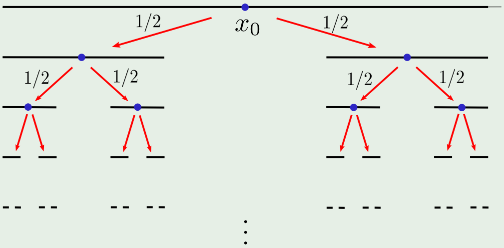
Then the associated self-similar measure ν satisfies

$$\nu = \frac{1}{2} f_1 \nu + \frac{1}{2} f_2 \nu,$$

whose support is the middle-$1/3$ Cantor set.

Example (Continued)

- The measure ν agrees with the probability distribution of the following random walk:



- Let $\Pi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be the coding map, let μ be the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ associated with the probability vector $p = (1/2, 1/2)$. Then we have

$$\nu = \Pi\mu.$$

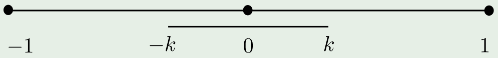
Iterated function systems with inverses (example)

Example (IFS with inverse)

- . Let $\Lambda = \{0, 1, 1^{-1}\}$, and let $p = (p_i)_{i \in \Lambda}$ be a probability vector. For $0 < k, l < 1$, define

$$f_0(x) = kx,\ \ f_1(x) = \frac{(1+l)x + 1 - l}{(1-l)x + 1 + l},\ \ f_{1^{-1}}(x) = \frac{(1+l)x - (1-l)}{-(1-l)x + 1 + l}.$$

  - We have $f_{1^{-1}} = f_1^{-1}$. It is easy to see that we have $f_0(0) = 0$, $f_1(-1) = -1$, $f_1(1) = 1$ and $f_0'(k) = k$, $f_1'(1) = l$, $f_1'(-1) = 1/l$.

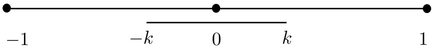


- It is well-known that there exists a unique Borel probability measure $\nu = \nu(k, l)$ that satisfies

$$\nu = \sum_{i \in \Lambda} p_i f_i \nu.$$

Reminder

- $\Lambda = \{0, 1, 1^{-1}\}$
- $f_0(x) = kx,\ f_1(x) = \frac{(1+l)x+1-l}{(1-l)x+1+l},\ f_{1^{-1}}(x) = \frac{(1+l)x-(1-l)}{-(1-l)x+1+l}$.



Example (Continued)

- We say that $\omega = \omega_1\omega_2\cdots \in \Lambda^{\mathbb{N}}$ is *reduced* if $\omega_i\omega_{i+1} \neq 11^{-1}, 1^{-1}1$.
- Fix $x_0 \in [-1, 1]$.

Key fact

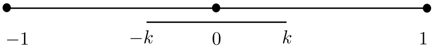
For any reduced sequence $\omega \in \Lambda^{\mathbb{N}}$, the limit

$$\lim_{n\to\infty} f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}(x_0).$$

exists.

Reminder

- $\Lambda = \{0, 1, 1^{-1}\}$
- $f_0(x) = kx,\ f_1(x) = \frac{(1+l)x+1-l}{(1-l)x+1+l},\ f_{1^{-1}}(x) = \frac{(1+l)x-(1-l)}{-(1-l)x+1+l}.$



Example (Continued)

- Therefore, one can define a natural projection map $\Pi : \{0, 1, 1^{-1}\} \to \mathbb{R}$ by

$$\Pi(\omega) = \lim_{n\to\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(x_0).$$

- let μ be the Bernoulli measure on $\{0, 1, 1^{-1}\}^{\mathbb{N}}$ associated with the probability vector $p = (p_0, p_1, p_{1^{-1}})$. Then we have

$$\nu = \Pi\mu.$$

Main result

- $\mathcal{F}_t = \{f_i^t\}_{i \in \Lambda}$: one-parameter family of IFS with inverse.
- $p = (p_i)_{i \in \Lambda}$: probability vector.
- $h_{RW} = h_{RW}(p)$: random walk entropy

Theorem ('22, T)

Assume that the transversality condition is satisfied. Then

(i) *For a.e.* $t \in I$,

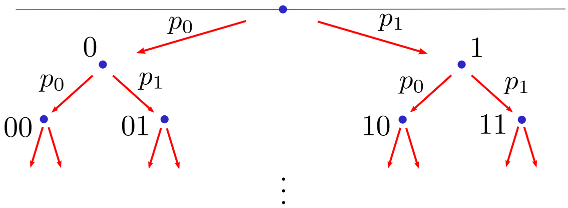
$$\dim \nu_t = \min \left\{ 1, \frac{h_{RW}}{\chi_t} \right\}.$$

(ii) *The measure* ν_t *is absolutely continuous for a.e.* t *in*

$$\left\{ t : \frac{h_{RW}}{\chi_t} > 1 \right\}.$$

Entropy and random walk entropy

- Let $\Lambda = \{0, 1\}$, and $p = (p_0, p_1)$ be a probability vector.

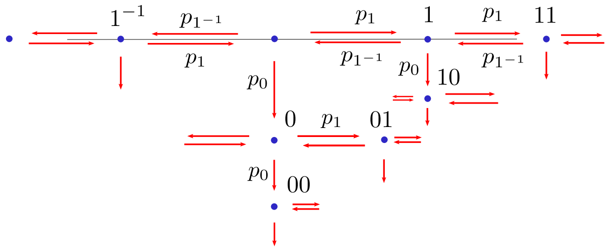


- For "typical" $\omega \in \Lambda^{\mathbb{N}}$, we have

$$p_{\omega_1} \cdots p_{\omega_n} \approx e^{-nh(p)}.$$

Entropy and random walk entropy

- Let $\Lambda = \{0, 1, 1^{-1}\}$, and $p = (p_0, p_1, p_{1^{-1}})$ be a probability vector.
- Let μ be a Bernoulli measure on $\Lambda^{\mathbb{N}}$ associated to p.



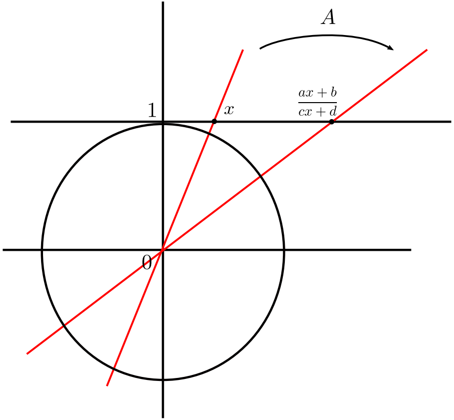
- For "typical" $\omega \in \Lambda^{\mathbb{N}}$, we have

$$\mu\left(\{v \in \Omega : \mathrm{red}(v|_n) = \mathrm{red}(\omega|_n)\}\right) \approx e^{-nh_{RW}}. \tag{1}$$

Part V: Furstenberg measure

Action of $SL_2(\mathbb{R})$ matrices

- Let $\mathbf{P}$ be the one-dimensional projective space.
- $A \in SL_2(\mathbb{R})$ acts naturally on $\mathbf{P}$.



Fustenberg measure

- Let $\mathcal{A} = \{A_i\}_{i\in\Lambda}$ be a finite collection of $SL_2(\mathbb{R})$ matrices, and let $p = (p_i)_{i\in\Lambda}$ be a probability vector.
- Assume that the semigroup generated by $\mathcal{A}$ is unbounded and totally irreducible.
- It is known that there exists a unique probability measure ν on $\mathbf{P}$ such that

$$\nu = \sum_{i\in\Lambda} p_i A_i \nu.$$

- The measure ν is called a *Fustenberg measure*.

Furstenberg measure

Example (Furstenberg measure)

- . Let $\Lambda = \{0, 1, 1^{-1}\}$, and let $p = (p_i)_{i\in\Lambda}$ be a probability vector. For $0 < k, l < 1$, define

$$A_0 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1+l & 1-l \\ 1-l & 1+l \end{pmatrix}, \ A_1 = \begin{pmatrix} 1+l & -(1-l) \\ -(1-l) & 1+l \end{pmatrix}.$$

- Let

$$f_0(x) = kx, \ \ f_1(x) = \frac{(1+l)x + 1 - l}{(1-l)x + 1 + l}, \ f_{1^{-1}}(x) = \frac{(1+l)x - (1-l)}{-(1-l)x + 1 + l}.$$

- Under the natural identification $\mathbf{P} \cong \mathbb{R} \cup \{\infty\}$, the associated Fustenberg measure agrees with the invariant measure ν that satisfies

$$\nu = \sum_{i\in\Lambda} p_i f_i \nu.$$

Natural (and probably very difficult) problem

- We say that a collection of $SL_2(\mathbb{R})$ matrices $\mathcal{A}$ is *symmetric* if $\mathcal{A} = \mathcal{A}^{-1}$. For example, the set $\mathcal{A} = (A, A^{-1}, B, B^{-1})$ is symmetric.

Problem

Show the following for some symmetric $\mathcal{A}_t$ $(t \in I)$:

(i) For a.e. $t \in I$,

$$\dim \nu_t = \min\left\{1, \frac{h_{RW}}{\chi_t}\right\}.$$

(ii) The measure ν_t is absolutely continuous for a.e. t in

$$\left\{t : \frac{h_{RW}}{\chi_t} > 1\right\}.$$

Thank you! :)