Diophantine approximation on random sets

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Thermodynamic Formalism:
Non-additive Aspects and **Related Topics**

UNIVERSITY OF OULU

MARIE CURIE ACTIONS

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Part 1: Khintchine's theorem

Given a function $\Psi : \mathbb{N} \to [0, \infty)$, we define

$$J(\Psi) = \{x \in \mathbb{R} : \|x - \tfrac{p}{q}\| \leq \Psi(q) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N}\}$$

An application of the Borel-Cantelli lemma shows that

$$\mathcal{L}(J(\Psi)) = 0 \quad \text{if} \quad \sum_{q \in \mathbb{N}} q \cdot \Psi(q) < \infty.$$

Khintchine proved a partial converse.

Theorem (Khintchine, 1926)

Assume that $\Psi : \mathbb{N} \to [0, \infty)$ is decreasing and

$$\sum_{q \in \mathbb{N}} q \cdot \Psi(q) = \infty.$$

Then, $\mathcal{L}$-almost every $x \in \mathbb{R}$ is in $J(\Psi)$.

Duffin-Schaeffer conjecture

The monotonicity assumption cannot be removed
(Duffin & Schaeffer, 1941)
motivating the (now proven) Duffin–Schaeffer conjecture:

Theorem (Koukoulopoulos & Maynard, 2020)

Let $\Psi : \mathbb{N} \to [0, \infty)$. Then,

$$\mathcal{L}\text{-a.e. } x \in \mathbb{R} \text{ is in } J(\Psi) \iff \sum_{q \in \mathbb{N}} \Psi(q)\phi(q) = \infty,$$

where $\phi(q)$ is the Euler totient function.

Monotonicity condition shows subtleties in the geometry of rational numbers. Different Ψ explore these.

This approach motivates "fractal Diophantine approximations".

101 on fractal sets.

Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}^n$ be all finite words over $\mathcal{A}$, and $\mathcal{A}^{\mathbb{N}}$ be all infinite words.

Let $\Phi = \{\phi_a\}_{a \in \mathcal{A}}$ be a (finite) collection of strict contractions on $\mathbb{R}^d$ indexed by $\mathcal{A}$. We write $\phi_{\mathtt{w}} = \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ for $\mathtt{w} = w_1 \dots w_n \in \mathcal{A}^n$.

There exists a unique, non-empty, compact set $X = X(\Phi) \subset \mathbb{R}^d$ that satisfies

$$X = \bigcup_{w \in \mathcal{A}} \phi_w(X).$$

The invariant set X is also called the attractor of X.

In fact, for any fixed $x \in \mathbb{R}^d$,

$$d_H \left(\bigcup_{\mathtt{w} \in \mathcal{A}^n} \phi_{\mathtt{w}}(x), X \right) \to 0 \quad \text{as} \quad n \to \infty.$$

Diophantine approximation on fractal sets

We emulate Diophantine approximation by replacing the role of rational numbers with those in the dynamical/iterative structure. Let $\Psi : \mathcal{A}^* \to [0, \infty)$ and $z \in \mathbb{R}^d$. We define

$$W_\Phi(z, \Psi) = \{x \in \mathbb{R}^d : \|x - \varphi_{\mathtt{w}}\| \leq \Psi(\mathtt{w}) \text{ for i.m. } \mathtt{w} \in \mathcal{A}^*\}.$$

We ask:

Motivating Question

Are there similar dichotomies with divergence conditions for the natural volume, e.g. does the following hold:

$$\sum_{n \in \mathbb{N}} \sum_{\mathtt{w} \in \mathcal{A}^n} \Psi(\mathtt{w})^{\dim_H X} = \infty \Longrightarrow \mathcal{H}^{\dim_H}(W_\Phi(z, \Psi)) = \mathcal{H}^{\dim_H X}(X)?$$

The implication holds, e.g. when φ_i are similarities or conformal mappings under separation conditions.

The behaviour above appears for suitable classes of Ψ in a variety of settings. It is closely linked to the general shrinking target problem. Recent progress: Allen and Bárány; Baker; Persson and Reeve; Levesly, Salp, and Velani; Baker and Koivusalo;. . . .

Studying the classes of Ψ for which such a statement holds provides information on how "spread out" the points in X are.

Similarity dimension, affinity dimension, etc.: The similarity dimension, affinity dimension, are the zero of a suitable pressure

$$P(s) = \lim_{n\to\infty} \log \sum_{\mathtt{w}\,\mathcal{A}^n} \sup_{x\in X} \|\phi'_{\mathtt{w}}(x)\|^s$$

[Replacing summand by the "singular value function" for affinities.]

Exceeding expectations

The zero of the pressure s_0 is the "best guess" to the dimension of the attractor X. Let $\Phi_{\mathbf{t}} = \{\phi_i(x) = \lambda_i \cdot O_i x + t_i\}_{i \in \mathcal{A}}$ be a finite collection of similarities/affinities on $\mathbb{R}^d$, where $\mathbf{t} = (t_i)_{i \in \mathcal{A}}$ is a collection of translation vectors. Write $X_{\mathbf{t}}$ for the invariant set.

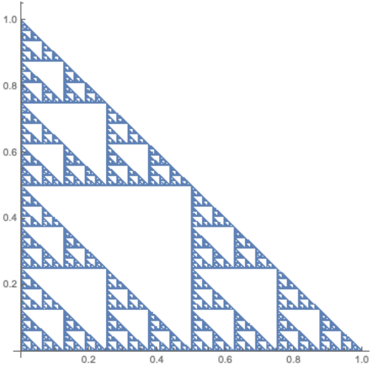
Theorem [Falconer '88, Solomyak '98, ...]

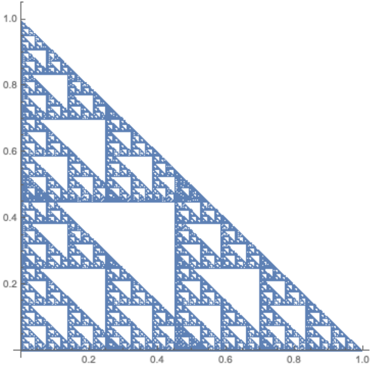
Let s_0 satisfy $P(s_0) = 0$. Then, for Lebesgue almost all $\mathbf{t} \in \mathbb{R}^{d \cdot \# \mathcal{A}}$,

$$\dim_H X_{\mathbf{t}} = \min\{s_0, d\}.$$

Further, if $s_0 > d$, the attractor satisfies $\mathcal{L}(X_{\mathbf{t}}) > 0$ for Lebesgue almost all $\mathbf{t}$.

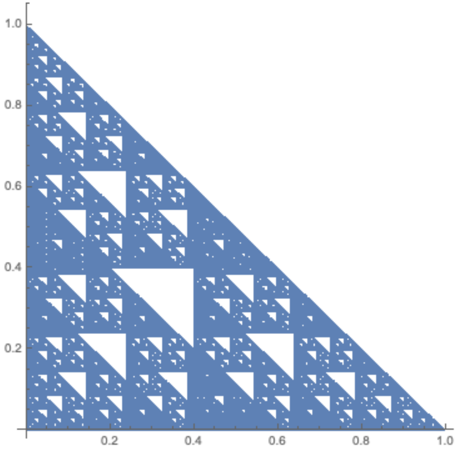
Some pictures





Sierpinski triangle for similarities with Lipschitz constants $1/2$ and $11/20$, and similarity dimensions $\log 3/\log 2 = 1.584\ldots$ and $\log 3/\log(20/11) = 1.837\ldots$, respectively.

Some pictures



Sierpinski triangle with Lipschitz constant $3/5$ and similarity dimension $\log 3/\log(5/3) = 2.150\ldots$.

Divergence on positive density

Let $B \subset \mathbb{N}$. Recall the upper density

$$\overline{d}(B) = \limsup_{n\to\infty} \frac{\#\{1 \leq j \leq n : j \in B\}}{n}$$

and write $G = \bigcup_{\gamma\in(0,1)} G_\gamma$, where

$$G_\gamma = \left\{ g : \mathbb{N} \to [0,\infty) \ : \ \sum_{n\in B} g(n) = \infty, \forall B \subseteq \mathbb{N} \text{ with } \overline{d}(B) > \gamma \right\}.$$

Heuristically, $g \in G$ is not summable on any positive density set.

A Diophantine fractal example

Assume additionally that the contractions are equicontractive.

Proposition (Baker, 2019)

Suppose $\log \# \mathcal{A} / \log(1/\lambda) > d$. Then, for Lebesgue almost every $\mathbf{t} \in \mathbb{R}^{\# \mathcal{A}\, d}$, for any $g \in G$ and $z \in X_{\mathbf{t}}$, the set

$$\left\{ x \in \mathbb{R}^d : |x - \phi_{\mathtt{w}}(z)| \leq \left(\frac{g(|\,\mathtt{w}\,|)}{(\# \mathcal{A})^{|\,\mathtt{w}\,|}} \right)^{1/d} \text{ for i.m. } \mathtt{w} \in \mathcal{A} \right\}$$

has positive Lebesgue measure.

Using different test functions g (such as $1/n$) gives information on the concentration of these typical attractors.

The positive Lebesgue measure of $X_{\mathbf{t}}$ is a consequence of the proposition for g a constant function.

Randomness to smooth things out

Observation: Randomisation "smooths" out intractable parts.

Jordan, Pollicott, and Simon (2007): self-affine attractors with random perturbations.

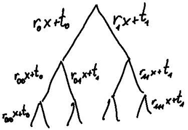
Peres, Simon, and Solomyak (2006): random self-similar constructions at every level (skew product).

Our Aim: Strengthen deterministic results through randomness.

Object: Stochastically self-similar and self-affine sets.

A stochastically self-similar/affine set F_ω, $(\omega \in \Omega)$ satisfies invariance in distribution:

$$F_\omega \equiv_d \bigcup_{i=1}^{N} \phi_{\omega,i}(F_{\omega',i})$$



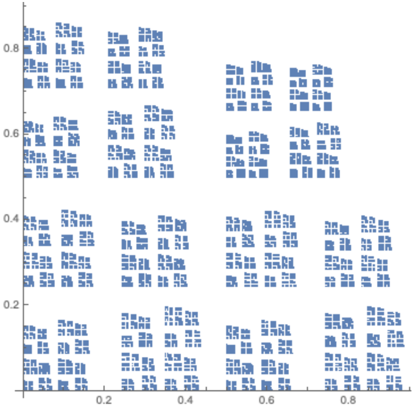
$$F_0 = \{0\}$$

$$F_1 = (r_{0,1} F_0 + t_0) \cup (r_{1,1} F_0 + t_0)$$

$$F_2 = (r_{0,2} F_1 + t_0) \cup (r_{1,2} F_1 + t_0)$$

$$\vdots$$

$$F = F_\infty$$



Stochastic self-similarity - (Slightly more) rigorous

Let M_d denote the set of invertible $d \times d$ matrices with $\|A\| < 1$ for all $A \in M_d$. Write $S_d \subset M_d$ for those which are similarities (scalar multiple of orthogonal matrices). For all $i \in \mathcal{A}$ we let $\Omega_i \subset M_d$ be a subset with measure η_i supported on Ω_i.

We define a product measure on $\Omega = \prod_{\mathtt{w} \in \mathcal{A}^*} \Omega_{\ell(\mathtt{w})}$ by $\eta = \prod_{\mathtt{w} \in \mathcal{A}^*} \mu_{\ell(\mathtt{w})}$ where $\ell(\mathtt{w})$ is the last letter of $\mathtt{w} \in \mathcal{A}^*$. A particular realisation $\omega \in \Omega$ is a collection of randomly chosen matrices, indexed by $\mathtt{w} \in \mathcal{A}^*$. We write $A_{\omega,\mathtt{w}}(x) = \omega_{\mathtt{w}} \cdot x$ to highlight the matrix/linear component associated with address $\mathtt{w}$ and realisation ω.

Note that for distinct $\mathtt{v}, \mathtt{w} \in \mathcal{A}^*$, the matrices $A_{\mathtt{v},\omega}$ and $A_{\mathtt{w},\omega}$ are independent though only identical in distribution if $\ell(\mathtt{w}) = \ell(\mathtt{v})$.

Random framework, continued

Let t_i for $i \in \mathcal{A}$ be a finite choice of distinct translations in $\mathbb{R}^d$. For every $\mathtt{w} \in \mathcal{A}^*$ we define the random maps

$$f_{\omega,\mathtt{w}}(x) = A_{\omega,\mathtt{w}}(x) + t_{\ell(\mathtt{w})}$$

and

$$\phi_{\omega,\mathtt{w}}(x) = f_{\omega,w_1} \circ \cdots \circ f_{\omega w_{|\mathtt{w}|}}.$$

Given a realisation $\omega \in \Omega$, and an infinite word $\mathtt{w} \in \mathcal{A}^{\mathbb{N}}$, we define its projection $\Pi_\omega(\mathtt{w}) : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}^d$ by

$$\Pi_\omega(\mathtt{w}) = \lim_{n\to\infty} \phi_{\omega,\mathtt{w}|_n}(0) = \lim_{n\to\infty} f_{\omega,w_1} \circ \cdots \circ f_{\omega w_n}(0)$$

and the random attractor by

$$F_\omega = \bigcup_{\mathtt{w}\in\mathcal{A}^N} \Pi_\omega(\mathtt{w}).$$

Random framework, continued

By definition, we have

$$F_\omega \equiv_d \bigcup_{i\in\mathcal{A}} f_{\omega',i}(F_{\omega_i''})$$

where $\omega, \omega', \omega_1'', \dots, \omega_{\#\mathcal{A}}''$ are independent realisations in (ω, η).

Given $\Psi : \mathcal{A}^* \to [0,\infty)$, $\mathtt{v} \in \mathcal{A}^{\mathbb{N}}$, and $\omega \in \Omega$ we want to investigate

$$W_\omega(\mathtt{v},\Psi) = \left\{x \in \mathbb{R}^d : |x - \Pi_\omega(\mathtt{w}\,\mathtt{v})| \le \Psi(\mathtt{w}) \text{ for infinitely many } \mathtt{w} \in \mathcal{A}^*\right\}$$

Doing this directly is difficult. Instead we consider an auxiliary family to deduce results about $W_{\omega(\mathtt{v},\Psi)}$.

Let μ be a slowly decaying measure defined on $\mathcal{A}^{\mathbb{N}}$ such that

$$\mu([w_1,\dots,w_{n+1})/\mu([w_1,\dots,w_n]) \ge c$$

for all n and μ almost all $\mathtt{w} \in \mathcal{A}^{\mathbb{N}}$.

Random framework, continued

Let $L_{\mu,n}$ be all the finite words $\mathtt{w}$ such that $\mu([\mathtt{w}]) \sim c^n$. We investigate

$$U_\omega(\mathtt{v},\mu,g) = \Big\{x \in \mathbb{R}^d : |x - \Pi_\omega(\mathtt{w}\,\mathtt{v})| \le (\mu([\mathtt{w}])g(n))^{1/d} \quad \text{for some } \mathtt{w} \in L_{\mu,n} \text{ for i.m. } n\Big\}.$$

We will also write

$$\lambda(\eta,\mu) = \sum_{i\in\mathcal{A}} \mu([i]) \cdot \lambda'(\eta_i),$$

where

$$\lambda'(\eta_i) = -\int_{\Omega_i} \log(|\operatorname{Det}(A)|)\, d\eta_i(A)$$

for the Lyapunov exponent of the random system with respect to μ.

Assumptions

Assumptions: Need Cramér's theorem on large deviations

$$\log \int_{\Omega_i} \exp(s \log|\mathrm{Det}(A)|) d\eta_i(A) < \infty.$$

We say that our RIFS is **non-singular** if there exists $C > 0$ such that for all $i \in \mathcal{A}, x \in \bigcup_{\omega \in \Omega} \prod_\omega(\mathcal{A}^{\mathbb{N}})$ and $B(y, r)$,

$$\eta_i(A \in \Omega_i : A \cdot x \in B(y, r)) \leq Cr^d.$$

We say that our RIFS is **distantly non-singular** if there exists $C > 0$ such that for all $i \in \mathcal{A}, x \in \bigcup_{\omega \in \Omega} \prod_\omega(\mathcal{A}^{\mathbb{N}})$ and $y \in \mathbb{R}^d \setminus B(0, \min_{i \neq j} |t_i - t_j|/8)$,

$$\eta_i(A \in \Omega_i : A \cdot x \in B(y, r)) \leq Cr^d.$$

Inspiration for condition

Peres, Solomyak, and Simon considered absolute continuity for random similarities in $\mathbb{R}$.

In our notation: $A_{\omega,\mathtt{w}} = Y_{|\mathtt{w}|} c_{\ell(\mathtt{w})}$, where c_i only depends on the last letter of $\mathtt{w} \in \mathcal{A}^*$ and Y is a random variable depending only on the length of the word $\mathtt{w} \in \mathcal{A}^*$.

Theorem (Peres, Simon, Solomyak 2006)

Let Y be an absolutely continuous random variable with distribution ν satisfying, for some $C > 0$,

$$\frac{d\nu}{dx} \leq C \frac{1}{x}.$$

Let μ be an ergodic shift invariant measure on $\mathcal{A}^{\mathbb{N}}$. Assume further that $h(\mu)/\lambda(\eta, \mu) > 1$. Then, F_ω has positive Lebesgue measure.

Result

Theorem (Baker-T., 2022)

Let $(\{\Omega_i\}_{i\in\mathcal{A}}, \{\eta_i\}_{i\in\mathcal{A}}, \{t_i\}_{i\in\mathcal{A}})$ be a RIFS and assume one of:

A. Assume $\Omega_i \subset S_d$ for all $i \in \mathcal{A}$ and the RIFS is distantly non-singular.

B. Assume $\Omega_i \subset M_d$ for all $i \in \mathcal{A}$ and the RIFS is non-singular.

Suppose μ is a slowly decaying shift invariant ergodic probability measure with $h(\mu)/\lambda(\eta, \mu) > d$. Then the following hold:

1. For any $\mathtt{v} \in \mathcal{A}^{\mathbb{N}}$, for η almost every $\omega \in \Omega$, for any $g \in G$, the set $U_\omega(\mathtt{v}, \mu, g)$ has positive Lebesgue measure.
2. For any $\mathtt{v} \in \mathcal{A}^{\mathbb{N}}$, for η almost every $\omega \in \Omega$, for any $\Psi : \mathcal{A}^* \to [0, \infty)$ the set $W_\omega(\mathtt{v}, \Psi)$ has positive Lebesgue measure if there exists $g \in G$ such that $\Psi(\mathtt{w}) \approx (m([\mathtt{w}])g(n))^{1/d}$.

Some Corollaries

Corollary

Let $(\{\Omega_i\}_{i\in\mathcal{A}}, \{\eta_i\}_{i\in\mathcal{A}}, \{t_i\}_{i\in\mathcal{A}})$ be a RIFS and assume one of:

A. Assume $\Omega_i \subset S_d$ for all $i \in \mathcal{A}$ and that the RIFS is distantly non-singular.

B. Assume $\Omega_i \subset M_d$ for all $i \in \mathcal{A}$ and the RIFS is non-singular.

Let $(p_i)_{i\in\mathcal{A}}$ be a probability vector satisfying $\frac{-\sum p_i \log p_i}{\sum p_i \lambda'(\eta_i)} > d$. Then for all $\mathtt{v} \in \mathcal{A}^{\mathbb{N}}$, for η-almost every $\omega \in \Omega$ the set

$$\left\{ x \in \mathbb{R}^d : |x - \Pi_\omega(\mathtt{w}\,\mathtt{v})| \leq \left(\frac{\prod_{k=1}^{|\mathtt{w}|} p_{w_k}}{|\mathtt{w}|} \right)^{1/d} \text{ for i.m. } \mathtt{w} \in \mathcal{A}^* \right\}$$

has positive Lebesgue measure.

Some Corollaries (cont.)

The compactness of F_ω implies that $U_\omega(\mathrm{v}, \mu, g) \subseteq F_\omega$ whenever g is bounded. This gives

Corollary

Let $(\{\Omega_i\}_{i\in\mathcal{A}}, \{\eta_i\}_{i\in\mathcal{A}}, \{t_i\}_{i\in\mathcal{A}})$ be a RIFS and assume one of:

A. Assume $\Omega_i \subset S_d$ for all $i \in \mathcal{A}$ and that the RIFS is distantly non-singular.

B. Assume $\Omega_i \subset M_d$ for all $i \in \mathcal{A}$ and the RIFS is non-singular.

If there exists a slowly decaying shift invariant ergodic probability measure μ satisfying $h(\mu)/\lambda(\eta, \mu) > d$, then for η-almost every $\omega \in \Omega$ the set F_ω has positive Lebesgue measure.

Examples

Self-similar. For each $i \in \mathcal{A}$ let $0 \le r_i^- < r_i^+ < 1$ and set

$$\Omega_i = \left\{ \lambda \cdot O : \lambda \in [r_i^-, r_i^+],\ O \in \mathcal{O}(d) \right\},$$

where $\mathcal{O}(d)$ is the set of orthogonal $d \times d$ matrices. For each $i \in \mathcal{A}$ let η_i be the product measure of the Haar measure and the Lebesgue measure, restricted and normalised to $[r_i^-, r_i^+]$.

Letting $r^+ > 0$ and $\# \mathcal{A}$ be sufficiently large, the uniform Bernoulli measure μ satisfies $h(\mu)/\lambda(\eta, \mu) > d$ and our Theorem and its Corollaries apply.

Affine Cases

Self-affine. Letting $Z_i \subset M_d$ be compact with

$$\Omega_i = \left\{ \lambda \cdot OB : \lambda \in [r_i^-, r_i^+],\ O \in \mathcal{O}(d),\ B \in Z_i \right\},$$

and assuming $\{t_i\}$ are large enough such that

$$B(0,\delta) \cap \bigcup_{\omega \in \Omega} \Pi_\omega(\mathcal{A}^{\mathbb{N}}) = \varnothing$$

we can apply our results under the non-singular condition.

Observations

In the stochastic self-similar / self-affine setting the "correct" Lyapunov exponent should be

$$\log \int_{\Omega_i} |\operatorname{Det}(A)| d\eta_i(A)$$

rather than

$$\int_{\Omega_i} \log |\operatorname{Det}(A)| d\eta_i(A).$$

Our use of large deviations suggests our results are "sharp": We suspect it is because of needing "level specific" information, as opposed to "eventually averaging".

Thank you for your attention