Relative pressure functions and their equilibrium states

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Question

(X, σ_X): one sided subshift on finitely many symbols

We say that a sequence of continuous functions $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on a subshift X is subadditive if $f_{n+m}(x) \leq f_n(x) f_m(\sigma^n x)$ for every $x \in X$, $n, m \in \mathbb{N}$.

[Question] Given a subadditive sequence of continuous functions $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on a subshift X, what are necessary and sufficient conditions for the existence of a continuous function h on X such that

$$\lim_{n \to \infty} \frac{1}{n} \int \log f_n d\mu = \int h d\mu \tag{1}$$

for every invariant Borel probability measure μ on X?

Remark: If such an h exists, the thermodynamic formalism for sequences $\mathcal{F}$ is reduced to the thermodynamic formalism for continuous functions.

Structure of the talk:
1. Background (motivation)
2. Results

Motivation

Relative pressure functions have representations by subadditive sequences.

1. To study the subadditive sequences associated to relative pressure functions is related to a study on the existence of compensation functions for factor maps between subshifts.

(Answering Question 1 for the subadditive sequences associated to relative pressure functions gives us a characterization of the existence of a compensation function)

2. We can apply the results to study factors of weak invariant Gibbs measures

Factor maps between subshifts on finitely many symbols

(X, σ_X): one sided subshift
$M(X, \sigma_X)$: the set of σ_X-invariant Borel probability measures
$h_\mu(\sigma_X)$: measure theoretic entropy of σ_X with respect to $\mu \in M(X, \sigma_X)$.

[Definition] Let (X, σ_X) and (Y, σ_Y) be (one-sided) subshifts on finite alphabets. A continuous function $\pi : X \to Y$ is a factor map if it is surjective and satisfies $\pi \circ \sigma_X = \sigma_Y \circ \pi$. A function $F \in C(X)$ is a *compensation function* for $(\sigma_X, \sigma_Y, \pi)$ if

$$\sup_{\mu \in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \int (F + \phi \circ \pi) d\mu\} = \sup_{m \in M(Y,\sigma_Y)} \{h_m(\sigma_Y) + \int \phi dm\} \quad (2)$$

for all $\phi \in C(Y)$. Furthermore, $G \circ \pi$ is a *saturated compensation function* if $F = G \circ \pi$ for some $G \in C(Y)$.

Boyle and Tuncel (1984), Walters (1986)

Shin(2001): A saturated compensation function does not always exist.

Antonioli(2016): existence of a compensation function for a factor map $\pi : X \to Y$ where X is an irreducible shift of finite type

Relation between relative pressure functions and compensation functions

Let $(X, \sigma_X), (Y, \sigma_Y)$ be subshifts and $\pi : X \to Y$ be a factor map. Let $f \in C(X)$, $n \in \mathbb{N}$ and $\delta > 0$. For each $y \in Y$, define

$$P_n(\sigma_X, \pi, f, \delta)(y) = \sup\{\sum_{x \in E} e^{(S_n f)(x)} : E \text{ is an } (n, \delta) \text{ separated subset of } \pi^{-1}(\{y\})\},$$

$$P(\sigma_X, \pi, f, \delta)(y) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(\sigma_X, \pi, f, \delta)(y),$$

$$P(\sigma_X, \pi, f)(y) = \lim_{\delta \to 0} P(\sigma_X, \pi, f, \delta)(y).$$

The function $P(\sigma_X, \pi, f) : Y \to \mathbb{R}$ is the *relative pressure* function of $f \in C(X)$ with respect to $(\sigma_X, \sigma_Y, \pi)$. In general it is merely Borel measurable.

Ledrappier and Walters, 1977, Relativised Variational Principle

Theorem. Let (X, σ_X) and (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a factor map. Let $f \in C(X)$. Then for $m \in M(Y, \sigma_Y)$,

$$\int P(\sigma_X, \pi, f) dm = \sup\{h_\mu(\sigma_X) - h_m(\sigma_Y) + \int f d\mu : \mu \in M(X, \sigma_X), \pi\mu = m\}.$$

Relation between relative pressure functions and compensation functions

[Lemma1] Given a function $f \in C(X)$, consider the relative pressure function $P(\sigma_X, \pi, f)$. Then there exists a function $h \in C(Y)$ such that

$$\int P(\sigma_X, \pi, f) dm = \int h dm,$$

for every $m \in M(Y, \sigma_Y)$ if and only if $f - h \circ \pi \in C(X)$ is a compensation function for π

Special case: Set $f = 0$. Let $G \in C(Y)$. Then

$$\int P(\sigma_X, \pi, 0) dm = \int G dm,$$

for every $m \in M(Y, \sigma_Y)$. if and only if $-G \circ \pi \in C(X)$ is a saturated compensation function for π.

Shin (2006) characterized the existence of a saturated compensation function for π by studying some properties of $P(\sigma_X, \pi, 0)$.

Factor maps between subshifts on finitely many symbols

Our question: What are necessary and sufficient conditions for the existence of a continuous function h on Y such that

$$\int P(\sigma_X, \pi, f)dm = \int hdm$$

for every invariant Borel probability measure m on Y?
(This will give us the existence of a certain type of compensation functions.)

Properties of $P(\sigma_X, \pi, f)$, $f \in C(X)$.
1. $P(\sigma_X, \pi, f)$ can be represented by a subadditive sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$, where g_n is a continuous function on Y:

$$\int P(\sigma_X, \pi, f)dm = \lim_{n \to \infty} \frac{1}{n} \int \log g_n d\mu$$

for every $m \in M(Y, \sigma_Y)$.
NOTE: In general, $\mathcal{G}$ is not asymptotically additive, not quasi-multiplicative.

Characterization of Asymptotically additive sequences

[Asymptotically additive sequence] Feng and Huang (2010) For every $\epsilon > 0$ there exists a continuous function g_ϵ such that

$$\limsup_{n\to\infty} \frac{1}{n} \|\log f_n - S_n g_\epsilon\|_\infty < \epsilon,$$

where $S_n g_\epsilon$ denotes the Birkhoff sums of g_ϵ and $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$.

Results by Cuneo (2020)

Theorem. Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an asymptotically additive sequence on X. Then there exists $f \in C(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \|\log f_n - S_n f\|_\infty = 0, \tag{3}$$

Hence if $\mathcal{F}$ is asymptotically additive, then there exists $f \in C(X)$ such that $\lim_{n\to\infty}(1/n)\int \log f_n d\mu = \int f d\mu$ for every $\mu \in M(X, \sigma_X)$.

NOTE: (3) implies that $\mathcal{F}$ is asymptotically additive.

Results: Subadditive sequences

$M(X, \sigma_X)$: the set of σ_X-invariant Borel probability measures
$Erg(X, \sigma_X)$: the set of ergodic members of $M(X, \sigma_X)$

[Proposition 1] Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a subadditive sequence on X. For $h \in C(X)$, the following conditions are equivalent.

1. $$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu = \int h d\mu$$
   for every $\mu \in M(X, \sigma_X)$.
2. $$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu = \int h d\mu$$
   for every $\mu \in Erg(X, \sigma_X)$.
3. $$\lim_{n\to\infty} \frac{1}{n} \log \left(\frac{f_n(x)}{e^{(S_n h)(x)}} \right) = 0$$
   μ-almost everywhere on X, for every $\mu \in Erg(X, \sigma_X)$.

Subadditive sequences

Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}$ be a sequence of continuous functions on a subshift X with tempered variation.

$B_n(X)$: the set of allowable words of length n of X.

[C1] $f_{n+m}(x) \leq f_n(x) f_m(\sigma^n x) e^C$ for some $C \geq 0$. (equivalently, $\{\log f_n e^C\}$ is subadditive)

[Condition A] There exist $k, N \in \mathbb{N}$ and a sequence $\{M_n\}_{n=1}^{\infty}$ of positive real numbers satisfying $\lim_{n\to\infty}(1/n)\log M_n = 0$ such that for given any $u \in B_n(X), n \geq N$, there exist $0 \leq q \leq k$ and $w \in B_q(X)$ such that $z := (uw)^\infty$ is a point in X satisfying

$$f_{j(n+q)}(z) \geq (M_n \sup\{f_n(x) : x \in [u]\})^j$$

for every $j \in \mathbb{N}$.

[Remark 1] Let (X, σ_X) be an irreducible shift of finite type with weak specification k. Then for each $u \in B_n(X)$ there exist $0 \leq q \leq k$ and $w \in B_q(X)$ such that $(uw)^\infty \in X$.

[Remark 2] There are subadditive sequences which are not asymptotically additive but satisfy [Condition A].

Subadditive sequences satisfying Condition A

[Theorem 1] Let (X, σ_X) be a subshift. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence on X satisfying [C1] with tempered variation and Condition A .Then the following statements are equivalent for $h \in C(X)$.

1. $\mathcal{F}$ is asymptotically additive on X satisfying

$$\lim_{n\to\infty} \frac{1}{n} \|\log\left(\frac{f_n}{e^{(S_n h)}}\right)\|_\infty = 0.$$

2.

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu = \int h d\mu$$

for every $\mu \in M(X, \sigma_X)$.

3.

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every periodic point $x \in X$.

4.

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every $x \in X$.

Relative pressure

Ideas: We want to apply Theorem 1 for the subadditive sequences associated to relative pressure functions.
Let (X, σ_X) be a subshift with the weak specification property, (Y, σ_Y) be a subshift and $\pi : X \to Y$ be a one-block factor map. For $y = (y_i)_{i=1}^{\infty}$, let $E_n(y)$ be a set consisting of exactly one point from each cylinder $[x_1 \dots x_n]$ in X such that $\pi(x_1 \dots x_n) = y_1 \dots y_n$.

Applying results by Petersen and Shin (2005) and Feng (2011), we obtain:

For $f \in C(X)$,

$$P(\sigma_X, \pi, f)(y) = \limsup_{n \to \infty} \frac{1}{n} \log g_n(y)$$

μ-almost everywhere for every invariant Borel probability measure μ on Y, where g_n is defined by

$$g_n(y) = \sup_{E_n(y)} \{ \sum_{x \in E_n(y)} e^{(S_n f)(x)} \}.$$

We call $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ a (subadditive) sequence associated to the relative pressure $P(\sigma_X, \pi, f)$.

Properties of relative pressure functions

(2) There exist $p \in \mathbb{N}$ and a positive sequence $\{D_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ such that given any $u \in B_n(X), v \in B_m(X)$, $n, m \in \mathbb{N}$, there exists $w \in B_k(X), 0 \leq k \leq p$ such that $uwv \in B_{n+m+k}$,

$$\sup\{g_{n+m+k}(x) : x \in [uwv]\} \geq D_{n,m} \sup\{g_n(x) : x \in [u]\} \sup\{g_m(x) : x \in [v]\},$$

where $\lim_{n\to\infty}(1/n)\log D_{n,m} = \lim_{m\to\infty}(1/m)\log D_{n,m} = 0$.

Not asymptotically additive, Not quasi-multiplicative

(3) the sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ satisfies [Condition A] if (X, σ_X) is an irreducible shift of finite type

Apply Theorem 1 and the work of Walters (1986) on the properties of compensation functions.

Some answers to Question

[Theorem 2] Let (X, σ_X) be an irreducible shift of finite type and (Y, σ_Y) be a subshift. Let $\pi : X \to Y$ be a one-block factor map and $f \in C(X)$. Then the following statements are equivalent for $h \in C(Y)$.

1. $P(\sigma_X, \pi, f - h \circ \pi)(y) = 0$ for every periodic point $y \in Y$, equivalently,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{g_n(y)}{e^{(S_n h)(y)}}\right) = 0$$

   for every periodic point $y \in Y$.

2. The function $f - h \circ \pi$ is a compensation function for π.
3. $$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{g_n(y)}{e^{(S_n h)(y)}}\right) = 0$$

   for every $y \in Y$.

4. The sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ is asymptotically additive on Y satisfying

$$\lim_{n\to\infty} \frac{1}{n} \|\log\left(\frac{g_n}{e^{(S_n h)}}\right)\|_\infty = 0.$$

5. $\int P(\sigma_X, \pi, f) dm = \int h dm$ for all $m \in M(Y, \sigma_Y)$.

Existence of saturated compensation functions

The existence of a saturated compensation function is equivalent to one of the statements of Theorem 3 with $f = 0$.

Remark: Shin (2006) gave a characterization on the existence of saturated compensation functions for factor maps between two sided shifts of finite type studying periodic points.

Applications -Factors of invariant weak Gibbs measures for continuous functions

Let (X, σ_X) and (Y, σ_Y) be subshifts on finitely many symbols and $\pi : X \to Y$ be a one block factor map.

Question: If μ is a shift-invariant Gibbs measure for some continuous function f, is $\pi\mu$ also a shift-invariant Gibbs measure for some continuous function g? What is the property of g?

Conditions for $\pi\mu$ to be Gibbs and related topics are studied by Chazottes-Ugalde (2003, 2011), Yoo (2010), Pollicott-Kempton (2011), Kempton (2011), Verbitskiy (2011), Piraino (2019), Hong (2020), etc.

We study
"Suppose μ is weak Gibbs for a continuous function. Then $\pi\mu$ is weak Gibbs for a continuous function $\iff$?"

Applications -Factors of invariant weak Gibbs measures for continuous functions

[Corollary 2] Let (X, σ_X) be an irreducible shift of finite type, Y be a subshift and $\pi : X \to Y$ be a one-block factor map. Suppose that μ is an invariant weak Gibbs measure for some $f \in C(X)$. Let $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y be the sequence of continuous functions associated to the relative pressure $P(\sigma_X, \pi, f)$. Then

(1) The measure $\pi\mu$ is an invariant weak Gibbs measure for $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y.

(2) The invariant measure $\pi\mu$ is a weak Gibbs measure for a continuous function on Y if and only if one of the equivalent statements in Theorem 2 holds.

(3) If there is no sequence $\{C_{n,m}\}_{n,m\in\mathbb{N}}$ satisfying

$$\frac{1}{C_{n,m}} \leq \frac{g_{n+m}(y)}{g_n(x)g_m(\sigma_Y^n y)} \leq C_{n,m}, \text{ where } \lim_{n\to\infty} \frac{1}{n}\log C_{n,m} = \lim_{m\to\infty} \frac{1}{m}\log C_{n,m} = 0,$$

then there exists no continuous function on Y for which $\pi\mu$ is an invariant weak Gibbs measure on Y.