Noncommutative Complex Structures on Quantum Homogeneous Spaces

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Abstract

A new framework for noncommutative complex geometry on quantum homogeneous spaces is introduced. The main ingredients used are covariant differential calculi and Takeuchi's categorical equivalence for quantum homogeneous spaces. A number of basic results are established, producing a simple set of necessary and sufficient conditions for noncommutative complex structures to exist. Throughout, the framework is applied to the quantum projective spaces endowed with the Heckenberger–Kolb calculus.

1 Introduction

Classical complex geometry is a subject of remarkable richness and beauty with deep connections to modern physics. Yet despite over twenty five years of noncommutative geometry, the development of noncommutative complex geometry is still in its infancy. What we do have is a large number of examples which demand consideration as noncommutative complex spaces. We cite, among others, noncommutative tori [6], noncommutative projective algebraic varieties [30], fuzzy flag manifolds [22], and (most importantly from our point of view) examples arising from the theory of quantum groups [9, 20].

Thus far, there have been two attempts to formulate a general framework for noncommutative complex geometry. The first, due to Khalkhali, Landi, and van Suijlekom [12], was introduced to provide a context for their work on the noncommutative complex geometry of the Podleś sphere. This followed on from earlier work of Majid [20], Schwartz and Polishchuk [28], and Connes [4, 3]. Khalkhali and Moatadelro [13, 14] would go on to apply this framework to D'Andrea and Dąbrowski's work [5] on the higher order quantum projective spaces.

Subsequently, Beggs and Smith introduced a second more comprehensive approach to noncommutative complex geometry in [1]. Their motive was to provide a framework for quantising the intimate relationship between complex differential geometry and complex projective geometry. They foresee that the rich interaction between algebraic and

*Supported by the Grant GACR P201/12/G028

analytic techniques occurring in the classical setting will carry over to the noncommutative world.

The more modest aim of this paper is to begin the development of a theory of noncommutative complex geometry for quantum group homogeneous spaces. This will be done very much in the style of Majid's noncommutative Riemannian geometry [20, 19], with the only significant difference being that here we will not need to assume that our quantum homogeneous spaces are Hopf–Galois extensions, while we will assume a faithful flatness condition. We first introduce the notion of a *covariant noncommutative complex structure* for a differential calculus. Then, by calling on our assumption of faithful flatness, we use Takeuchi's categorical equivalence to establish a simple set of necessary and sufficient conditions for such noncommutative complex structures to exist. In subsequent work, it is intended to build upon these results and formulate noncommutative generalisations of Hodge theory and Kähler geometry for quantum homogeneous spaces [25]. Indeed, the first steps in this direction have already been taken [26].

For this undertaking to be worthwhile, however, it will need to be applicable to a good many interesting examples. Recall that classically one of the most important classes of homogeneous complex manifolds is the family of generalised flag manifolds. As has been known for a long time, these spaces admit a direct q-deformation in terms of the Drinfeld–Jimbo quantum groups [17, 32, 34]. Somewhat more recently, it was shown by Heckenberger and Kolb [9] that the Dolbeault double complex of the irreducible flag manifolds survives this q-deformation intact. This result gives us one of the most important families of noncommutative complex structures that we have, and as such, provides an invaluable testing ground for any newly proposed theory of noncommutative complex geometry.

In this paper we show that, for the special case of the quantum projective spaces, the work of Heckenberger and Kolb can be understood in terms of our general framework for noncommutative complex geometry. This allows for a significant simplification of the required calculations, and helps identify some of the underlying general processes at work. It is foreseen that this work will prove easily extendable to all the irreducible quantum flag manifolds. Moreover, it is hoped to extend it even further to include all the quantum flag manifolds, and in so doing, produce new examples of noncommutative complex structures.

The paper is organised as follows: In section 2 we introduce some well-known material about quantum homogeneous spaces, Takeuchi's categorical equivalence, covariant differential calculi, almost complex structures, and complex structures.

In Section 3 we discuss the quantum special unitary group, and the quantum projective spaces, as well as the Heckenberger–Kolb calculus for these spaces.

In Section 4 we introduce one of the basic results of the paper **Proposition 4.1**: It shows that for a special subcategory of Mod^H_M, the monoidal structure induced on it by the canonical monoidal structure of ${}^G_M\mathrm{Mod}_M$ (through Takeuchi's equivalence) is equivalent to the vector space tensor product.

In Section 5, **Theorem 5.7** shows how to find an explicit description of the maximal prolongation of a covariant first-order differential calculus in terms of a certain ideal $I_M \subseteq M^+$.

In Section 6 we introduce the notion of factorisability for almost complex structures, and establish a simple set of necessary and sufficient conditions for factorisable almost complex structures to exist.

Finally, in Section 7, **Proposition 7.1** gives a simple method for verifying that an almost complex structure is a complex structure.

Throughout, the family of quantum projective spaces, endowed with the Heckenberger–Kolb calculus, is taken as the motivating set of examples. In each section, the newly constructed general theory is applied to these examples in detail, building up to an explicit presentation of their q-deformed Dolbeault double complexes.

2 Preliminaries and First Results

In this section we recall Takeuchi's categorical equivalence for quantum homogeneous spaces, some of its applications to the theory of covariant differential calculi, and finally the definition of a complex structure.

2.1 Quantum Homogeneous Spaces

Let G be a Hopf algebra with comultiplication Δ, counit ε, antipode S, unit 1, and multiplication m. Throughout, we use Sweedler notation, as well as denoting $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ = V \cap \ker(\varepsilon)$, for V a subspace of G. For a right G-comodule V with coaction Δ_R, we say that an element $v \in V$ is *coinvariant* if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by V^G, and call it the *coinvariant subspace* of the coaction. For H a Hopf algebra, a *homogeneous* right H-coaction on G is a coaction of the form $(\mathrm{id} \otimes \pi) \circ \Delta$, where $\pi : G \to H$ is a Hopf algebra map. The coinvariant subspace of such a coaction is a subalgebra [33, Proposition 1].

Definition 2.1. We call the coinvariant subalgebra $M := G^H$ of a homogeneous coaction a *quantum homogeneous space* if G is faithfully flat as a right module over M, which is to say if the functor $G \otimes_M - : {}_M\mathrm{Mod} \to {}_{\mathbb{C}}\mathrm{Mod}$, from the category of left M-modules to the category of complex vector spaces, maps a sequence to an exact sequence if and only if the original sequence is exact.

In this paper we will *always* use the symbols G, H, π and M in this sense. We also note that G is itself a trivial example of a quantum homogeneous space, where $\pi = \varepsilon$. Moreover, the coproduct of G restricts to a right G-coaction on M, and

$$\pi(m) = \varepsilon(m)1_H, \qquad \text{for all } m \in M. \tag{1}$$

If G and H are Hopf $*$-algebras, and π is a Hopf $*$-algebra map, then M is a $*$-subalgebra of G.

2.2 Some Categories

We now define the abelian categories ${}^G_M\mathrm{Mod}_M$ and Mod^H_M. The objects in ${}^G_M\mathrm{Mod}_M$ are M-bimodules $\mathcal{E}$ (with left and right actions denoted by juxtaposition) endowed with a left G-coaction Δ_L such that

$$\Delta_L(mem') = m_{(1)}e_{(-1)}m'_{(1)} \otimes m_{(2)}e_{(0)}m'_{(2)}, \qquad \text{for all } \ m, m' \in M, e \in \mathcal{E}. \tag{2}$$

The morphisms in ${}^G_M\mathrm{Mod}_M$ are the M-bimodule homomorphisms that are also homomorphisms of left G-comodules. The objects in Mod^H_M are right M-modules V (with right action denoted by $\triangleleft$) endowed with a right H-coaction Δ_R such that

$$\Delta_R(v \triangleleft m) = v_{(0)} \triangleleft m_{(2)} \otimes S(\pi(m_{(1)}))v_{(1)}, \qquad \text{for all } \ v \in V, m \in M. \tag{3}$$

The morphisms in Mod^H_M are the M-module homomorphisms that are also homomorphisms of right H-comodules.

Next we introduce a subcategory of ${}^G_M\mathrm{Mod}_M$, and a subcategory of Mod^H_M, that play important roles in the paper. The definition of the latter requires the following technical lemma.

Lemma 2.2 *For Mod^H the category of right H-comodules, we have a fully faithful embedding*

$$\mathrm{Mod}^H \to \mathrm{Mod}^H_M, \qquad V \mapsto (V, \triangleleft), \tag{4}$$

where $\triangleleft$ is the the trivial right M-module structure, $v \triangleleft m = \varepsilon(m)v$, for $v \in V, m \in M$.

Proof. To show that $(V, \triangleleft)$ is well-defined as an object in Mod^H_M, we need to show that (3) is satisfied. This is implied by (1) as follows:

$$\begin{aligned} v_{(0)} \triangleleft m_{(2)} \otimes S(\pi(m_{(1)}))v_{(1)} &= v_{(0)}\varepsilon(m_{(2)}) \otimes \varepsilon(m_{(1)})v_{(1)} = \varepsilon(m)v_{(0)} \otimes v_{(1)} \\ &= \Delta_R(\varepsilon(m)v) = \Delta_R(v \triangleleft m). \end{aligned}$$

Moreover, since any comodule map between V and W is trivially a module map with respect to $\triangleleft$, it is clear that (4) defines a fully faithful functor. □

Definition 2.3. Denote by ${}^G_M\mathrm{Mod}_0$ the full subcategory of ${}^G_M\mathrm{Mod}_M$ whose objects $\mathcal{E}$ satisfy $\mathcal{E}M^+ \subseteq M^+\mathcal{E}$, and denote by Mod^H_0 the image of Mod^H under the embedding in (4).

2.3 Takeuchi's Categorical Equivalence

If $\mathcal{E} \in {}^G_M\mathrm{Mod}_M$, then $\mathcal{E}/(M^+\mathcal{E})$ becomes an object in Mod^H_M with the obvious right M action, and the right H-coaction

$$\Delta_R(\overline{e}) = \overline{e_{(0)}} \otimes S(\pi(e_{(-1)})), \qquad e \in \mathcal{E}. \tag{5}$$

We define a functor $\Phi : {}^G_M\mathrm{Mod}_M \to \mathrm{Mod}^H_M$ as follows: $\Phi(\mathcal{E}) := \mathcal{E}/(M^+\mathcal{E})$, and if $g : \mathcal{E} \to \mathcal{F}$ is a morphism in ${}^G_M\mathrm{Mod}_M$, then $\Phi(g) : \Phi(\mathcal{E}) \to \Phi(\mathcal{F})$ is the map to which g descends on $\Phi(\mathcal{E})$.

If $V \in \mathrm{Mod}^H_M$, then $G \,\square_H V := (G \otimes V)^H$ (where $G \otimes V$ has the obvious tensor product H-comodule structure) becomes an object in ${}^G_M\mathrm{Mod}_M$ with M-bimodule structure

$$m(\sum_i g^i \otimes v^i) = \sum_i mg^i \otimes v^i, \qquad (\sum_i g^i \otimes v^i)m = \sum_i g^i m_{(1)} \otimes (v^i \triangleleft m_{(2)}),$$

and left-G-coaction

$$\Delta_L(\sum_i g^i \otimes v^i) = \sum_i g^i_{(1)} \otimes g^i_{(2)} \otimes v^i.$$

We define a functor $\Psi : \mathrm{Mod}^H_M \to {}^G_M\mathrm{Mod}_M$ as follows: $\Psi(V) := G \,\square_H V$, and if γ is a morphism in Mod^H_M, then $\Psi(\gamma) := \mathrm{id} \otimes \gamma$.

Theorem 2.4 *[33, Theorem 1] An equivalence of categories between* ${}^G_M\mathrm{Mod}_M$ *and* Mod^H_M*, which we call* Takeuchi's equivalence*, is given by the functors* Φ *and* Ψ *and the natural transformations*

$$\mathrm{C} : \Phi \circ \Psi(V) \to V, \qquad \overline{\sum_i g^i \otimes v^i} \mapsto \sum_i \varepsilon(g^i)v^i, \tag{6}$$

$$\mathrm{U} : \mathcal{E} \to \Psi \circ \Phi(\mathcal{E}), \qquad e \mapsto e_{(-1)} \otimes \overline{e_{(0)}}. \tag{7}$$

We define the *dimension* of an object $\mathcal{E} \in {}^G_M\mathrm{Mod}_M$ to be the vector space dimension of $\Phi(\mathcal{E})$.

We now present an explicit formula for the inverse of U. We do so in a number of steps, so as to highlight some results that will be of use to us later.

Lemma 2.5 *An isomorphism is given by*

$$u : G \otimes_M \mathcal{E} \to G \otimes \Phi(\mathcal{E}), \qquad g \otimes_M e \mapsto ge_{(-1)} \otimes \overline{e_{(0)}}.$$

Moreover, the inverse of u *acts according to* $u^{-1}(g \otimes \overline{e}) \mapsto gS(e_{(-1)}) \otimes_M e_{(0)}$.

Proof. We begin by showing that u^{-1} is well-defined. For $g \otimes me$ an element in $G \otimes \mathcal{E}$, with $m \in M^+$, we have

$$\begin{aligned} gS((me)_{(-1)}) \otimes_M (me)_{(0)} &= gS(m_{(1)}e_{(-1)}) \otimes_M m_{(2)}e_{(0)} \\ &= gS(e_{(-1)})S(m_{(1)})m_{(2)} \otimes_M e_{(0)} \\ &= \varepsilon(m)gS(e_{(-1)}) \otimes_M e_{(0)} = 0. \end{aligned}$$

That u^{-1} is indeed the inverse of u follows from

$$\begin{aligned} u^{-1} \circ u(g \otimes_M e) = u^{-1}(ge_{(-1)} \otimes \overline{e_{(0)}}) &= ge_{-2}S(e_{(-1)}) \otimes_M e_{(0)} \\ &= g\varepsilon(e_{(-1)}) \otimes_M e_{(0)} = g \otimes_M e, \end{aligned}$$

and the corresponding calculation for $u \circ u^{-1}$. □

Proposition 2.6 *For $\mathcal{E} \in {}^{G}_{M}\mathrm{Mod}_M$, the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{1 \otimes_M \mathrm{id}} & G \otimes_M \mathcal{E} \\ {\scriptstyle \mathrm{U}}\downarrow & & \downarrow{\scriptstyle u} \\ \Psi \circ \Phi(\mathcal{E}) & \hookrightarrow & G \otimes \Phi(\mathcal{E}), \end{array}$$

where the inclusion in the bottom row is the obvious one. Hence, $1 \otimes_M \mathrm{id}$ *is an embedding.*

Proof.
It follows directly from the definitions of U and u that the diagram is commutative. Hence, since u is an isomorphism, $1 \otimes_M$ id must be an embedding. □

Corollary 2.7 *The inverse of* U *is given by*

$$\mathrm{U}^{-1} : \Psi \circ \Phi(\mathcal{E}) \to G \otimes_M \mathcal{E}, \qquad \sum_i g^i \otimes \overline{e^i} \mapsto \sum_i g^i S(e^i_{(-1)}) \otimes_M e^i_{(0)}.$$

Finally, we turn to the question of how Takeuchi's equivalence behaves upon restricting to the two subcategories ${}^{G}_{M}\mathrm{Mod}_0$ and Mod_0^H.

Lemma 2.8 *Takeuchi's equivalence restricts to an equivalence between the subcategories* ${}^{G}_{M}\mathrm{Mod}_0$ *and* Mod_0^H.

Proof. If $\mathcal{E}$ is an object in ${}^{G}_{M}\mathrm{Mod}_0$, then for any $e \in \mathcal{E}$, and $m \in M^+$, the fact that $\mathcal{E}M^+ \subseteq M^+\mathcal{E}$ implies that $\overline{e} \triangleleft m = 0$. Hence, for any $n \in M$, we have

$$\overline{e} \triangleleft n = \overline{e} \triangleleft (n^+ + \varepsilon(n)1) = \overline{e} \triangleleft n^+ + \overline{e} \triangleleft (\varepsilon(n)1) = \varepsilon(n)\overline{e},$$

showing us that $\Phi_M(\mathcal{E})$ is well-defined as an object in Mod_0^H. Conversely, if V is an object in Mod_0^H, then for any element $\sum_i f^i \otimes v^i$ in $\Psi(V)$,

$$(\sum_i f^i \otimes v^i)m = \sum_i f^i m_{(1)} \otimes (v^i \triangleleft m_{(2)}) = \sum_i f^i m_{(1)} \otimes \varepsilon(m_{(2)})v^i = \sum_i f^i m \otimes v^i.$$

If $m \in M^+$, then $\sum_i f^i m \otimes v^i \in \ker(\mathrm{C})$. But $\ker(\mathrm{C}) = M^+\Psi_M(V)$ so $(\sum_i f^i \otimes v^i)m \in M^+\Psi_M(V)$. Hence $\Psi_M(V)$ belongs to ${}^{G}_{M}\mathrm{Mod}_0$. This establishes the second assertion of the lemma. □

Remark 2.9 *Roughly speaking, we view* ${}^{G}_{M}\mathrm{Mod}_M$ *as generalising the category of equivariant vector bundles over a homogeneous space;* Mod_M^H *as generalising the category of representations of the isotropy subgroup; and Takeuchi's adjunction as generalising the well known equivalence between these categories [29, §1].*

2.4 First Order Differential Calculi

Let A be a unital algebra (in what follows all algebras are assumed to be unital). A *first-order differential calculus* over A is a pair (Ω^1, d), where Ω^1 is an A-A-bimodule and $\mathrm{d} : A \to \Omega^1$ is a linear map for which the *Leibniz rule* holds

$$\mathrm{d}(ab) = a(\mathrm{d}b) + (\mathrm{d}a)b, \qquad a, b, \in A,$$

and for which $\Omega^1 = \mathrm{span}_{\mathbb{C}}\{a\mathrm{d}b \,|\, a, b \in A\}$. (Where no confusion arises, we will drop explicit reference to d and denote a calculus by its bimodule Ω^1 alone.) We call an element of Ω^1 a *one-form*. An *isomorphism* between two first-order differential calculi $(\Omega^1(A), \mathrm{d}_\Omega)$ and $(\Gamma^1(A), \mathrm{d}_\Gamma)$ is a bimodule isomorphism $\varphi : \Omega^1(A) \to \Gamma^1(A)$ such that $\varphi \circ \mathrm{d}_\Omega = \mathrm{d}_\Gamma$. The *direct sum* of two first-order differential calculi $(\Omega^1(A), \mathrm{d}_\Omega)$ and $(\Gamma^1(A), \mathrm{d}_\Gamma)$ is the calculus $(\Omega^1(A) \oplus \Gamma^1(A), \mathrm{d}_\Omega + \mathrm{d}_\Gamma)$.

The *universal first-order differential calculus* over A is the pair $(\Omega^1_u(A), \mathrm{d}_u)$, where $\Omega^1_u(A)$ is the kernel of the product map $m : A \otimes A \to A$ endowed with the obvious bimodule structure, and d_u is defined by

$$\mathrm{d}_u : A \to \Omega^1_u(A), \qquad a \mapsto 1 \otimes a - a \otimes 1.$$

By [35, Proposition 1.1], every first-order differential calculus over A is of the form $(\Omega^1_u(A)/N_A, \mathrm{proj} \circ \mathrm{d}_u)$, where N_A is a A-sub-bimodule of $\Omega^1_u(A)$, and $\mathrm{proj} : \Omega^1_u(A) \to \Omega^1_u(A)/N_A$ is the canonical projection. Moreover, this gives a bijective correspondence between calculi and sub-bimodules.

We say that a differential calculus $\Omega^1(M)$, over a quantum homogeneous space M, is *covariant* if there exists a (necessarily unique) map $\Delta_L : \Omega^1(M) \to G \otimes \Omega^1(M)$, such that

$$\Delta_L(m\mathrm{d}n) = \Delta(m)(\mathrm{id} \otimes \mathrm{d})\Delta(n), \qquad m, n \in M.$$

Any covariant calculus $\Omega^1(M)$ is naturally an object in ${}^G_M\mathrm{Mod}_M$. Moreover, the universal calculus over M is covariant, and covariance of any $\Omega^1(M) \simeq \Omega^1_u(M)/N_M$ is equivalent to N_M being a sub-object of $\Omega^1_u(M)$ in ${}^G_M\mathrm{Mod}_M$. (Note that d is not a morphism in ${}^G_M\mathrm{Mod}_M$.)

The following theorem is a special case of more general results originally established by Hermisson [10, Theorem 2], and Majid [20, Theorem 2.1].

Theorem 2.10 *For a quantum homogeneous space M, considering M^+ as an object in Mod^H_M according to its obvious right M-module structure, and the right H-comodule structure $\Delta_R(m) = m_{(2)} \otimes S(\pi(m_{(1)}))$, for $m \in M^+$, it holds that:*

1. *Covariant first-order differential calculi over M are in bijective correspondence with sub-objects of M^+.*

2. *The ideal corresponding to $\Omega^1(M)$ is given by*

$$I_M := \{\sum_i \varepsilon(m_i)m_i^+ \mid \sum_i m_i\mathrm{d}n_i = 0\}.$$

3. *Denoting* $V_M := M^+/I_M$, *(which we call the* cotangent space *of* $\Omega^1(M)$*) we have an isomorphism*

$$\sigma : \Phi(\Omega^1(M)) \to V_M, \qquad \overline{m\mathrm{d}n} \mapsto \overline{\varepsilon(m)m^+}.$$

Proof. Applying the functor Φ to the collection of sub-objects of $\Omega^1_u(M)$ gives a correspondence between left-covariant calculi over M and sub-objects of $\Phi(\Omega^1_u(M))$. The theorem then follows from the easily verifiable fact that an isomorphism is given by

$$\Phi(\Omega^1_u(M)) \to M^+, \qquad \overline{m\mathrm{d}n} \mapsto \varepsilon(m)n^+.$$

□

For the special case of a trivial quantum homogeneous space, this result reduces to Woronowicz's celebrated theorem classifying left-covariant calculi over a Hopf algebra G [35, Theorem 1.5]. For such a calculus $\Omega^1(G)$, we follow the standard convention of denoting its cotangent space by Λ^1_G, and calling it the *space of left-invariant one forms* of the calculus.

Finally, we note that for $\Omega^1(G)$ any calculus on G, the bimodule $\Omega^1(M) := \{m\mathrm{d}n \,|\, m, n \in M\}$ has the natural structure of a first-order differential calculus over M. We call it the *restriction* of $\Omega^1(G)$ to M.

2.5 Differential Calculi

For $(S, +)$ a commutative semigroup, an *S-graded algebra* is an algebra A equipped with a decomposition $A = \bigoplus_{s \in S} A^s$, where each A^s is a linear subspace of A, and $A^s A^t \subseteq A^{s+t}$, for all $s, t \in S$. If $a \in A^s$, then we say that a is a *homogeneous element of degree s*. A *homogenous mapping of degree t* on A is a linear mapping $L : A \to A$ such that if $a \in A^s$, then $L(a) \in A^{s+t}$. We say that a subspace B of A is *homogeneous* if it admits a decomposition $B = \oplus_{s \in S} B^s$, with $B^s \subseteq A^s$, for all $s \in S$.

A pair (A, d) is called a *complex* if A is an $\mathbb{N}_0$-graded algebra, and d is a homogenous mapping of degree 1, such that $\mathrm{d}^2 = 0$. A triple $(A, \partial, \overline{\partial})$ is called a *double complex* if A is an $\mathbb{N}_0^2$-graded algebra, ∂ is homogeneous mapping of degree $(1, 0)$, $\overline{\partial}$ is homogeneous mapping of degree $(0, 1)$, and

$$\partial^2 = \overline{\partial}^2 = 0, \qquad \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial.$$

Observe that we can associate to any double complex $(A, \partial, \overline{\partial})$ the complex $(A, \partial + \overline{\partial})$.

A complex (A, d) is called a *differential graded algebra* if d is a *graded derivation*, which is to say, if it satisfies the *graded Leibniz rule*

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^n a\mathrm{d}b, \qquad \text{for all } a \in A^n,\ b \in A.$$

The operator d is called the *differential* of the differential graded algebra.

Definition 2.11. A *differential calculus* over an algebra A is a differential algebra $(\Omega(A), \mathrm{d})$ such that $\Omega^0 = A$, and

$$\Omega^k = \mathrm{span}_{\mathbb{C}}\{a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_k \,|\, a_0, \ldots, a_k \in A\}. \tag{8}$$

We use $\wedge$ to denote the multiplication between elements of a differential calculus when both are of order greater than or equal to 1, otherwise we use juxtaposition.

For any A-A-bimodule $\mathcal{E}$, we denote $\mathcal{T}(\mathcal{E}) := \bigoplus_{k=0}^{\infty} \mathcal{E}^{\otimes_A k}$. Endowed with the obvious structure of a graded algebra, we call $\mathcal{T}(\mathcal{E})$ the *tensor algebra* of $\mathcal{E}$. Any first-order differential calculus $\Omega^1(A)$ can be extended to a differential calculus: For $N_A \subseteq \Omega^1_u(A)$ the sub-bimodule corresponding to $\Omega^1(A)$, denote

$$\Omega^\bullet(A) := \mathcal{T}(\Omega^1(A)) / \langle \mathrm{d}N_A \rangle \,, \tag{9}$$

where $\langle \mathrm{d}N_A \rangle$ is the ideal of $\mathcal{T}(\Omega^1(A))$ generated by $\mathrm{d}N_A$, and by abuse of notation, $\mathrm{d}N_A$ is the image in $(\Omega^1(A))^{\otimes_A 2})$ of $d_u N_A$ under the canonical projection $(\Omega^1_u(A))^{\otimes_A 2} \to (\Omega^1(A))^{\otimes_A 2}$. The exterior derivative d is easily seen to have a unique extension $\mathrm{d} : \Omega^\bullet(A) \to \Omega^\bullet(A)$ such that $(\Omega^\bullet(A), \mathrm{d})$ is a differential calculus. We call this differential calculus the *maximal prolongation* of $(\Omega^1(A), \mathrm{d})$. The maximal prolongation is unique in the sense that any other calculus extending $(\Omega^1(A), \mathrm{d})$ can be obtained as a quotient of the maximal prolongation.

If $\Omega^1(M)$ is a covariant first order calculus, and Δ_L extends to a (necessarily unique) algebra map $\Delta_L : \Omega^\bullet(M) \to G \otimes \Omega^\bullet(M)$, then we say that $\Omega^\bullet(M)$ is *covariant*. Clearly, this implies that $\Omega^k \in {}^G_M\mathrm{Mod}_M$, for all $k \in \mathbb{N}_0$. As is easy to see, the maximal prolongation of a covariant first order calculus is covariant, see [15, §12.2.3] for details.

2.6 Differential Calculi over $*$-Algebras

A *first-order differential $*$-calculus* $(\Omega^1(A), \mathrm{d})$ over a $*$-algebra A is a differential calculus over A such that the involution of A extends to an involutive conjugate-linear map $*$ on $\Omega^1(A)$ for which $(a\mathrm{d}b)^* = (\mathrm{d}b^*)a^*$, for all $a, b \in A$. If $\Omega^1(G)$ is a $*$-calculus, then it is easy to see that the restriction of $\Omega^1(G)$ to M will also be a $*$-calculus.

We define $*_\sigma$ to be the mapping for which the following diagram is commutative:

$$\begin{array}{ccc}
\Omega^1(M) & \xleftarrow{\ \mathrm{U}^{-1} \circ (\mathrm{id} \otimes \sigma^{-1})\ } & G \,\Box_H V_M \\
\Big\downarrow{*} & & \Big\downarrow{*_\sigma} \\
\Omega^1(M) & \xrightarrow{\ (\mathrm{id} \otimes \sigma) \circ \mathrm{U}\ } & G \,\Box_H V_M.
\end{array}$$

As is routinely verified, an explicit formula for $*_\sigma$ is given by

$$*_\sigma\Big(\sum_i m^i \otimes \overline{n^i}\Big) = -\sum_i (m^i_{(1)})^* \otimes \overline{S(n^i)^*(m^i_{(2)})^*}. \tag{10}$$

We call a differential calculus $(\Omega^\bullet(A), \mathrm{d})$ over a $*$-algebra A a $*$*-differential calculus* if the involution of A extends to an involutive conjugate-linear map on $\Omega^\bullet$, for which $(\mathrm{d}\omega)^* = \mathrm{d}\omega^*$, for all $\omega \in \Omega$, and

$$(\omega_p \omega_q)^* = (-1)^{pq} \omega_q^* \omega_p^*, \qquad \text{for all } \omega_p \in \Omega^p, \ \omega_q \in \Omega^q.$$

If $\Omega^1(A)$ is a first order $*$-calculus, then its maximal prolongation is a $*$-calculus, see [15, §12.2.3] for details.

2.7 Complex Structures

In this section we introduce a reformulation of Beggs and Smith's definition [1, Definition 2.6] of an almost complex structure (see the remark below) which highlights alternative aspects of the structure more relevant to this and subsequent papers [25]. We also recall Beggs and Smith's generalisation of integrability to the noncommutative setting.

Definition 2.12. An *almost complex structure* for a differential $*$-calculus $\Omega^\bullet(A)$ over a $*$-algebra A, is an $\mathbb{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbb{N}_0^2} \Omega^{(a,b)}$ for $\Omega^\bullet(A)$ such that, for all $(a,b) \in \mathbb{N}_0^2$:

1. $\Omega^k(A) = \bigoplus_{a+b=k} \Omega^{(a,b)}$;
2. $*(\Omega^{(a,b)}) = \Omega^{(b,a)}$.

We call an element of $\Omega^{(a,b)}$ an (a,b)*-form*.

Let ∂, and $\overline{\partial}$ be the unique order $(1,0)$, and $(0,1)$ respectively, homogeneous operators

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

where $\mathrm{proj}_{\Omega^{(a+1,b)}}$, and $\mathrm{proj}_{\Omega^{(a,b+1)}}$, are the projections from $\Omega^{a+b+1}(A)$ onto $\Omega^{(a+1,b)}$, and $\Omega^{(a,b+1)}$, respectively.

Lemma 2.13 *[1, §3.1] If $\bigoplus_{(a,b)\in\mathbb{N}_0^2} \Omega^{(a,b)}$ is an almost-complex structure for a differential calculus $\Omega^\bullet(A)$ over an algebra A, then the following conditions are equivalent:*

1. $\mathrm{d} = \partial + \overline{\partial}$;
2. $\partial^2 = 0$;
3. $\overline{\partial}^2 = 0$;
4. *the triple* $(\bigoplus_{(a,b)\in\mathbb{N}^2} \Omega^{(a,b)}, \partial, \overline{\partial})$ *is a double complex;*
5. $\mathrm{d}(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}$;
6. $\mathrm{d}(\Omega^{(0,1)}) \subseteq \Omega^{(1,1)} \oplus \Omega^{(0,2)}$.

Definition 2.14. When the conditions in Lemma 2.13 hold for an almost-complex structure, then we say that it is *integrable*. We call an integrable almost-complex structure a *complex structure*, and we call the double complex $(\bigoplus_{(a,b)\in\mathbb{N}^2}\Omega^{(a,b)},\partial,\overline{\partial})$ the *Dolbeault double complex* of the complex structure

The following useful lemma shows that Beggs and Smith's definition of a complex structure is equivalent to the definition given by Khalkhali, Landi, and van Suijlekom in [12, Definition 2.1].

Lemma 2.15 *[1, §3] If an almost complex structure $\bigoplus_{(a,b)\in\mathbb{N}_0^2}\Omega^{(a,b)}$ is integrable, then*

1. *$\partial(a^*) = (\overline{\partial}a)^*$, and $\overline{\partial}(a^*) = (\partial a)^*$, for all $a \in A$;*
2. *both ∂ and $\overline{\partial}$ satisfy the graded Leibniz rule.*

Remark 2.16. In [1, Definition 2.6] an almost complex structure, for a differential $*$-calculus $\Omega^\bullet(A)$, is defined to be a zero-order derivation $J : \Omega^\bullet(A) \to \Omega^\bullet(A)$ such that, for all $a \in A$, $J(a) = 0$, and, for all $\omega \in \Omega^1(A)$, $J^2(\omega) = -\omega$ and $J(\omega^*) = (J(\omega))^*$. In [1, §2.5], it is shown that, for any such J, an almost complex structure in the sense of Definition 2.12 is uniquely determined by $J(\omega) = (a-b)i\omega$, for $\omega \in \Omega^{(a,b)}$. That the reverse construction is well-defined follows from the second part of Theorem 6.1.

3 Quantum Projective Space

We introduce quantum projective N-space $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ as a $\mathbb{C}_q[U_{N-1}]$-covariant subalgebra of the quantum special unitary group $\mathbb{C}_q[SU_N]$, and describe its Heckenberger–Kolb first-order differential calculus $\Omega^1_q(\mathbb{C}P^{N-1})$.

3.1 The Quantum Special Unitary Group $\mathbb{C}_q[SU_N]$

We begin by fixing notation and recalling the various definitions and constructions needed to introduce the quantum unitary group and the quantum special unitary group. (Where proofs or basic details are omitted we refer the reader to [15, §9.2].)

For $q \in (0,1]$ and $\nu := q - q^{-1}$, let $\mathbb{C}_q[GL_N]$ be the quotient of the free noncommutative algebra $\mathbb{C}\left\langle u^i_j, \det^{-1} \mid i,j = 1,\ldots,N\right\rangle$ by the ideal generated by the elements

$$\begin{array}{lll} u^i_k u^j_k - q u^j_k u^i_k, & u^k_i u^k_j - q u^k_j u^k_i, & 1 \le i < j \le N, 1 \le k \le N; \\ u^i_l u^j_k - u^j_k u^i_l, & u^i_k u^j_l - u^j_l u^i_k - \nu u^i_l u^j_k, & 1 \le i < j \le N,\ 1 \le k < l \le N; \\ \det_N \det_N^{-1} - 1, & \det_N^{-1} \det_N - 1, & \end{array}$$

where $\det_N$, the *quantum determinant*, is the element

$$\det_N := \sum\nolimits_{\pi\in S_N} (-q)^{\ell(\pi)} u^1_{\pi(1)} u^2_{\pi(2)} \cdots u^N_{\pi(N)}$$

with summation taken over all permutations π of the set $\{1,\ldots,N\}$, and $\ell(\pi)$ the length of π. As is well-known, $\det_N$ is a central and grouplike element of the bialgebra.

A bialgebra structure on $\mathbb{C}_q[GL_N]$ with coproduct Δ, and counit ε, is uniquely determined by $\Delta(u^i_j) := \sum_{k=1}^N u^i_k \otimes u^k_j$; $\Delta(\det_N^{-1}) = \det_N^{-1} \otimes \det_N^{-1}$; and $\varepsilon(u^i_j) := \delta_{ij}$; $\varepsilon(\det_N^{-1}) = 1$. Moreover, we can endow $\mathbb{C}_q[GL_N]$ with a Hopf algebra structure by defining

$$S(\det_N^{-1}) = \det_N, \qquad S(u^i_j) = (-q)^{i-j} \sum\nolimits_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u^{k_1}_{\pi(l_1)} u^{k_2}_{\pi(l_2)} \cdots u^{k_{N-1}}_{\pi(l_{N-1})} \det_N^{-1},$$

where $\{k_1,\ldots,k_{N-1}\} = \{1,\ldots,N\}\backslash\{j\}$, and $\{l_1,\ldots,l_{N-1}\} = \{1,\ldots,N\}\backslash\{i\}$ as ordered sets. A Hopf $*$-algebra structure is determined by $(\det_N^{-1})^* = \det_N$, and $(u^i_j)^* = S(u^j_i)$. We denote the Hopf $*$-algebra by $\mathbb{C}_q[U_N]$, and call it the *quantum unitary group of order N*. We denote the Hopf $*$-algebra $\mathbb{C}_q[U_N]/\left\langle \det_N - 1 \right\rangle$ by $\mathbb{C}_q[SU_N]$, and call it the *quantum special unitary group of order N*.

3.2 The Quantum Projective Spaces $\mathbb{C}_q[\mathbb{C}P^{N-1}]$

Following the description introduced in [21, §3], we present quantum $(N-1)$-projective space as the coinvariant subalgebra of a $\mathbb{C}_q[U_{N-1}]$-coaction on $\mathbb{C}_q[SU_N]$. (This subalgebra is a q-deformation of the coordinate algebra of the complex manifold SU_N/U_{N-1}. Recall that $\mathbb{C}P^{N-1}$ is isomorphic to SU_N/U_{N-1}.)

Definition 3.1. Let $\alpha_N : \mathbb{C}_q[SU_N] \to \mathbb{C}_q[U_{N-1}]$ be the surjective Hopf algebra map defined by setting $\alpha_N(u^1_1) = \det_{N-1}^{-1}$, $\alpha_N(u^1_i) = \alpha_N(u^i_1) = 0$, for $i = 2,\cdots,N$, and $\alpha_N(u^i_j) = u^{i-1}_{j-1}$, for $i,j = 2,\ldots,N$. *Quantum projective $(N-1)$-space* $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ is defined to be the quantum homogeneous space of the corresponding homogeneous coaction $\Delta_{SU_N,\alpha_N} = (\mathrm{id} \otimes \alpha_N) \circ \Delta$.

As is well known, $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ is generated as a $\mathbb{C}$-algebra by the set $\{z_{ij} := u^i_1 S(u^1_j) \mid i,j = 1,\ldots,N\}$ (see [15, §11.6] for more details). Moreover, $\mathbb{C}_q[SU_N]$ is faithfully flat as a right module over $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ [23], and so, $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ is a quantum homogeneous space.

An important family of objects in ${}^G_M\mathrm{Mod}_M$ is the *quantum line bundles $\mathcal{E}_p$*, for $p \in \mathbb{Z}$, where $\mathcal{E}_p := \Psi(\mathbb{C})$, with $\mathbb{C}$ considered as an object in Mod^H_M according to the $\mathbb{C}[U_{N-1}]$-coaction $\lambda \mapsto \lambda \otimes \det^p_{N-1}$, for $\lambda \in \mathbb{C}$. Clearly, we have that $\mathcal{E}_0 = \mathbb{C}_q[\mathbb{C}P^{N-1}]$. (In the $q = 1$ case, these modules are the modules of sections of the line bundles over $\mathbb{C}P^{N-1}$, see Remark 2.9.)

3.3 The Heckenberger–Kolb Calculus $\Omega^1_q(\mathbb{C}P^{N-1})$

In this subsection we recall the first-order differential calculi introduced by Heckenberger and Kolb, and its realization as the restriction to $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ of a certain calculus on $\mathbb{C}_q[SU_N]$.

A left-covariant first-order calculus over an algebra A is called *irreducible* if it does not possess any non-trivial quotients by a left-covariant A-bimodule.

Theorem 3.2 *[8] There exist exactly two non-isomorphic finite-dimensional irreducible left-covariant first-order differential calculi over quantum projective $(N-1)$-space. We call the direct sum of these two calculi the* Heckenberger–Kolb *calculus for* $\mathbb{C}_q[\mathbb{C}P^{N-1}]$.

In general, it proves very useful to realise a calculus on a quantum homogeneous space as the restriction of a calculus over G. The following proposition recalls some details about a calculus over $\mathbb{C}_q[SU_N]$ that restricts to the Heckenberger-Kolb calculus. The technical formulae presented here will be play a crucial role in later calculations.

Proposition 3.3 *[24, §4, §5] For $q \neq 1$, there exists a covariant $*$-calculus $\Omega^1_q(SU_N)$ over $\mathbb{C}_q[SU_N]$ such that:*

1. *For $i = 1, \ldots, N-1$, a basis for $\Lambda^1_q(SU_N)$ is given by $e^0 := \overline{u^1_1 - 1}$, and*

$$e^+_i := \overline{z_{i+1,1}} = q^{-1+\frac{2}{N}} \overline{u^{i+1}_1} = -q^{-\frac{2}{N}} \overline{S(u^{i+1}_1)},$$
$$e^-_i := \overline{z_{1,i+1}} = -q^{1-2i+\frac{2}{N}} \overline{u^1_{i+1}} = q^{2-\frac{2}{N}} \overline{S(u^1_{i+1})}.$$

2. *For $i, j = 2, \ldots, N$, it holds that $\overline{z_{ij}} = \overline{u^i_j} = \overline{S(u^i_j)} = 0$.*

3. *For $i < j$, all the non-zero actions of the generators on the basis elements $e^\pm_i$ are given by*

$$e^+_i \triangleleft u^j_{i+1} = q^{-\frac{2}{N}} \nu e^+_{j-1}, \qquad e^-_i \triangleleft u^{i+1}_j = q^{-\frac{2}{N}} \nu e^-_{j-1}, \tag{11}$$

$$e^\pm_i \triangleleft u^{j+1}_{j+1} = q^{\delta_{1j}+\delta_{ij}-\frac{2}{N}} e^\pm_i. \tag{12}$$

4. *For $i < j$, all non-zero actions of the antipodes of the generators are given by*

$$e^+_i \triangleleft S(u^j_{i+1}) = -q^{\frac{2}{N}} \nu e^+_{j-1}, \qquad e^-_i \triangleleft S(u^{i+1}_j) = -q^{2(i-j+1+\frac{1}{N})} \nu e^-_{j-1}, \tag{13}$$

$$e^\pm_i \triangleleft S(u^j_j) = q^{\frac{2}{N}-\delta_{j1}-\delta_{ji+}} e^\pm_i. \tag{14}$$

The following proposition recalls some important facts about the restriction of $\Omega^1_q(SU_N)$ to $\mathbb{C}_q[\mathbb{C}P^{N-1}]$, principal among them that it is in fact equal to the Heckenberger-Kolb calculus.

Proposition 3.4 *[24, §5] For $\Omega^1_q(\mathbb{C}P^{N-1})$ the restriction of $\Omega^1_q(SU_N)$ to a $*$-calculus over $\mathbb{C}_q[\mathbb{C}P^{N-1}]$, it holds that:*

1. *The right ideal $I_{\mathbb{C}P^{N-1}} \subseteq \mathbb{C}_q[\mathbb{C}P^{N-1}]^+$, corresponding to $\Omega^1_q(\mathbb{C}P^{N-1})$, is generated by the elements*

$$\{z_{ij},\, z_{i1}z_{kl},\, z_{1i}z_{kl} \mid i,j = 2,\ldots,N,\ k,l = 1,\ldots,N,\ (k,l) \neq (1,1)\}, \tag{15}$$

and so, $\Omega^1_q(\mathbb{C}P^{N-1})$ is an object in the subcategory ${}^G_M\mathrm{Mod}_0$.

2. *A decomposition of $V_{\mathbb{C}P^{N-1}}$ in the category Mod^H_M is given by*

$$V_{\mathbb{C}P^{N-1}} = V^{(1,0)} \oplus V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\{e^+_i \mid i = 2,\ldots,N\} \oplus \mathrm{span}_{\mathbb{C}}\{e^-_i \mid i = 2,\ldots,N\},$$

and we denote the corresponding decomposition in ${}^G_M\mathrm{Mod}_M$ by

$$\Omega^1_q(\mathbb{C}P^{N-1}) := \Omega^{(1,0)}_q \oplus \Omega^{(0,1)}_q.$$

3. *The two calculi $\Omega^{(1,0)}_q$ and $\Omega^{(0,1)}_q$ are non-isomorphic, and are the calculi identified in Theorem 3.2.*

4 Monoidal Structures and Equivalences

We use Takeuchi's categorical equivalence to transfer the canonical monoidal structure on ${}^G_M\mathrm{Mod}_M$ to a monoidal structure on Mod^H_M. We then show that for the subcategory Mod^H_0 it has a particularly simple from.

4.1 Monoidal Structures on ${}^G_M\mathrm{Mod}_M$ and Mod^H_M

Let us first recall the standard monoidal structure for ${}^G_M\mathrm{Mod}_M$. For $\mathcal{E}, \mathcal{F}$ two objects in ${}^G_M\mathrm{Mod}_M$, we define $\mathcal{E} \otimes_M \mathcal{F}$ to be the usual bimodule tensor product endowed with the standard left G-comodule structure

$$\Delta_L : \mathcal{E} \otimes_M \mathcal{F} \to G \otimes \mathcal{E} \otimes_M \mathcal{F}, \qquad e \otimes_M f \mapsto e_{(-1)}f_{(-1)} \otimes e_{(0)} \otimes_M f_{(0)}. \tag{16}$$

Clearly, $\mathcal{E} \otimes_M \mathcal{F}$ is well-defined as an object in ${}^G_M\mathrm{Mod}_M$.
The equivalence between ${}^G_M\mathrm{Mod}_M$ and Mod^H_M can be used to induce a monoidal structure on Mod^H_M: For $V, W \in \mathrm{Mod}^H_M$, we define

$$V \odot W := \Phi(\Psi(V) \otimes_M \Psi(W)).$$

4.2 The Restriction of the Monoidal Structure of Mod^H_M to the Subcategory Mod^H_0

The explicit presentation of $\odot$ is somewhat cumbersome. However, upon restricting to the subcategory Mod^H_0 introduced in Section 2.2, a significant simplification occurs.

Proposition 4.1 *Denoting by* $(\mathrm{Mod}_0^H, \otimes)$ *the monoidal category for which* $V \otimes W$ *is the standard right* H*-comodule tensor product of* V *and* W*, endowed with the trivial right action, a monoidal equivalence between* $(\mathrm{Mod}_0^H, \otimes)$ *and* $(\mathrm{Mod}_0^H, \odot)$ *is given by*

$$\mu : V \odot W \to V \otimes W, \qquad \overline{(\sum_i f_i \otimes v_i) \otimes_M (\sum_j g_j \otimes w_j)} \mapsto \sum_{i,j} \varepsilon(f_i)\varepsilon(g_j) v_i \otimes w_j.$$

Proof. The defining property of the subcategory ${}^G_M\mathrm{Mod}_0$ implies that an isomorphism $\Phi(\Psi(V) \otimes_M \Psi(W)) \to \Phi(\Psi(V)) \otimes \Phi(\Psi(W))$ is given by

$$\overline{(\sum_i f_i \otimes v_i) \otimes_M (\sum_j g_j \otimes w_j)} \mapsto \overline{(\sum_i f_i \otimes v_i)} \otimes \overline{(\sum_j g_j \otimes w_j)}.$$

Composing this isomorphism with $\mathrm{U}^{\otimes 2}$ gives the map μ. □

Corollary 4.2 *For any covariant first-order differential calculus* $\Omega^1(M)$ *contained in* ${}^G_M\mathrm{Mod}_0$*, an isomorphism is given by*

$$\sigma^k : \Phi(\Omega^1(M)^{\otimes_M k}) \to V_M^{\otimes k}, \qquad \overline{m_0 \mathrm{d}m_1 \otimes \cdots \otimes \mathrm{d}m_k} \mapsto \varepsilon(m_0)\overline{(m_1)^+} \otimes \cdots \otimes \overline{(m_k)^+},$$

where V_M *is the cotangent space of* $\Omega^1(M)$*.*

Proof. The monoidal equivalence between $(\mathrm{Mod}_0^H, \odot)$ and $(\mathrm{Mod}_0^H, \otimes)$ induces a unique isomorphism $\Phi(\Omega^1(M)^{\otimes_M k}) \simeq \Phi(\Omega^1(M))^{\otimes k}$. Composing this isomorphism with $\sigma^{\otimes k}$ gives σ^k as described. □

Example 4.3. In this subsection we verify that the general framework introduced in this section can be applied to our motivating set of examples.

5 Describing the Maximal Prolongation of a Covariant First-Order Calculus

We give explicit descriptions of the maximal prolongation of a covariant first-order differential calculus, over a quantum homogeneous space M, in terms of the corresponding submodule $N_M \subseteq \Omega^1_u(M)$, and in terms of the corresponding ideal $I_M \subseteq M^+$. The second presentation is then applied to the Heckenberger-Kolb calculus for $\mathbb{C}_q[\mathbb{C}P^{N-1}]$. Throughout this section we will assume that $\Omega^1(M) \in {}^G_M\mathrm{Mod}_0$.

5.1 Describing the Maximal Prolongation in Terms of a Certain Submodule $I_M^{(2)} \subseteq V_M^{\otimes 2}$

We show that the task of finding an explicit description of the maximal prolongation of a first-order calculus can be reduced to the problem of finding an explicit description of a certain submodule $I_M^{(2)} \subseteq V_M^{\otimes 2}$.

Lemma 5.1 *Denote* $V_M^k := V_M^{\otimes k}/I_M^{(k)}$, *where* $I_M^{(k)}$ *is the degree* k *component of the ideal of* $\mathcal{T}(V_M)$ *generated by* $I_M^{(2)} := \sigma^2(\Phi(\mathrm{d}N_M))$. *An isomorphism is given by*

$$\sigma^{\wedge k} : \Phi(\Omega^k(M)) \to V_M^k, \qquad \overline{m_0 \mathrm{d}m_1 \otimes \cdots \otimes \mathrm{d}m^k} \mapsto \varepsilon(m_0)\overline{(m_1)^+} \wedge \cdots \wedge \overline{(m_k)^+},$$

where we use $\wedge$ *to denote multiplication in* $V_M^\bullet := \bigoplus_k V_M^k$.

Proof. For $\omega \in N_M$, and $\mathrm{d} : \Omega^1_u(M) \to \Omega^1(M) \otimes_M \Omega^1(M)$, it follows from

$$\mathrm{d}(m\omega) = \mathrm{d}m \otimes \omega + m\mathrm{d}\omega = m\mathrm{d}\omega, \qquad \mathrm{d}(\omega m) = (\mathrm{d}\omega)m + \omega \otimes \mathrm{d}m = (\mathrm{d}\omega)m,$$

that $\mathrm{d}N_M$ is well-defined as an object in ${}^G_M\mathrm{Mod}_M$. Denoting by $\langle \mathrm{d}N_M \rangle^k$ the k^{th}-component of the ideal of $\mathcal{T}(\Omega^1(M))$ generated by $\mathrm{d}N_M$, exactness of Φ implies that

$$\Phi(\Omega^k(M)) = \Phi(\Omega^1(M)^{\otimes_M k}/\left\langle \mathrm{d}N_M \right\rangle^k) \simeq \Phi(\Omega^1(M)^{\otimes_M k})/\Phi(\langle \mathrm{d}N_M \rangle^k).$$

The map σ^k now induces an isomorphism

$$\sigma^k : \Phi(\Omega^k(M)) \simeq V_M^{\otimes k}/\sigma^{\otimes k}(\Phi(\langle \mathrm{d}N_M \rangle^k)).$$

The lemma now follows from the fact that $\sigma^k(\Phi(\langle \mathrm{d}N_M \rangle^k))$ is equal to the degree k part of the ideal of $\mathcal{T}(V_M)$ generated by $I_M^{(2)}$. □

The following lemma is a first step towards finding a workable description of $I_M^{(2)}$.

Lemma 5.2 *It holds that*

$$I_M^{(2)} = \{\sum_i \overline{m_i^+} \otimes \overline{n_i^+} \mid \sum_i m^i \mathrm{d}n^i = 0\}. \tag{17}$$

Proof. From the definition of the maximal prolongation, we have

$$\Phi_M(\mathrm{d}N_M) = \{\sum_i \overline{\mathrm{d}m^i \otimes_M \mathrm{d}n^i} \mid \sum_i m^i \mathrm{d}n^i = 0\}. \tag{18}$$

Operating on (18) by σ^2 then gives us (17). □

5.2 Describing $I_M^{(2)}$ in Terms of I_M

While we now have an explicit description of $I_M^{(2)}$ in terms of N_M, it proves more useful in practice to have a description of $I_M^{(2)}$ in terms of I_M. In this section we use a certain type of first-order calculus on G to produce just such a description.

Definition 5.3. For any first-order differential calculus $\Omega^1(M)$ over M, a *framing calculus* $\Omega^1(G)$ is a first-order differential calculus over G such that:

1. $\Omega^1(M)$ is the restriction of $\Omega^1(G)$ to M;
2. $\Omega^1(M)G \subseteq G\Omega^1(M)$.

Some consequences of the definition of a framing calculus, under the assumption of covariance, are presented in the following lemma.

Lemma 5.4 *For $\Omega^1(M)$ a covariant first-order differential calculus over M:*

1. *If $\Omega^1(G)$ is a covariant framing calculus, the map $\iota : V_M \to \Lambda_G$, $\overline{m} \mapsto \overline{m}$ is an embedding which induces a right G-action on V_M.*
2. *If $\Omega^1(M)$ is finite dimensional, then a linear isomorphism is defined by*

$$\gamma : V_M^{\otimes 2} \to V_M^{\otimes 2}, \qquad \overline{m^1} \otimes \overline{m^2} \mapsto \overline{m^1 m^2_{(1)}} \otimes \overline{(m^2_{(2)})^+}, \tag{19}$$

with inverse given by $\gamma^{-1}(\overline{m^1} \otimes \overline{m^2}) = \overline{m^1 S(m^2_{(1)})} \otimes \overline{(m^2_{(2)})^+}$.

Proof. The fact that ι is an embedding follows from commutativity of the diagram

$$\begin{array}{ccc} G \,\square_H V_M & \xrightarrow{\mathrm{id}\otimes\iota} & G \otimes \Lambda_G \\ \simeq \uparrow & & \uparrow \simeq \\ \Omega^1(M) & \hookrightarrow & \Omega^1(G). \end{array}$$

Condition 2 in the definition of a framing calculus implies that $\iota(V_M)$ is a G-submodule of Λ_G.

To establish the third part of the lemma, we work in the category of vector spaces. Define γ to be the map for which the following diagram is commutative

$$\begin{array}{ccc} V_M^{\otimes 2} & \xrightarrow{\gamma} & \Lambda_G^{\otimes 2} \\ (\sigma^{\otimes 2})^{-1} \downarrow & & \uparrow \tau \\ \Phi((\Omega^1(M))^{\otimes 2} & \xrightarrow[E]{} & (\Omega^1(G))^{\otimes_G 2}/(G^+(\Omega^1(G))^{\otimes_G 2}), \end{array}$$

where E is the obvious linear map (well-defined since $\Omega^1(M) \in {}^G_M\mathrm{Mod}_0$), and τ is the isomorphism

$$\tau(\overline{g^0 \mathrm{d}g^1 \otimes_G \mathrm{d}g^2}) = \varepsilon(g^0)\overline{(g^1)^+ g^2_{(1)}} \otimes \overline{(g^2_{(2)})^+}.$$

(The fact that τ is well-defined is implied by the canonical isomorphism $\Omega^1(G) \otimes_G \Omega^1(G) \simeq G \otimes \Lambda_G^{\otimes 2}$.) To see that γ coincides with the map defined in (19) we note that

$$\begin{aligned} \tau \circ E \circ (\sigma^{\otimes 2})^{-1}(\overline{m^1} \otimes \overline{m^2}) &= \tau \circ E(\overline{\mathrm{d}m^1} \otimes \overline{\mathrm{d}m^2}) = \tau(\overline{\mathrm{d}m^1 \otimes_G \mathrm{d}m^2}) \\ &= \overline{m^1 m^2_{(1)}} \otimes \overline{(m^2_{(2)})^+}. \end{aligned}$$

Since we are assuming V_M to be finite-dimensional, γ is an isomorphism if, and only if, it is surjective. This is implied by the following calculation, as is the given formula for γ^{-1}:

$$\gamma(\overline{m^1 S(m^2_{(1)})} \otimes \overline{(m^2_{(2)})^+}) = \overline{m^1 S(m^2_{(1)}) m^2_{(2)}} \otimes \overline{(m^2_{(3)})^+} = \overline{m^1} \otimes \overline{m^2}.$$

□

The following useful result is needed for the proof of the theorem below, and for the proof of Proposition (7.1).

Lemma 5.5 *Let* d_σ *be the unique map for which the diagram commutes*

$$\begin{array}{ccc} \Omega^2(M) & \xrightarrow{(\mathrm{id}\otimes\sigma^{\wedge 2})\circ \mathrm{U}} & G\,\square_H V_M^2 \\ \uparrow \mathrm{d} & & \uparrow \mathrm{d}_\sigma \\ \Omega^1(M) & \xleftarrow[\mathrm{U}^{-1}\circ(\mathrm{id}\otimes\sigma^{-1})]{} & G\,\square_H V_M. \end{array}$$

For any $\sum_i f^i \otimes \overline{m^i} \in G\,\square_H V_M$*, the map* d_σ *acts according to*

$$\mathrm{d}_\sigma(\sum_i f^i \otimes \overline{m^i}) = \sum_i f^i_{(1)} \otimes \overline{(f^i_{(2)} S(m^i_{(1)}))^+} \wedge \overline{(m^i_{(2)})^+}.$$

***Proof*.** As a direct consequence of Proposition 2.7 and Lemma 5.1, it holds that

$$\begin{aligned} &(\mathrm{id} \otimes \sigma^{\wedge 2}) \circ \mathrm{U} \circ \mathrm{d} \circ \mathrm{U}^{-1} \circ (\mathrm{id} \otimes \sigma^{-1})(\sum_i f^i \otimes \overline{m^i}) \\ =& (\mathrm{id} \otimes \sigma^{\wedge 2}) \circ \mathrm{U} \circ \mathrm{d}(\sum_i f^i S(m^i_{(1)}) \mathrm{d} m^i_{(2)}) \\ =& (\mathrm{id} \otimes \sigma^{\wedge 2}) \circ \mathrm{U}(\sum_i \mathrm{d}(f^i S(m^i_{(1)})) \wedge \mathrm{d} m^i_{(2)}) \\ =& (\mathrm{id} \otimes \sigma^{\wedge 2})(\sum_i f^i_{(1)} S(m^1_{(2)}) m^i_{(3)} \otimes \overline{\mathrm{d}(f^i_{(2)} S(m^i_{(1)})) \wedge \mathrm{d} m^i_{(4)}}) \\ =& (\mathrm{id} \otimes \sigma^{\wedge 2})(\sum_i f^i_{(1)} \otimes \overline{\mathrm{d}(f^i_{(2)} S(m^i_{(1)})) \wedge \mathrm{d} m^i_{(2)}}) \\ =& \sum_i f^i_{(1)} \otimes \overline{(f^i_{(2)} S(m^i_{(1)}))^+} \wedge \overline{(m^i_{(2)})^+}. \end{aligned}$$

□

A direct consequence of the lemma is the following result.

Corollary 5.6 *It holds that*

$$I^{(2)}_M = \{\sum_i \overline{(f^i S(z^i_{(1)}))^+} \otimes \overline{(z^i_{(2)})^+} \mid \sum f^i \otimes z^i \in G\,\square_H I_M\}.$$

We now come to the central result of this section, which gives us a workable description of the $I_M^{(2)}$.

Theorem 5.7 *Let $\Omega^1(G)$ be a framing calculus for $\Omega^1(M)$, we have the equality*

$$\iota^{\otimes 2}(I_M^{(2)}) = \mathrm{span}_{\mathbb{C}}\{\overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+} \mid z \in \mathrm{Gen}(I_M)\},$$

where $\mathrm{Gen}(I_M)$ *is any subset of* I_M *that generates it as a right* M*-module.*

Proof. Using the elementary identity $(fg)^+ = f^+g + \varepsilon(f)g^+$, for $f, g \in G$, we see that

$$\begin{aligned}
&\sum_i \gamma^{-1} \circ \gamma(\overline{(f^iS(z^i_{(1)}))^+} \otimes \overline{(z^i_{(2)})^+}) \\
&= \sum_i \gamma^{-1} \circ \gamma(\overline{(f^i)^+S(z^i_{(1)})} \otimes \overline{(z^i_{(2)})^+} + \varepsilon(f^i)\overline{(S(z^i_{(1)}))^+} \otimes \overline{(z^i_{(2)})^+})) \\
&= \sum_i \gamma^{-1}(\overline{(f^i)^+S(z^i_{(1)})(z^i_{(2)})} \otimes \overline{(z^i_{(3)})^+} + \varepsilon(f^i)\overline{(S(z^i_{(1)}))^+z^i_{(2)}} \otimes \overline{(z^i_{(3)})^+}) \\
&= \sum_i \gamma^{-1}(\overline{(f^i)^+} \otimes \overline{(z^i)^+} - \varepsilon(f^i)\overline{(z^i_{(1)})^+} \otimes \overline{(z^i_{(2)})^+}) \\
&= -\sum_i \gamma^{-1}(\varepsilon(f^i)\overline{(z^i_{(1)})^+} \otimes \overline{(z^i_{(2)})^+}) = -\sum_i \varepsilon(f^i)\overline{(z^i_{(1)})^+S(z^i_{(1)})} \otimes \overline{(z^i_{(2)})^+} \\
&= \sum_i \varepsilon(f^i)\overline{(S(z^i_{(1)}))^+} \otimes \overline{(z^i_{(2)})^+} = \sum_i \varepsilon(f^i)\overline{S((z^i)_{(1)})} \otimes \overline{(z^i_{(2)})^+}.
\end{aligned}$$

It now follows from the corollary above that $I_M^{(2)} \subseteq \mathrm{span}_{\mathbb{C}}\{\overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+} \mid z \in Gen(I_M)\}$.

For the opposite inclusion, take any $z \in I_M$, and choose a representative element in $G \,\square_H I_M$ for the class $\mathrm{C}^{-1}(z)$. An elementary basis argument shows that the representative can be written in the form $1 \otimes z + \sum_i g^i \otimes z^i$, for some $g^i \in G^+, z^i \in I_M$. Hence, by the above calculation, an element of $I_M^{(2)}$ is given by

$$\overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+} + \sum_i \varepsilon(g^i)\overline{S(z^i_{(1)})} \otimes \overline{(z^i_{(2)})^+} = \overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+}.$$

□

5.3 Framing Calculi for the Heckenberger–Kolb Calculus

We now apply the general theory developed in this section to our motivating set of examples. First, we take $\Omega^1_q(SU_N)$ as a framing calculus for $\Omega^1_q(\mathbb{C}P^{N-1})$, and use it to produce a description of $I_M^{(2)}$. Then we take the famous three-dimensional Woronowicz calculus $\Gamma^1_q(SU_2)$ as a framing calculus for $\Omega^1_q(SU_2)$.

5.3.1 The Calculus $\Omega^1_q(SU_N)$ as a Framing Calculus for $\Omega^1_q(\mathbb{C}P^{N-1})$

In this subsection we show that $\Omega^1_q(SU_N)$ is a framing calculus for $\Omega^1_q(\mathbb{C}P^{N-1})$, and use Theorem 5.7 to calculate the maximal prolongation of $\Omega^1_q(\mathbb{C}P^{N-1})$.

Proposition 5.8 *The calculus $\Omega^1_q(SU_N)$ is a framing calculus for $\Omega^1_q(\mathbb{C}P^{N-1})$, with respect to which the subspace $I^{(2)}_{\mathbb{C}P^{N-1}}$ is spanned by the elements*

$$e^-_i \otimes e^+_j + q^{-1} e^+_j \otimes e^-_i, \qquad e^+_i \otimes e^-_i + q^2 e^-_i \otimes e^+_i - q\nu \sum_{a=i+1}^{N-1} e^-_a \otimes e^+_a, \tag{20}$$

$$e^-_i \otimes e^-_h + q^{-1} e^-_h \otimes e^-_i, \quad e^+_i \otimes e^+_h + q e^+_h \otimes e^+_i, \quad e^+_i \otimes e^+_i, \quad e^-_i \otimes e^-_i, \tag{21}$$

for $h, i, j = 1, \ldots, N-1$, $i \neq j$, and $h < i$.

Proof. To see that $\Omega^1_q(SU_N)$ is a framing calculus $\Omega^1_q(\mathbb{C}P^{N-1})$, we first recall that $\Omega^1_q(SU_N)$ restricts to $\Omega^1_q(\mathbb{C}P^{N-1})$ on $\mathbb{C}_q[\mathbb{C}P^{N-1}]$. The generating set for I_{SU_N} given in definition (**??**) shows that $V_{\mathbb{C}P^{N-1}}$ is a right submodule of $\Lambda^1_{SU_N}$. Hence, $\Omega^1_q(SU_N)$ is a framing calculus for $\Omega^1_q(\mathbb{C}P^{N-1})$.

For sake of convenience, let us recall the generating set of $I_{\mathbb{C}P^{N-1}}$ $\{z_{ij}, z_{i1}z_{kl}, z_{1i}z_{kl} \,|\, i,j = 2, \ldots, N, i \neq j, (k,l) \neq (1,1)\}$ given in the second part of Proposition 3.3. For z_{ij}, we have

$$\overline{S((z_{ij})_{(1)})} \otimes \overline{(z_{ij})_{(2)}} = \sum_{a,b=1}^{N} \overline{S(u^i_a S(u^b_j))} \otimes \overline{u^a_1 S(u^1_b)} = \sum_{a,b=1}^{N} \overline{S^2(u^b_j) S(u^i_a)} \otimes \overline{u^a_1 S(u^1_b)}$$
$$= \sum_{a,b=1}^{N} q^{2(b-j)} \overline{u^b_j S(u^i_a)} \otimes \overline{u^a_1 S(u^1_b)}.$$

From Proposition ?.? we can conclude that the summand $\overline{u^b_j S(u^i_a)} \otimes \overline{u^a_1 S(u^1_b)}$ is non-zero only if $a = i, b = 1$, or $a = 1, b = j$. Thus, we have

$$\begin{aligned}\overline{S((z_{ij})_{(1)})} \otimes \overline{(z_{ij})_{(2)}} &= q^{2(1-j)} \overline{u^1_j S(u^i_i)} \otimes \overline{u^i_1 S(u^1_1)} + \overline{u^j_j S(u^i_1)} \otimes \overline{u^1_1 S(u^1_j)} \\ &= q^{2(1-j)} \overline{u^1_j S(u^i_i)} \otimes e^+_{i-1} + \overline{S(u^i_1) u^j_j} \otimes e^-_{j-1} &(22)\\ &= q^{2(1-j)+2j-3} e^-_{j-1} \otimes e^+_{i-1} - e^+_{i-1} \otimes e^-_{j-1} &(23)\\ &= -q^{-1} e^-_{j-1} \otimes e^+_{i-1} - e^+_{i-1} \otimes e^-_{j-1}, &(24)\end{aligned}$$

where we have used Proposition ?.?, the standard relation $u^j_j S(u^i_1) = S(u^i_1) u^j_j$ [27, Theorem 1]. This gives us the first element in (20). A similar analysis will show that $\overline{S((z_{i1}z_{1j})_{(1)})} \otimes \overline{(z_{i1}z_{1j})_{(2)}}$, and $\overline{S((z_{1j}z_{i1})_{(1)})} \otimes \overline{(z_{1j}z_{i1})_{(2)}}$, are both also equal to scalar multiples of $e^-_{j-1} \otimes e^+_{i-1} + q e^+_{i-1} \otimes e^-_{j-1}$.

If we now assume that $i < j$, then for $z_{i1}z_{j1}$, and $z_{j1}z_{i1}$, we have that $\overline{S((z_{i1}z_{j1})_{(1)})} \otimes \overline{(z_{i1}z_{j1})_{(2)}}$, and $\overline{S((z_{j1}z_{i1})_{(1)})} \otimes \overline{(z_{j1}z_{i1})_{(2)}}$, are both equal to linear multiples of the element

$$e^+_{i-1} \otimes e^+_{j-1} + q\, e^+_{j-1} \otimes e^+_{i-1}.$$

While for $z_{1i}z_{1j}$, and $z_{1j}z_{i1}$, we have that $\overline{S((z_{1i}z_{1j})_{(1)})} \otimes \overline{(z_{1i}z_{1j})_{(2)}}$, and $\overline{S((z_{1j}z_{1i})_{(1)})} \otimes \overline{(z_{1j}z_{1i})_{(2)}}$, are both equal to linear multiples of the element

$$e^-_{j-1} \otimes e^-_{i-1} + q^{-1} e^-_{i-1} \otimes e^-_{j-1}.$$

The generators z_{ii}, $z_{i1}z_{1i}$, and $z_{1i}z_{i1}$, give in all three cases a linear multiple of

$$e^+_{i-1} \otimes e^-_{i-1} + q^2 e^-_{i-1} \otimes e^+_{i-1} - q\nu \sum_{a=i+1}^{N-1} e^-_{a-1} \otimes e^+_{a-1}.$$

Similarly, the generators $z_{i1}z_{i1}$, and $z_{1i}z_{1i}$ give scalar multiples of $e^+_{i-1} \otimes e^+_{i-1}$, and $e^-_{i-1} \otimes e^-_{i-1}$ respectively. Finally, for $k, l \neq 1$, we get that

$$\overline{S((z_{i1}z_{kl})_{(1)})} \otimes \overline{(z_{i1}z_{kl})_{(2)}} = \overline{S((z_{1i}z_{kl})_{(1)})} \otimes \overline{(z_{1i}z_{kl})_{(2)}} = 0.$$

□

5.3.2 The Woronowicz $3D$-Calculus on $\Omega^1_q(SU_2)$ as a Framing Calculus

In this subsection we specialise to the case of $\mathbb{C}_q[\mathbb{C}P^1]$, and use the famous Woronowicz $3D$-calculus $\Gamma^1_q(SU_2)$ on $\mathbb{C}_q[SU_2]$ as a framing calculus for the Heckenberger–Kolb calculus. This serves to highlight the fact that there can exist more than one framing calculus for a calculus on a quantum homogeneous space. Following standard convention, we use the following notation for the special case of $\mathbb{C}_q[SU_2]$: $a := u^1_1, b := u^1_2, c := u^2_1$, and $d := u^2_2$.

Definition 5.9. The *Woronowicz $3D$-calculus* $\Gamma^1_q(SU_2)$ is the left-covariant first-order differential calculus on $\mathbb{C}_q[SU_2]$ corresponding to the ideal of $\mathbb{C}_q[SU_2]^+$ generated by the elements

$$a + q^{-2}d - (1 + q^{-2}),\ bc,\ b^2,\ c^2,\ (a-1)b,\ (a-1)c. \tag{25}$$

Using $\Gamma^1_q(SU_2)$ as a framing calculus, we recalculate $I^{(2)}_{\mathbb{C}P^1}$, and see that it agrees with Proposition 5.8 for the case of $N = 2$.

Lemma 5.10 *The calculus $\Gamma^1_q(SU_2)$ is a framing calculus for $\Omega^1_q(\mathbb{C}P^1)$, and*

$$I^{(2)}_{\mathbb{C}P^{N-1}} = \text{span}_{\mathbb{C}}\{e^+ \otimes e^+,\ e^- \otimes e^-,\ e^+ \otimes e^- + q^{-2}e^- \otimes e^+\}. \tag{26}$$

Proof. For the case of $N = 2$, the description of the Heckenberger-Kolb ideal given in the second part of Proposition 3.3 reduces to $I_{\mathbb{C}P^1} = \langle (ab)^2, bc, (cd)^2 \rangle$. Hence, we have a well-defined map $V_{\mathbb{C}P^1} \to \Lambda^1_{SU_2}$. As is well-known [35] and easily checked, the elements $\overline{b}$ and $\overline{c}$ are linearly independent in $\Lambda^1_q(SU_2)$. From (25) we have that $e^+ = q\overline{c}$, and $e^- = -q^{-1}\overline{b}$, and so, this map is an inclusion. Moreover, since it is also clear from (25) that $V_{\mathbb{C}P^1}$ is a right $\mathbb{C}_q[SU_2]$-submodule of $\Lambda^1_{SU_2}$, we have that $\mathbb{C}_q[SU_2]$ is a framing calculus for $\Omega^1_q(\mathbb{C}P^1)$.

We now come to the calculation of $I^{(2)}_{\mathbb{C}P^1}$: For bc, we have that

$$\begin{aligned}\overline{S((bc)_{(1)})} \otimes \overline{((bc)_{(2)})^+} &= \overline{S(ac)} \otimes \overline{ba} + \overline{S(ad)} \otimes \overline{bc} + \overline{S(bc)} \otimes \overline{(da)^+} + \overline{S(bd)} \otimes \overline{dc} \\ &= -q\,\overline{cd} \otimes \overline{ba} - q^{-1}\overline{ab} \otimes \overline{dc} = qe^+ \otimes e^- + q^{-1}e^- \otimes e^+ \\ &= q(e^+ \otimes e^- + q^{-2}e^- \otimes e^+).\end{aligned}$$

Analogous calculations will show that $S(((ab)^2)_{(1)}) \otimes ((ab)^2)_{(2)}$ is equal to a scalar multiple of $e^- \otimes e^-$, and $S(((cd)^2)_{(1)}) \otimes ((cd)^2)_{(2)}$ is equal to a scalar multiple of $e^- \otimes e^-$. □

6 Covariant Almost Complex Structures

We begin this section by providing sufficient and necessary conditions for a bimodule decomposition of a first-order differential calculus to extend to a complex structure on its maximal prolongation. The notion of a factorisable complex structure is then introduced, and a convenient formulation of the concept at the level of 1-forms is produced. Finally, we apply our results to the Heckenberger–Kolb calculus for the quantum projective spaces. ?

6.1 Extending 1-Form Decompositions to Almost Complex Structures

For any smooth manifold M, every decomposition of the cotangent bundle into a direct sum of sub-bundles of equal dimension extends to an almost complex structure on M. As the following proposition shows, things are more complicated in the noncommutative setting.

Proposition 6.1 *For $\Omega^1(A) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ a decomposition of $\Omega^1(A)$ into sub-bimodules:*

1. *the decomposition has at most one extension, satisfying condition (1), to an $\mathbb{N}^2_0$-grading of the maximal prolongation of $\Omega^1(A)$;*

2. *such an extension exists if, and only if, $\mathrm{d}N_A$ is homogeneous with respect to the decomposition*

$$(\Omega^1)^{\otimes_A 2} = \Omega^{\otimes(2,0)} \oplus \Omega^{\otimes(1,1)} \oplus \Omega^{\otimes(0,2)}, \tag{27}$$

where $\Omega^{\otimes(\bullet,\bullet)}$ denotes the unique $\mathbb{N}_0^2$-grading, of the tensor algebra of $\Omega^1(A)$, extending the decomposition of $\Omega^1(A)$.

3. *Moreover, condition 2 holds if, and only if, $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$, or equivalently if, and only if, $*(\Omega^{(0,1)}) = \Omega^{(1,0)}$.*

Proof. Since $\langle \mathrm{d}N_A \rangle$ is generated as an ideal by $\mathrm{d}N_A$, it is clear that homogeneity of $\mathrm{d}N_A$, with respect to the decomposition in (27), will imply homogeneity of $\langle \mathrm{d}N_A \rangle$ with respect to the grading $\Omega^{\otimes(\bullet,\bullet)}$. Hence, $\Omega^{\otimes(\bullet,\bullet)}$ will descend to a grading on the maximal prolongation. Conversely, if $\mathrm{d}N_A$ is not homogeneous with respect to to (27), then $\Omega^{\otimes(\bullet,\bullet)}$ obviously cannot descend to a grading on the maximal prolongation.

Next we show that this grading is the only possible $\mathbb{N}_0^2$-grading on the maximal prolongation extending the decomposition of $\Omega^1(A)$: For another distinct grading $\Gamma^{(\bullet,\bullet)}$ to exist, there would have to be an element $\omega \in \Omega^{\otimes(a,b)}$, for some $(a,b) \in \mathbb{N}_0^2$, such that the image of ω in $\Omega^\bullet(A)$ was not contained in $\Gamma^{(a,b)}$. Now every element of $\Omega^{\otimes(a,b)}$ is of the form $\omega := \sum_{i=1} \omega_1^i \otimes \cdots \otimes \omega_{a+b}^i$, where each summand $\omega_1^i \otimes \cdots \otimes \omega_{a+b}^i$ has exactly a of its factors contained in $\Omega^{(1,0)}$, and b of its factors contained in $\Omega^{(0,1)}$. The general properties of a graded algebra imply that the image of such an element in $\Omega^\bullet(A)$ must be contained in $\Gamma^{(a,b)}$. Hence, we must conclude that there exists no other grading on the maximal prolongation extending the decomposition $\Omega^1(A)$. This gives us the first and second parts of the proposition.

For the third and final part of the theorem, note that since the $*$-map is involutive, assuming $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$ is equivalent to assuming $*(\Omega^{(0,1)}) = \Omega^{(1,0)}$. Every element of $\Omega^{(a,b)}$ is of the form $\omega := \sum_{i=1} \omega_1^i \wedge \cdots \wedge \omega_{a+b}^i$, where each summand $\omega_1^i \wedge \cdots \wedge \omega_{a+b}^i$ has exactly a of its factors contained in $\Omega^{(1,0)}$, and b of its factors contained in $\Omega^{(0,1)}$. The properties of a graded $*$-algebra imply that

$$\omega^* = \sum_{i=1} (\omega_1^i \wedge \cdots \wedge \omega_{a+b}^i)^* = \sum_{i=1} (-1)^{\frac{(a+b)(a+b-1)}{2})} (\omega_{a+b}^i)^* \wedge \cdots \wedge (\omega_1^i)^*. \tag{28}$$

Our two equivalent assumptions, and the properties of a graded algebra, now imply that ω^* must be contained in $\Omega^{(b,a)}$, giving us that $*(\Omega^{(a,b)}) \subseteq \Omega^{(b,a)}$. The opposite inclusion is established analogously, giving us the desired equality. □

6.2 Covariant Almost Complex Structures

In this subsection, we introduce the notion of a covariant almost complex structure, and find a set of simple conditions for such a structure to exist.

Definition 6.2. An almost complex structure $\Omega^{(\bullet,\bullet)}$ for a covariant differential $*$-calculus $\Omega^\bullet(M)$ is *left-covariant* if $\Omega^{(a,b)}$ is a sub-object of Ω^k in ${}^G_M\mathrm{Mod}_M$, for all $(a,b) \in \mathbb{N}_0^2$.

The following theorem is an easy consequence of Proposition 6.1 and Proposition 4.1, so we omit the proof.

Theorem 6.3 *For a covariant differential $*$-calculus $\Omega^\bullet(M)$:*

1. *An almost complex structure is covariant if, and only if, $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are objects in ${}^G_M\mathrm{Mod}_M$.*
2. *If $\Omega^1(M) \in {}^G_M\mathrm{Mod}_0$, then such a decomposition extends to an $\mathbb{N}_0^2$-grading of the maximal prolongation of $\Omega^1(M)$ if, and only if, $I_M^{(2)}$ is homogeneous with respect to the decomposition*
$$V_M^{\otimes 2} = V_M^{\otimes(2,0)} \oplus V_M^{\otimes(1,1)} \oplus V_M^{\otimes(0,2)}, \tag{29}$$
where $V_M^{\otimes(\bullet,\bullet)}$ is the obvious grading on the tensor algebra of V_M induced by σ^k.

We finish this subsection by deriving a simple pair of sufficient conditions for the second axiom of an almost complex structure to hold.

Lemma 6.4 *It holds that $G(G \,\square_H \Phi(\mathcal{E})) = G \otimes \Phi(\mathcal{E})$.*

***Proof*.** Since $ge_{(-1)} \otimes \overline{e_{(0)}}$ is clearly contained in $G(G \,\square_H \Phi(\mathcal{E}))$, we must have $u(G \otimes_M \mathcal{E}) \subseteq G(G \,\square_H \Phi(\mathcal{E}))$. The required equality now follows from the fact that u is an isomorphism. $\square$

Proposition 6.5 *Let $\Omega^1(M) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ be a decomposition in ${}^G_M\mathrm{Mod}_M$, and let $\Omega^1(G)$ be a $*$-calculus on G that frames $\Omega^1(M)$. If*
$$\Omega^{(1,0)}G \subseteq G\Omega^{(1,0)}, \qquad \text{and} \qquad \Omega^{(0,1)}G \subseteq G\Omega^{(0,1)}, \tag{30}$$
then we have $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$ if, and only if,
$$\{\overline{S(m)^*} \,|\, \overline{m} \in V_M^{(1,0)}\} = V_M^{(0,1)}, \quad \text{or equivalently} \quad \{\overline{S(m)^*} \,|\, \overline{m} \in V_M^{(0,1)}\} = V_M^{(1,0)}. \tag{31}$$

***Proof*.** It is clear from (10) that if the first inclusion in (30) holds, then the first equality in (31) implies $*(\Omega^{(1,0)}) \subseteq \Omega^{(0,1)}$. That the second equality in (31) is equivalent to the first follows from the identity $S(S(g)^*)^* = g$, for all $g \in G$. This similarly implies that $*(\Omega^{(0,1)}) \subseteq \Omega^{(1,0)}$, giving the required equality $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$.
It follows from the lemma above that
$$m \circ u^{-1} \circ (\mathrm{id} \otimes \sigma^{-1})(G \otimes V_M^{(1,0)}) = G\Omega^{(1,0)}.$$
Hence, we have the commutative diagram
$$\begin{array}{ccc}
G \otimes V_M^{(1,0)} & \xrightarrow{\;m \circ u^{-1} \circ (\mathrm{id}\otimes\sigma^{-1})\;} & G\Omega^{(1,0)} \\
\downarrow{\scriptstyle *_\sigma} & & \downarrow{\scriptstyle *} \\
G \otimes V_M \supseteq G \otimes V_M^{(0,1)} & \xleftarrow[\;(\mathrm{id}\otimes\sigma) \circ u\;]{} & G\Omega^{(0,1)} \supseteq \Omega^{(0,1)}G.
\end{array}$$
Hence, $*_\sigma(1 \otimes \overline{m}) = 1 \otimes \overline{S(m^*)} \in G \otimes V_M^{(0,1)}$ giving $\{\overline{S(m)^*} \,|\, \overline{m} \in V_M^{(1,0)}\} \subseteq V_M^{(0,1)}$. The inclusion $\{\overline{S(m)^*} \,|\, \overline{m} \in V_M^{(0,1)}\} \subseteq V_M^{(1,0)}$ is established analogously. That these two inclusions are equalities now follows from the identity $S(S(g)^*)^* = g$. $\square$

6.3 Factorisable Almost Complex Structures

In this section we introduce the property of factorisability for an almost complex structure. The Dolbeault double complex of every complex manifold automatically satisfies this property [11, §1.2], as do the Heckenberger-Kolb calculi for the all irreducible flag manifolds [9, Proposition 3.11].

Definition 6.6. An *almost complex structure* for a differential $*$-calculus $\Omega^\bullet(A)$ over a $*$-algebra A, is called *factorisable* if we have isomorphisms

$$\wedge : \Omega^{(a,0)} \otimes_A \Omega^{(0,b)} \simeq \Omega^{(a,b)}, \qquad \text{and} \qquad \wedge : \Omega^{(0,b)} \otimes_A \Omega^{(a,0)} \simeq \Omega^{(a,b)}. \tag{32}$$

The following proposition establishes a simple set of necessary and sufficient criteria for an almost complex structure to be factorisable.

Proposition 6.7 *An almost complex structure is factorisable if, and only if, we have isomorphisms*

$$\wedge : \Omega^{(1,0)} \otimes_A \Omega^{(0,1)} \to \Omega^{(1,1)}, \qquad \wedge : \Omega^{(0,1)} \otimes_A \Omega^{(1,0)} \to \Omega^{(1,1)}. \tag{33}$$

***Proof*.** Surjectivity of the first map in (33) means that for any $\omega^+ \in \Omega^{(1,0)}$, and $\omega^- \in \Omega^{(0,1)}$, there exist forms $\omega_i^+ \in \Omega^{(1,0)}$, and $\omega_i^- \in \Omega^{(0,1)}$ such that $\omega^- \wedge \omega^+ = \sum_i \omega_i^+ \wedge \omega_i^-$. This easily implies surjectivity of the first map in (32). The proof in the other direction is trivial. That surjectivity of the second map in (32) is equivalent to surjectivity of the second map in (33) is established analogously.

The first map in (32) is injective if, for all $(a,b) \in \mathbb{N}_0^2$,

$$\langle \mathrm{d}N \rangle \cap (\Omega^{\otimes(a,0)} \otimes_A \Omega^{\otimes(0,b)}) = \langle \mathrm{d}N \rangle^{(a,0)} \otimes_A \Omega^{\otimes(0,b)} + \Omega^{\otimes(a,0)} \otimes_A \langle \mathrm{d}N \rangle^{(0,b)}, \tag{34}$$

where $\langle \mathrm{d}N \rangle^{(a,0)}$, and $\langle \mathrm{d}N \rangle^{(0,b)}$, are respectively the $\otimes(a,0)$, and $\otimes(0,b)$, homogeneous components of $\langle \mathrm{d}N \rangle$. Now when the first mapping in (33) is injective

$$\mathrm{d}N \cap (\Omega^{(1,0)} \otimes_A \Omega^{(0,1)}) = \{0\}.$$

Thus, for a general element $\sum_i \nu_i \otimes \omega_i \otimes \nu_i' \in \langle \mathrm{d}(N) \rangle$, where $\nu_i, \nu_i' \in \mathcal{T}(\Omega^1(A))$, and $w_i \in \mathrm{d}N$ is a homogeneous element of $\mathrm{d}N$, when $\sum_i \nu_i \otimes \omega_i \otimes \nu_i' \in \Omega^{\otimes(a,0)} \otimes_A \Omega^{\otimes(0,b)}$ we must have that $\omega^i \in \Omega^{\otimes(2,0)}$, or $\omega^i \in \Omega^{\otimes(0,2)}$. Hence (34) holds, and the first map in (33) is injective. The proof in the other direction is trivial. That injectivity of the second map in (32) is equivalent to injectivity of the second map in (33) is established analogously. □

Finally, we specialise to the case of a covariant almost complex structure, such that $\Omega^1(M) \in {}^G_M\mathrm{Mod}_0$, and find a very useful reformulation of (33). We omit the proof which follows directly from the above proposition and Proposition 4.1 and ??monoidal??.

Corollary 6.8 *If* $\Omega^1(M) \in {}^G_M\mathrm{Mod}_0$, *then the almost complex structure is factorisable if, and only if, we have isomorphisms*

$$\wedge : V_M^{(1,0)} \otimes V_M^{(0,1)} \to V_M^{(1,1)}, \qquad \wedge : V_M^{(0,1)} \otimes V_M^{(1,0)} \to V_M^{(1,1)}. \tag{35}$$

6.4 A Factorisable Almost Complex Structure for the Maximal Prolongation of the Heckenberger–Kolb Calculus

We now apply the general theory developed in this section to the Heckenberger-Kolb calculus. (Note that the isomorphisms in (36) are direct generalisations of well known classical results [11, §2.4]. In particular, the third isomorphism generalises orientability of $\mathbb{C}_q[\mathbb{C}P^{N-1}]$.)

Proposition 6.9 *The decomposition* $\Omega^1_q(\mathbb{C}P^{N-1}) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ *(as presented in Proposition 3.3) extends to a covariant factorisable almost complex structure on the maximal prolongation of* $\Omega^1(\mathbb{C}P^{N-1})$.

***Proof*.** That $I^{(2)}_{\mathbb{C}P^{N-1}}$ is homogeneous with respect to the decomposition (29) in Theorem 6.3 follows directly from Proposition 5.8, as does the fact that the maps (35) in Corollary 6.8 are isomorphisms. The inclusions (30) in Proposition 6.5 follow directly from the third part of Proposition 3.3. Moreover, (31) follows from the fact that, for $i = 2, \ldots, N$, we have

$$\begin{aligned}\overline{S(z_{i1})^*} &= \overline{S(u^i_1 S(u^1_1))^*} = \overline{(u^1_1 S(u^i_1))^*} = \overline{S(u^i_1)^*(u^1_1)^*} = \overline{S^{-1}((u^i_1)^*)S(u^1_1)}\\ &= \overline{S^{-1}\circ S(u^1_i)S(u^1_1)} = \overline{u^1_i S(u^1_1)} = q^{-4+2i} e^-_{i-1} \in V^{(0,1)}_M,\end{aligned}$$

where we have used the standard Hopf $*$-algebra identity $* \circ S = S^{-1} \circ *$. Thus, the decomposition extends to an almost complex structure. □

Corollary 6.10 *The vector space dimension of* $V^{(a,b)}_{\mathbb{C}P^{N-1}}$ *is* $\binom{N-1}{a}\binom{N-1}{b}$*, and a basis given by*

$$\{e^+_{i_1} \wedge \cdots \wedge e^+_{i_a} \wedge e^-_{j_1} \wedge \cdots \wedge e^-_{j_b} \,|\, i_1 < \cdots < i_a;\, j_1 < \cdots < j_b\}.$$

***Proof*.** It is obvious from the description of the generators of $I^{(2)}_M$ given in **??** that the proposed basis spans $V^\bullet_M$. Hence, we only need to establish linear independence. We begin with the case of $V^{(N-1,0)}_M$, where this amounts to showing that we have a non-zero vector space. To this end, we define a function $f : (V^{(1,0)})^{\otimes N-1} \to \mathbb{C}$ by specifying its values on the basis $\{e^+_{i_1} \otimes \cdots \otimes e^+_{i_N} \,|\, i_1, \ldots, i_{N-1} = 1, \ldots, N-1\}$. For basis elements of the form $e^+_{\pi(1)} \otimes \cdots \otimes e^+_{\pi(N-1)}$, where π is a permutation of the set $\{1, \ldots, N-1\}$, we define

$$f(e^+_{\pi(1)} \otimes \cdots \otimes e^+_{\pi(N-1)}) := (-q)^{-\ell(\pi)},$$

where ℓ is the length of the permutation. On all other basis elements we set f to zero. It follows from our description of the generators of $I^{(2)}_M$, and the definition of $I^{(N)}_M$, that f descends to a non-zero map on the quotient $V^{(N-1,0)}_M = (V^{(1,0)})^{\otimes N-1}/I^{(N-1)}_M$. Hence, $V^{(N-1,0)}_M \neq 0$.

We now move on to the case of $V_M^{(a,0)}$, for $a = 2, \dots, N-2$. Suppose we have a linear combination of the proposed basis vectors, with each summand of degree $(a,0)$, and for which

$$\sum_{i_1<\cdots<i_a} \lambda_{i_1,\dots,i_a} e^+_{i_1} \wedge \cdots \wedge e^+_{i_a} = 0.$$

For any $1 \le j_1 \le \cdots \le j_a \le N$, if we denote by $\{k_1, \dots, k_{N-a}\}$ the complement of $\{j_1, \dots, j_a\}$ in the set $\{1, \dots, N\}$, then

$$\begin{aligned}
&\Big(\sum_{i_1<\cdots<i_a} \lambda_{i_1,\dots,i_a} e^+_{i_1} \wedge \cdots \wedge e^+_{i_a}\Big) \wedge (e^+_{k_1} \wedge \cdots \wedge e^+_{k_{N-a}}) \\
&= \lambda_{j_1,\dots,j_a} e^+_{j_1} \wedge \cdots \wedge e^+_{j_k} \wedge e^+_{k_1} \wedge \cdots \wedge e^+_{k_{N-k}} \\
&= \lambda_{j_1,\dots,j_a} (-q)^l e^+_1 \wedge \cdots \wedge e^+_{N-1},
\end{aligned}$$

for some $\lambda \in \mathbb{Z}$. Since $e^+_1 \wedge \cdots \wedge e^+_{N-1} \neq 0$, we must have $\lambda_{j_1,\dots,j_a} = 0$. Thus, the proposed basis elements which are contained in $V_M^{(\bullet,0)}$ are linearly independent.

A similar argument will establish linear independence of the basis elements contained in $V_M^{(0,\bullet)}$. Linear independence of all the proposed basis elements now follows from the fact that the calculus is factorisable □

The above corollary tells us that the bundles $\Omega^{(N-1,0)}$, $\Omega^{(0,N-1)}$, and $\Omega^{(N-1,N-1)}$ are non-trivial line bundles. The lemma below identifies these line bundles with respect to the classification of equivariant line bundles in terms of integers.

Lemma 6.11 *It holds that*

$$\Omega_q^{(N-1,0)} \simeq \mathcal{E}_{-N}, \qquad \Omega_q^{(0,N-1)} \simeq \mathcal{E}_N, \qquad \Omega_q^{(N-1,N-1)} \simeq \mathbb{C}_q[\mathbb{C}P^{N-1}]. \tag{36}$$

Proof. The coaction on $V_M^{(N-1,0)} \simeq \mathbb{C}e^+_1 \wedge \cdots \wedge e^+_{N-1}$, which we denote by Δ_M^{N-1}, acts according to

$$\begin{aligned}
\Delta_M^{N-1}(e^+_1 \wedge \cdots \wedge e^+_{N-1}) &= \sum_{l=1}^{N-1} \sum_{k_l=1}^{N-1} e^+_{k_1} \wedge \cdots \wedge e^+_{k_{N-1}} \otimes S(u^1_{k_1}) \cdots S(u^{N-1}_{k_{N-1}}) \det{}^{N-1}_{N-1} \\
&= \sum_{l=1}^{N-1} \sum_{k_l=1}^{N-1} e^+_{k_1} \wedge \cdots \wedge e^+_{k_{N-1}} \otimes S(u^{N-1}_{k_{N-1}} \cdots u^1_{k_1}) \det{}^{N-1}_{N-1}.
\end{aligned}$$

Since any summand with a repeated basis element in the first tensor factor will be zero,

$$\Delta_M^{N-1}(e^+_1 \wedge \cdots e^+_{N-1}) = \sum_{\pi \in S_{N-1}} e^+_{\pi(1)} \wedge \cdots \wedge e^+_{\pi(N-1)} \otimes S(u^{N-1}_{\pi(N-1)} \cdots u^1_{\pi(1)}) \det{}^{N-1}_{N-1}.$$

Moreover, $e^+_{\pi(1)} \wedge \cdots \wedge e^+_{\pi(N-1)} = (-q)^{-\mathrm{sgn}(\pi)} e^+_1 \wedge \cdots \wedge e^+_{N-1}$, for any $\pi \in S_{N-1}$. Since

$$\sum_{\pi \in S_{N-1}} (-q)^{-\mathrm{sgn}(\pi)} u^{N-1}_{\pi(N-1)} \cdots u^1_{\pi(1)} = \det{}_{N-1},$$

we must have that

$$\Delta_M^{N-1}(e_1^+ \wedge \cdots e_{N-1}^+) = e_1^+ \wedge \cdots \wedge e_{N-1}^+ \otimes S(\det_{N-1}) \det_{N-1}^{N-1}$$
$$= e_1^+ \wedge \cdots \wedge e_{N-1}^+ \otimes \det_{N-1}^{-N},$$

which in turn implies that $\Omega^{(N-1,0)} \simeq \mathcal{E}_{-N}$.

An analogous argument will establish that $\Omega^{(N-1,0)}$ is isomorphic to $\mathcal{E}_N$. It follows as a direct consequence of these two results that $\Omega^{(N-1,N-1)}$ is isomorphic to $\mathbb{C}_q[\mathbb{C}P^{N-1}]$. $\square$

7 Complex Structures

In this section we give a simple set of sufficient criteria for a covariant almost complex structure to be integrable, find an interesting connection between integrability and the maximal prolongations of $\Omega^{(0,1)}$ and $\Omega^{(0,1)}$, and show that the Heckenberger–Kolb calculus for $\mathbb{C}_q[\mathbb{C}P^{N-1}]$ satisfies these criteria.

7.1 Integrability for a Covariant Almost Complex Structure

We use the assumption of covariance to find a simple set of sufficient criteria for an almost-complex structure to be integrable. Throughout this subsection $\Omega^{(\bullet,\bullet)}$ denotes a covariant almost-complex structure, such that each $\Omega^{(a,b)}$ is an object in ${}^G_M\mathrm{Mod}_0$.

Proposition 7.1 *The almost-complex structure $\Omega^{(\bullet,\bullet)}$ is integrable if, for any linear projection $P : \Lambda_G^{\otimes 2} \to V_M^{\otimes 2}$, it holds that*

$$P(\overline{(S(m_{(1)}))^+} \otimes \overline{(m_{(2)})^+}) \in \iota^{\otimes 2}(V_M^{\otimes(2,0)}) \oplus \iota^{\otimes 2}(V_M^{\otimes(1,1)}), \quad \text{for all } \overline{m} \in V_M^{(1,0)}, \tag{37}$$

where $\iota : V_M \to \Lambda_G$ is the embedding introduced in Lemma 5.4. Integrability also follows from the corresponding condition for $V_M^{(0,1)}$.

Proof. Denote by $K : V_M^2 \to \Lambda_G^{\otimes 2}/I_M^{(2)}$ the obvious linear embedding, and $\overline{v} \wedge \overline{w} := \mathrm{proj}(\overline{v} \otimes \overline{w})$, where $\mathrm{proj} : \Lambda_G^{\otimes 2} \to \Lambda_G^{\otimes 2}/I_M^{(2)}$ is the canonical projection. By Lemma 5.1, for any $\sum_i f^i \otimes \overline{m^i} \in G \,\square_H V_M^{(1,0)}$, we have that

$$\begin{aligned}
&(\mathrm{id} \otimes K) \circ \mathrm{d}_\sigma(\sum_i f^i \otimes \overline{m^i}) \\
&= (\mathrm{id} \otimes K)(\sum_i f^i_{(1)} \otimes \overline{(f^i_{(2)} S(m^i_{(1)}))^+} \wedge \overline{(m^i_{(2)})^+}) \\
&= \sum_i f^i_{(1)} \otimes \varepsilon(f^i_{(2)})\overline{(S(m^i_{(1)}))^+} \wedge \overline{(m^i_{(2)})^+} + \sum_i f^i_{(1)} \otimes \overline{(f^i_{(2)})^+ S(m^i_{(1)})} \wedge \overline{(m^i_{(2)})^+} \\
&= \sum_i f^i \otimes \overline{(S(m^i_{(1)}))^+} \wedge \overline{(m^i_{(2)})^+} + \sum_i f^i_{(1)} \otimes \overline{(f^i_{(2)})^+ \varepsilon(S(m^i_{(1)}))} \wedge \overline{(m^i_{(2)})^+} \\
&= \sum_i f^i \otimes \overline{(S(m^i_{(1)}))^+} \wedge \overline{(m^i_{(2)})^+} + \sum_i f^i_{(1)} \otimes \overline{(f^i_{(2)})^+} \wedge \overline{(m^i)^+},
\end{aligned}$$

where in the third line we have used the standard identity $(fg)^+ = \varepsilon(f)g^+ + f^+g$, for $f, g \in G$, and in the fourth line we have used the assumption that $\Omega^{(a,b)} \in {}^{G}_{M}\mathrm{Mod}_0$, for all $(a,b) \in \mathbb{N}_0^2$. Thus, if (37) holds, then $\mathrm{d}_\sigma(\sum_i f^i \otimes \overline{m^i}) \in G \,\square_H (V_M^{(2,0)} \oplus V_M^{(1,1)})$, which in turn implies that $\mathrm{d}(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}$.

That integrability follows from the corresponding condition for $V_M^{(0,1)}$ is established analogously. $\square$

7.2 Integrability and the Maximal Prolongations of $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$

In this subsection we give an alternative characterisation of integrability for a (not necessarily covariant) almost complex structure $\Omega^{(\bullet,\bullet)}$:The pairs $(\Omega^{(1,0)}, \partial)$ and $(\Omega^{(0,1)}, \overline{\partial})$ are each first order differential calculi with their own respective maximal prolongations. Let us denote the k-forms of the maximal prolongation of $\Omega^{(1,0)}$, and $\Omega^{(0,1)}$, by $(\Omega^{(1,0)})^k$, and $(\Omega^{(0,1)})^k$ respectively.

Lemma 7.2 *For an almost complex structure $\Omega^{(\bullet,\bullet)}$, the equalities*

$$(\Omega^{(1,0)})^k = \Omega^{(k,0)}, \qquad \textit{and} \qquad (\Omega^{(0,1)})^k = \Omega^{(0,k)}, \tag{38}$$

are equivalent to each other, and to integrability.

***Proof*.** Let $\{\omega_i^-\}_i$, be a subset of $\Omega^1_u(M)$, such that $\mathrm{span}_{\mathbb{C}}\{\omega_i^-\} = \Omega^{(0,1)}$, (where we use the same symbol for ω_i^- as for its coset in $\Omega^1(A)$). If N_A is the sub-bimodule of $\Omega^1_u(A)$ corresponding to $\Omega^1(A)$, then it is clear that the sub-bimodule of $\Omega^1_u(A)$ corresponding to $(\Omega^{(1,0)}, \partial)$ is given by

$$N_A^+ := N_A + \mathrm{span}_{\mathbb{C}}\{\omega_i^-\}_i.$$

Since $(\Omega^{(1,0)})^{\otimes_A k} = (\Omega^1(A))^{\otimes(k,0)}$, it is clear that the lemma would follow from the equality $\partial N_A^+ = (\mathrm{d}N_A)^{\otimes(2,0)}$ (where by abuse of notation we mean d and ∂ in the sense of (9)). But $\partial B = (\mathrm{d}B)^{\otimes(2,0)}$, for any bimodule $B \subseteq \Omega^1_u(A)$. Hence, having $\partial N_A^+ = (\mathrm{d}N_A)^{\otimes(2,0)}$ is equivalent to having $\partial\omega_i^- = 0$, for all i, which is in turn equivalent to the almost complex structure being integrable.

That the second equality in (38) is equivalent to integrability is established analogously. $\square$

7.3 Integrability of the Heckenberger-Kolb Calculus

We now apply the general results developed in this section to our motivating set of examples.

Proposition 7.3 *The almost-complex structure $\Omega_q^{(\bullet,\bullet)}(\mathbb{C}P^{N-1})$ is integrable.*

Proof. For $\overline{z_{i1}} = \overline{u_1^i S(u_1^1)} \in V_M^{(1,0)}$, with $i = 2, \ldots, N$, we have:

$$\begin{aligned}
&\overline{(S((z_{i1})_{(1)}))^+} \otimes \overline{((z_{i1})_{(2)})^+} \\
&= \sum_{a=2}^{N} \overline{(S(u_a^i S(u_1^b)))^+} \otimes \overline{(u_1^a S(u_b^1))^+} = \sum_{a=1}^{N} q^{2(b-1)} \overline{(u_1^b S(u_a^i))^+} \otimes \overline{(u_1^a S(u_b^1))^+} \\
&= \sum_{a=2}^{N} \overline{(u_1^1 S(u_a^i))^+} \otimes \overline{(u_1^a S(u_1^1))^+} + \sum_{b=2}^{N} q^{2(b-1)} \overline{(u_1^b S(u_1^i))^+} \otimes \overline{(u_1^1 S(u_b^1))^+} \\
&= \sum_{a=2}^{N} \overline{(u_1^1 S(u_a^i))^+} \otimes \overline{(u_1^a S(u_1^1))^+} = \overline{(u_1^1 S(u_i^i))^+} \otimes \overline{u_1^i S(u_1^1)} = \overline{(u_1^1 S(u_i^i))^+} \otimes e_{i-1}^+.
\end{aligned}$$

Thus, for any linear projection $P : \Lambda_G^{\otimes 2} \to V_M^{\otimes 2}$, requirement (37) of Proposition (7.1) will be satisfied. □

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