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ON TOPOLOGICAL QUANTUM FIELD THEORY
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PREFACE

The axiomatic approach to quantum field theories as functors from geometric bordism categories to vector spaces can be viewed as a representation of spacetime on algebra. Over the past three decades, this way of thinking has been deeply tied with several interrelated disciplines in mathematics and theoretical physics. These include topological invariants, representation theory, homological mirror symmetry, the geometric Langlands programme, (∞, n)-categories, topological quantum computation and topological phases of matter.

The abstract scheme admits explicit realisations in low-dimensional field theories in the presence of a topological charge in terms of distinguished differential characters. These appear naturally in a variety of physical models, including the Lagrangean formulation of critical string theory, effective field theories of spin-chain excitations and the Hamiltonian description of certain condensed-matter systems, and thus demonstrate the applicability of the arsenal of methods and intuitions of TQFT in the physical setting.

The present set of lecture notes, originally prepared for the *Advanced School on Topological Quantum Field Theory* which was held in Warsaw in December 2015, aims to provide an introduction to the basic theory of low-dimensional TQFTs, as well as to *some* of the more recent developments. More precisely, a brief outline is as follows:

Chapter 1 (*Introductory lectures on topological quantum field theory*) reviews Atiyah's original axioms for (closed, oriented) TQFTs and relates them in detail to the definition in terms of symmetric monoidal functors that has arguably since become standard. The relevant category-theoretic notions are introduced, and TQFTs in one and two spacetime dimensions are discussed in some detail. Some aspects of 3-dimensional "extended" TQFTs are sketched—hopefully putting the reader in a good position to benefit more from the existing literature.

Chapter 2 (*Lecture notes on two-dimensional defect TQFT*) introduces "defect TQFTs" in two dimensions as functors on stratified, decorated bordisms. Such TQFTs naturally give rise to an algebraic structure called "pivotal 2-category". This relation is described in detail, and several examples are mentioned. Furthermore, it is discussed how "open/closed" 2-dimensional TQFTs are special cases of defect TQFTs, and their algebraic description in terms of commutative Frobenius algebras, Calabi–Yau categories, and the Cardy condition is explained.

Chapter 3 (*An invitation to 2D TQFT and quantization of Hitchin spectral curves*) uncovers the relations between TQFT and quantum curves, whose theory has been recently actively developed. In the first part of this chapter, notions such as ribbon graphs, Frobenius algebras, and cohomological field theories are introduced, and their relations to 2-dimensional TQFT are discussed. In the second part of this chapter, the quantum curves are presented from the viewpoint of quantization of Hitchin spectral curves to opers.

Chapter 4 (*Bundle gerbes for topological insulators*) bridges the gap between the abstract and the concrete, even phenomenological in the context of interest and puts TQFT in a modern physical perspective. This is achieved by guiding the reader through a hands-on construction of several so-called Kane–Mele $\mathbb{Z}_2$-valued topological invariants for 2- and 3-dimensional static resp. periodically driven (Floquet) topological insulators with time-reversal symmetry. The construction is based on a suitable adaptation of the concept of surface holonomy associated with a geometrisation of a class in the 3rd de Rham cohomology group of a manifold, termed the “bundle gerbe (with curving and connection)” and known from the study of 2d sigma models with a topological charge. Implementation of the time-reversal involution on the gerbe leads to “equivariantisation” of the geometric construction, discussed at length in Lecture 1 in the physically relevant (unitary) Lie-group setting. Lecture 2 introduces the condensed-matter systems with crystalline symmetry and an energy gap around the Fermi level whose quantum-mechanical (Hamiltonian) description leads to the emergence of the topological invariants mentioned, with a clear-cut geometric interpretation presented in the text. The invariants are subsequently reexpressed in terms of (square roots of) surface holonomies of involution-equivariant gerbes computed along maps—determined by the Hamiltonian spectral projectors of the quantum crystal—to (the double cover of) the unitary-group manifold. Finally, a holographic conduction mechanism in finite-size topological insulators is indicated which renders the invariants amenable to direct experimental measurements.

The lecture notes collected in this volume are by no means exhaustive on the subject of TQFTs and their applications. In particular, fully extended TQFTs are hardly covered at all, and most of the applications—some of them mentioned above—are entirely omitted. A similar fate befell natural extensions of the more physically oriented components of the course. For them, we refer to the rich and quickly growing repertoire of review articles, lecture notes, books and front-line papers, which can be considered the main course after the appetisers offered here.

The Editors