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Perfect Gaussian integers

by

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1. Introduction. Let $\eta = \varepsilon\prod\pi_i^{k_i}$ be a Gaussian integer, with ε a unit, $\operatorname{Re}\pi_i > 0$ and $\operatorname{Im}\pi_i \geqslant 0$. In 1961, Spira [3] defined a sum of divisors function σ in the ring $\mathfrak{G}$ of Gaussian integers by

$$\sigma(\eta) = \prod (\pi_i^{k_i+1}-1)/(\pi_i-1),$$

and extended the concepts of even and odd, Mersenne primes, and perfect numbers to $\mathfrak{G}$ as follows:

(i) η is defined to be an *even* Gaussian integer if $(1+i)|\eta$ and an *odd* integer if $(1+i)\nmid\eta$.

(ii) The sum $\sigma\big((1+i)^{k-1}\big) = -i[(1+i)^k-1] = M_k$ is called a *complex Mersenne prime* if M_k is prime.

(iii) η is a *perfect number* if $\sigma(\eta) = (1+i)\eta$ and η is a *norm-perfect number* if $\|\sigma(\eta)\| = 2\|\eta\|$. (For any complex number z, $\|z\| = zz^*$, where z^* denotes the complex conjugate of z.)

If a (norm-) perfect number η does not have a (norm-) perfect number as a proper divisor, we will say that η is *primitive*.

We characterize the even primitive norm-perfect numbers in this paper and establish as our principal result the following analog of the corresponding Euclid–Euler theorem in the ring $\mathcal{Z}$ of rational integers:

MAIN THEOREM. *Let M_p be a complex Mersenne prime and ε a unit. If $p\equiv 1 \pmod 8$, $\eta = \varepsilon(1+i)^{p-1}M_p$ is a primitive norm-perfect number; if $p\equiv -1 \pmod 8$, $\eta = \varepsilon(1+i)^{p-1}M_p^*$ is a primitive norm-perfect number. Conversely, if η is an even primitive norm-perfect number, then, for some unit ε, either $\eta = \varepsilon(1+i)^{p-1}M_p$, where M_p is a complex Mersenne prime with $p\equiv 1 \pmod 8$, or $\eta = \varepsilon(1+i)^{p-1}M_p^*$, where M_p is a complex Mersenne prime with $p \equiv -1 \pmod 8$.*

Our theorem has the following

COROLLARY. *η is an even primitive perfect number iff there exists a rational prime $p\equiv 1 \pmod 8$ such that $\eta = (1+i)^{p-1}M_p$, where M_p is a complex Mersenne prime.*

With the exception of M_k, Greek letters will be used throughout to designate Gaussian integers, and p will always denote a rational prime. In the interest of simplifying the notation we will write $\sigma(1+i)^{k-1}$ for $\sigma\big((1+i)^{k-1}\big)$.

2. An inequality. Spira [3] showed that if $\eta\in\mathfrak{G}$ and $\operatorname{Re}\eta\geqslant 1$, then

$$|\eta^{n+1}-1|\geqslant|\eta^n|\cdot|\eta-1|,$$

with equality holding iff $\eta=1$. A proof for $\eta=z$, an arbitrary complex number, may be found in Mitrinović [2], p. 140.

We now improve this inequality for $|z|\geqslant\sqrt{5}$.

THEOREM 1. *Let $z=x+iy$, $|z|\geqslant\sqrt{5}$, and $x\geqslant 1$. For any positive integer n,*

$$\|1+z+\ldots+z^n\|>\|z^n\|\cdot[1+(2x-1.4)/\|z\|],$$

and if, moreover, $|y|\leqslant x-1$,

$$\|1+z+\ldots+z^n\|>\|z^n\|\cdot[1+2x/\|z\|].$$

Proof. If $n=1$, we have

$$\|1+z\|=(1+x)^2+y^2=\|z\|\cdot[1+(2x+1)/\|z\|].$$

If $n=2$,

$$\begin{aligned}\|1+z+z^2\|&=\|z\|\cdot\|z^{-1}+1+z\|\\&=\|z\|\cdot[\|z\|+2x+1+(2x^2+2x+1-2y^2)/\|z\|]\\&>\begin{cases}\|z^2\|\cdot[1+(2x-1)/\|z\|], & \text{for arbitrary } y,\\ \|z^2\|\cdot[1+(2x+1)/\|z\|], & \text{for } |y|\leqslant x-1.\end{cases}\end{aligned}$$

Now, writing $z=re^{i\theta}$, we have

$$\begin{aligned}\|1+z+\ldots+z^n\|&=\|(z^{n+1}-1)/(z-1)\|\\&=(z^{n+1}-1)[(z^*)^{n+1}-1]/(z-1)(z^*-1)\\&=\{\|z^{n+1}\|+1-[z^{n+1}+(z^*)^{n+1}]\}/(r^2-2x+1)\\&=\|z^n\|\cdot[r^2+r^{-2n}-2r^{1-n}\cos(n+1)\theta]/(r^2-2x+1).\end{aligned}$$

If $n\geqslant 3$, this quotient is

$$>\|z^n\|\cdot(r^2-2r^{-2})/(r^2-2x+1).$$

Using $r\geqslant\sqrt{5}$, an easy calculation shows this last quotient to be $\geqslant\|z^n\|\times\times[1+(2x-1.4)/\|z\|]$; another short calculation shows that if $|y|\leqslant x-1$, the quotient is $>\|z^n\|\cdot[1+2x/\|z\|]$.

We shall refer to a Gaussian prime π as a first-quadrant prime if $\operatorname{Re}\pi>0$ and $\operatorname{Im}\pi\geqslant 0$. It should be noted that every Gaussian integer is uniquely representable as the product of a unit and powers of first-quadrant primes. Theorem 1 has the following

COROLLARY. *If π is an odd Gaussian prime, then*

$$\|\sigma(\pi^t)/\pi^t\|>(\|\pi\|+2x-1.4)/\|\pi\|,$$

where $x+iy$ is the first-quadrant prime associate of π; if $y\leqslant x-1$, then

$$\|\sigma(\pi^t)/\pi^t\|>(\|\pi\|+2x)/\|\pi\|.$$

Proof. We need only note that for the first-quadrant prime $x+iy$ and unit ε such that $\varepsilon\pi=x+iy$, $\sigma(\pi^t)$ is equal to $1+(\varepsilon\pi)+\ldots+(\varepsilon\pi)^t$, and then apply Theorem 1.

3. Norm-perfect numbers. Let $\eta=(1+i)^{k-1}\mu$, with μ odd and $k>1$. Spira [3] has noted that if $k\not\equiv 0,\pm1\pmod 8$, then $|\sigma(\eta)|>|(1+i)^k\mu|$, that is, $\|\sigma(\eta)\|>2\|\eta\|$; so η is norm-perfect only if $k\equiv 0,\pm1\pmod 8$.

We let $A_k=\|M_k\|$, and note the following:

(1) If $k\equiv 0\pmod 8$,

$$M_k=-i(2^{k/2}-1)\quad\text{and}\quad A_k=2^k-2^{(k+2)/2}+1;$$

(2) if $k\equiv\pm1\pmod 8$,

$$M_k=\pm2^{(k-1)/2}-(2^{(k-1)/2}-1)i\quad\text{and}\quad A_k=2^k-2^{(k+1)/2}+1.$$

LEMMA 1. *Suppose η is a norm-perfect number, π is an odd prim divisor of η, and $\varepsilon\pi$ is the first-quadrant associate of π.*

(a) *If $k\equiv 0,\pm1\pmod 8$ and $k\geqslant 11$, then $\|\pi\|>A_k^{1/3}$;*

(b) *if $k\equiv\pm1\pmod 8$, then $\|\pi\|>.3\,[2^{(k+1)/2}-1]$;*

(c) *if $k\equiv\pm1\pmod 8$ and $\operatorname{Re}\varepsilon\pi\neq 1$, then $\|\pi\|>A_k^{1/2}$.*

Proof. Let a be the largest rational integer such that $\pi^a|\eta$. Since, for any prime power δ^b, $\|\sigma(\delta^b)/\delta^b\|>1$, and since σ is a multiplicative function,

$$1=\|\sigma(\eta)\|/2\|\eta\|\geqslant\|\sigma(1+i)^{k-1}\|\cdot\|\sigma(\pi^a)\|/(2^k\|\pi^a\|).$$

By the Corollary to Theorem 1, this is greater than $A_k(\|\pi\|+c)/(2^k\|\pi\|)$, where $c=.6$ if $\operatorname{Re}\varepsilon\pi=1$ and $c=2.6$ if $\operatorname{Re}\varepsilon\pi>1$. Solving for $\|\pi\|$, we have

$$\|\pi\|>cA_k/(2^k-A_k).\tag{3}$$

(a). If $k\equiv 0,\pm1\pmod 8$, $A_k\geqslant 2^k-2^{(k+2)/2}+1$, by (1) and (2). Hence, from (3),

$$\|\pi\|\geqslant .6\,[2^k-2^{(k+2)/2}+1]/[2^{(k+2)/2}-1]>.6\,[2^{(k-2)/2}-1],$$

which, for $k\geqslant 11$, is $>2^{k/3}>A_k^{1/3}$.

(b) and (c). If $k \equiv \pm 1 \pmod 8$, $A_k = 2^k - 2^{(k+1)/2} + 1 > [2^{(k+1)/2} - 1]^2/2$. Hence, from (3),

$$\|\pi\| > c[2^{(k+1)/2} - 1]^2/[2(2^{(k+1)/2} - 1)] = (c/2)[2^{(k+1)/2} - 1].$$

This product is clearly greater than or equal to $.3[2^{(k+1)/2} - 1]$; if $\operatorname{Re} \varepsilon\pi \neq 1$, the product equals $1.3[2^{(k+1)/2} - 1]$ which is greater than $A_k^{1/2}$.

LEMMA 2. *Suppose $\eta = (1+i)^{k-1}\mu$ is a norm-perfect number with $k \equiv \pm 1 \pmod 8$, and $\sigma(1+i)^{k-1} = \varepsilon\pi\varrho$, where ε is a unit and π and ϱ are first-quadrant primes. If $\pi = a+bi$ and $\varrho = c+di$, then at most one of the rational integers a, b, c and d is equal to 1.*

Proof. If $k = 7$, $\sigma(1+i)^{k-1} = -(8+7i)$ is not the product of two primes. If $k = 9$, η is not norm-perfect, since $\sigma(1+i)^8 = -(2+3i)(1+6i)$, and $\|2+3i\| = 13 < \sqrt{481} = A_9^{1/2}$, contrary to Lemma 1(c). The remaining values of k are $\geqslant 15$.

Neither a and b nor c and d can both be 1, since $\sigma(1+i)^{k-1}$ is an odd number. Suppose one element from each set $\{a, b\}$ and $\{c, d\}$ is equal to 1, and assume, w.l.o.g., that $\|\pi\| \leqslant \|\varrho\|$. Then $\|\pi\| = 1+g^2$, where g is a or b, and $\|\varrho\|$ may be similarly expressed. Now, $\pi|\eta$ or $\pi^*|\eta$, since η norm-perfect implies that

$$2\eta\eta^* = 2\|\eta\| = \|\sigma(\eta)\| = \|\pi\varrho\sigma(\mu)\| = \pi\pi^*\|\varrho\sigma(\mu)\|.$$

By Lemma 1(b),

$$1+g^2 = \|\pi\| = \|\pi^*\| > .3[2^8-1] > 76.$$

So, $g \geqslant 9$, which implies that $0 < \arg\pi < 7°$ or $83° < \arg\pi < 90°$. Since $\|\varrho\| \geqslant \|\pi\|$, $0 < \arg\varrho < 7°$ or $83° < \arg\varrho < 90°$, also. However, for $k \geqslant 15$, $\arg\sigma(1+i)^{k-1}$ is within one degree of $-45°$ if $k \equiv 1 \pmod 8$ or within one degree of $-135°$ if $k \equiv -1 \pmod 8$. We now have a contradiction, since $\arg\sigma(1+i)^{k-1} = \arg\varepsilon + \arg\pi + \arg\varrho$.

LEMMA 3. *If $\eta = (1+i)^{k-1}\mu$ is norm-perfect, then $M_k = \sigma(1+i)^{k-1}$ is prime and k is a rational prime congruent to $\pm 1 \pmod 8$.*

Proof. Let π denote the first-quadrant prime factor of $\sigma(1+i)^{k-1}$ of least norm and assume that η is norm-perfect. As noted in the proof of Lemma 2, either $\pi|\eta$ or $\pi^*|\eta$. By Spira's result, $k \equiv 0, \pm 1 \pmod 8$.

If $k \equiv 0 \pmod 8$, say $k = 8t$, then $A_k = A_{8t} = (2^{4t}-1)^2$. If, now, $t = 1$, the Gaussian prime 3 divides A_k which implies that $3^r|\eta$ for some $r \geqslant 1$; but, using the Corollary to Theorem 1,

$$\begin{aligned} 1 &= \|\sigma(\eta)/[(1+i)\eta]\| \\ &\geqslant \|\sigma(1+i)^7 \cdot \sigma(3^r)/[(1+i)^8 \cdot 3^r]\| > \frac{225}{256} \cdot \frac{15}{9} > 1. \end{aligned}$$

On the other hand, if $t > 1$, then $\|\pi\| \leqslant 2^t - 1 = 2^{k/8} - 1 < A_k^{1/3}$, contrary to Lemma 1(a). We will therefore assume for the remainder of the proof that $k \equiv \pm 1 \pmod 8$.

If $k = 7$, the lemma is true, since $\sigma(1+i)^6 = -(8+7i)$ is a prime; if $k = 9$, the hypothesis is not satisfied, as noted in the proof of Lemma 2. If $k \geqslant 15$, then, by Lemma 1(a), $\|\pi\| > A_k^{1/3}$ which implies that $\sigma(1+i)^{k-1}$ has at most two prime factors (not necessarily distinct). Suppose $\sigma(1+i)^{k-1}$ has exactly two prime factors, and let ϱ denote the second first-quadrant prime factor. Now, we may assume that $\operatorname{Re}\pi = 1$, since if $\operatorname{Re}\pi \neq 1$, then $\|\pi\| > A_k^{1/2}$, by Lemma 1(c), and hence $\sigma(1+i)^{k-1}$ is prime. By Lemma 2, then, $\operatorname{Re}\varrho \neq 1$. π is the prime factor of $\sigma(1+i)^{k-1}$ of least norm, so $\|\varrho\| \geqslant A_k^{1/2}$. Since $A_k = 2^k - 2^{(k+1)/2} + 1$, $\|\varrho\| > .7[2^{(k+1)/2} - 1]$, and, by Lemma 1(b), $\|\pi\| > .3[2^{(k+1)/2} - 1]$. These two inequalities and the Corollary to Theorem 1 yield, for some rational integers c and d,

$$\begin{aligned} 1 &= \|\sigma(\eta)/[(1+i)\eta]\| \\ &\geqslant \|\sigma(1+i)^{k-1} \cdot \sigma(\pi^c) \cdot \sigma(\varrho^d)\| \cdot \|(1+i)^k \pi^c \varrho^d\|^{-1} \\ &> \|\pi\varrho\|(\|\pi\| + .6)(\|\varrho\| + 2.6)[2^k \cdot \|\pi\| \cdot \|\varrho\|]^{-1} \\ &> [2^k + (.2) \cdot 2^{(k+1)/2} + 1.36] \cdot 2^{-k} > 1. \end{aligned}$$

Hence, $\sigma(1+i)^{k-1}$ is prime.

To see that k is prime, we assume that $k = qs$, q a rational prime and $q \leqslant s$. Then $\sigma(1+i)^{k-1} = -i[(1+i)^k - 1]$ has the proper divisor $[(1+i)^q - 1]$, contradicting our result that $\sigma(1+i)^{k-1}$ is prime.

THEOREM 2. *If η is an even norm-perfect number, then there exists a Mersenne prime M_p such that, for some positive integer t and odd Gaussian integer δ, either*

(a) *$\eta = (1+i)^{p-1} M_p^t \cdot \delta$, with $p \equiv 1 \pmod 8$,*

or

(b) *$\eta = (1+i)^{p-1} (M_p^*)^t \cdot \delta$, with $p \equiv -1 \pmod 8$.*

Proof. Assume that η is an even norm-perfect number. Then $\|\sigma(\eta)\| = 2\|\eta\|$, so $\sigma(\eta)[\sigma(\eta)]^* = 2\eta\eta^*$. Since $\eta = (1+i)^{p-1}\mu$, $\sigma(\eta) = \sigma(1+i)^{p-1} \times \sigma(\mu)$, where, by Lemma 3, $\sigma(1+i)^{p-1} = M_p$ is a Mersenne prime and $p \equiv \pm 1 \pmod 8$. Since $M_p | \sigma(\eta)$, either $M_p|\eta$ or $M_p^*|\eta$. Now, if $p \equiv 1 \pmod 8$, $M_p^* = 2^{(p-1)/2} + (2^{(p-1)/2} - 1)i$. Suppose $(1+i)^{p-1}(M_p^*)^t|\eta$, for some t. Noting that $\operatorname{Im} M_p^* = \operatorname{Re} M_p^* - 1$ and using the Corollary to Theorem 1, we have

$$\begin{aligned} \|\sigma(\eta)/\eta\| &\geqslant \|\sigma(1+i)^{p-1}/(1+i)^{p-1}\| \cdot \|\sigma(M_p^*)^t/(M_p^*)^t\| \\ &> \|M_p\|(\|M_p^*\| + 2^{(p+1)/2})/(2^{p-1}\|M_p^*\|) \\ &= (2^p+1)/2^{p-1} > 2. \end{aligned}$$

Since this implies that η is not norm-perfect, $M_p^* \nmid \eta$. We find, similarly, that if $p \equiv -1 \pmod 8$, then $M_p \nmid \eta$. The proof is complete.

4. Primitive norm-perfect and perfect numbers.

DEFINITION. Let η be a (norm-) perfect number. If there exists a (norm-) perfect number δ such that $\delta | \eta$ and $\delta \neq \varepsilon\eta$ for any unit ε, then η is said to be an *imprimitive* (norm-) perfect number. If η is divisible by no (norm-) perfect number other than the associates of η, then η is said to be a *primitive* (norm-) perfect number.

In the ring of rational integers, all perfect numbers are primitive. This is a consequence of the following property of the σ function as defined on $\mathscr{Z}$: If $a|b$, then $\sigma(a)/a < \sigma(b)/b$. The analogous property ($\|\sigma(\alpha)/\alpha\| < \|\sigma(\beta)/\beta\|$) holds in the ring of Gaussian integers if $\beta = \alpha\gamma$ and α and γ are relatively prime, but not necessarily if $(\alpha, \gamma) \neq 1$ (for example, if $\alpha = 1+2i$ and $\beta = (1+2i)^2$, $\|\sigma(\alpha)/\alpha\| = 40/25 > 37/25 = \|\sigma(\beta)/\beta\|$).

Using Theorem 2, it is now possible to characterize the even primitive norm-perfect numbers.

Proof of the Main Theorem. If $p \equiv 1 \pmod 8$ and $\eta = \varepsilon(1+i)^{p-1}M_p$, where M_p is a Mersenne prime, then

$$\|\sigma(\eta)\| = \|\sigma(1+i)^{p-1}\cdot\sigma(M_p)\| = \|M_p\cdot 2^{(p-1)/2}(1+i)\| = 2\|\eta\|.$$

η is, similarly, norm-perfect if $p \equiv -1 \pmod 8$, and in either case η is obviously primitive.

Conversely, if η is an even primitive norm-perfect number, then, by Theorem 2, there exists a Mersenne prime M_p such that η is divisible by $(1+i)^{p-1}M_p$ or $(1+i)^{p-1}M_p^*$, according as $p \equiv 1$ or $-1 \pmod 8$; since η is primitive, η is equal to a unit times one of these divisors.

Proof of the Corollary. The sufficiency was proved by Spira [3]. The necessity is an easy consequence of Theorem 2 and the Main Theorem. Since every perfect number is norm-perfect, $\eta = (1+i)^{p-1}\mu$, for some odd integer μ and prime $p \equiv \pm 1 \pmod 8$, by Lemma 3. However, since $(1+i)\eta = \sigma(\eta) = M_p\cdot\sigma(\mu)$, M_p divides η, and it was shown in the proof of Theorem 2 that if $p \equiv -1 \pmod 8$, and $M_p|\eta$, then η is not norm-perfect. Hence, $p \equiv 1 \pmod 8$, and, by Theorem 2, η is divisible by $(1+i)^{p-1}M_p$. Since

$$\sigma\big(\varepsilon(1+i)^{p-1}M_p\big) = (1+i)^p\cdot M_p, \qquad \eta = (1+i)^{p-1}M_p.$$

A table of factors of A_p for $p < 1200$ was published in 1962 by Brillhart [1]. The primality of A_p for all $p < 809$ was determined. Examination of the table shows that M_p is a Mersenne prime for 12 primes $p < 809$, $p \equiv \pm 1 \pmod 8$: 7, 47, 73, 79, 113, 151, 167, 239, 241, 353, 367 and 457. Each prime is of course associated with a primitive norm-perfect number, and the five primes congruent to 1 modulo 8 (73, 113, 241, 353, 457) determine the even primitive perfect numbers $\eta = (1+i)^{p-1}M_p$ for $p < 809$.

5. Imprimitive norm-perfect numbers. Do imprimitive norm-perfect numbers exist? The answer to this question is "yes" and the simplest example of an imprimitive norm-perfect number is

$$\eta = (1+i)^6(7+8i)^2(7+120i).$$

While the form (or existence) of such numbers, beyond that given in Theorem 2 and Theorem 3 below, is not known, the following surprising result shows that an even imprimitive norm-perfect number exists for each prime $p \equiv 1 \pmod 8$ for which both M_p and $\sigma(M_p^2)$ are prime, and for each prime $p \equiv -1 \pmod 8$ for which both M_p and $\sigma\big((M_p^*)^2\big)$ are prime.

THEOREM 3. *Let ε be a unit.*

(a) *If $p \equiv 1 \pmod 8$ and M_p and $\sigma(M_p^2)$ are prime, then*

$$\eta = \varepsilon(1+i)^{p-1}M_p^2[\sigma(M_p^2)]^*$$

is an imprimitive norm-perfect number.

(b) *If $p \equiv -1 \pmod 8$ and M_p and $\sigma\big((M_p^*)^2\big)$ are prime, then*

$$\eta = \varepsilon(1+i)^{p-1}(M_p^*)^2\big[\sigma\big((M_p^*)^2\big)\big]^*$$

is an imprimitive norm-perfect number.

Proof of (a). Assume that $p \equiv 1 \pmod 8$ and M_p and $\sigma(M_p^2)$ are prime. Let $h = (p-1)/2$. By (2), $M_p = 2^h-(2^h-1)i = -i(2^h-1+2^hi)$. We find that

$$[\sigma(M_p^2)]^* = -[(2^h-1)+2^h(2^{h+1}-1)i].$$

Since $\sigma(M_p^2)$ is prime, so is $[\sigma(M_p^2)]^*$. Then,

$$\sigma\big(\sigma(M_p^2)\big)^* = 2^h+2^h(2^{h+1}-1)i = 2^h(1+i)[2^h+(2^h-1)i] = (1+i)^pM_p^*.$$

Now, if $\eta = \varepsilon(1+i)^{p-1}M_p^2[\sigma(M_p^2)]^*$, then

$$\sigma(\eta) = M_p\cdot\sigma(M_p^2)\cdot\sigma\big(\sigma(M_p^2)\big)^* = M_p\cdot\sigma(M_p^2)\cdot(1+i)(1+i)^{p-1}M_p^*.$$

It follows that $\|\sigma(\eta)\| = 2\|\eta\|$. η is obviously imprimitive since $(1+i)^{p-1}M_p$ is a proper divisor of η. The proof of (b) is similar to that of (a).

Whether imprimitive perfect numbers exist is not known. Examination of the proof of Theorem 3 reveals that the imprimitive norm-perfect numbers shown to exist in that proof are not perfect numbers. It is the author's conjecture that all perfect numbers in $\mathfrak{G}$ are primitive.

6. Conclusions. We have shown, in this paper, that one of the best known theorems involving the sum of divisors function, as defined in $\mathscr{Z}$ has an analog, based on Spira's definition of σ, in $\mathfrak{S}$. While it is not absolutely certain that an alternate definition of σ would not yield comparable or even better results, the case now seems to be quite strong for the validity of Spira's definition of the sum of divisors function in $\mathfrak{S}$.

Our examination of the properties of $\sigma(\eta)/\eta$ suggests that the answers (or lack of them) to nearly all the questions which could be asked concerning the existence or the structure of odd perfect numbers in $\mathfrak{S}$, and concerning whether there exists a finite number of even and odd perfect numbers in $\mathfrak{S}$, may be similar to the answers to the same questions when posed about perfect numbers in $\mathscr{Z}$. The fact that if π is a prime and $a < b$, then $\|\sigma(\pi^a)/\pi^a\|$ is not necessarily less than $\|\sigma(\pi^b)/\pi^b\|$, and the resultant implication that imprimitive perfect numbers may exist, certainly suggests additional questions. A characterization of the even imprimitive perfect numbers or a proof that all perfect numbers in $\mathfrak{S}$ are primitive would be of considerable interest.

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$\Gamma^*(8)$

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1. Introduction. For each positive integer k, $\Gamma^*(k)$ is defined as the least s such that the congruence

$$(*) \qquad a_1x_1^k+\ldots+a_sx_s^k \equiv 0 \pmod{p^n}$$

has a primitive solution for all non zero integers $a_1, \ldots, a_s$ and all prime powers p^n.

Except for $k = 8$, $\Gamma^*(k)$ is known for $1 \leqslant k \leqslant 12$ and also when k is of the form $p-1$ or $p(p-1)$ where p is a prime ([3] and [5]). These results are given in Table 2. In all cases for which $\Gamma^*(k)$ is known it is true that $\Gamma^*(k) \equiv 1 \pmod k$ and Norton [5] has conjectured that $\Gamma^*(k) \equiv 1 \pmod k$ for all k. In this paper we show that $\Gamma^*(8) = 39$, disproving this conjecture.

Throughout this paper we use the same notation as Dodson in [3].

2. Lemma 1. *Let n be a positive integer and suppose that for $i = 0, \ldots, n$, $F_i = \sum_{j=1}^{v_i} a_{ij}x_{ij}$ with all the a_{ij} odd and with $\sum_{i=0}^{k-1} v_i \geqslant 2^k$ for each $k = 1, \ldots, n$. Then for any positive integer $N > n$, $\sum_{i=0}^{n} 2^iF_i$ represents at least $\min(\sum_{i=0}^{n} v_i, 2^N)$ different residue classes* $(\bmod\ 2^N)$ *where the $x_{ij} = 0$ or 1 and with at least one of the $x_{0j} = 1$.*

Proof. The proof is by induction on n. For $n = 0$ the result follows from Chowla's theorem on the addition of residue classes ([1] or [4], p. 49, Theorem 15).

Assume that the result is true for $n-1$. It is given that $\sum_{i=0}^{k-1} v_i \geqslant 2^k$, $k = 1, \ldots, n$ and so $\sum_{i=0}^{n-1} 2^iF_i$ represents at least 2^n different residue classes $(\bmod\ 2^n)$ by the induction hypothesis, and at least $\sum_{i=0}^{n-1} v_i = N_0$ say, different residue classes $(\bmod\ 2^N)$.

Let us represent these residue classes by numbers of the form $s+2^nY_{st}$, $t = 1, \ldots, u_s$; $s = 0, \ldots, 2^n-1$, where $1 \leqslant u_s \leqslant 2^{N-n}$ for each s.