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Simultaneous quadratic inequalities

by

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1. Introduction. H. Davenport and H. Heilbronn [6] proved that if

$$Q(\boldsymbol{x}) = \sum_{i=1}^{5} \nu_i x_i^2 \tag{1}$$

is an indefinite quadratic form with real coefficients, such that at least one of the ratios ν_i/ν_j is irrational, then for every $\varepsilon > 0$ there exist integers $x_1, \ldots, x_5$, not all zero, such that

$$|Q(\boldsymbol{x})| < \varepsilon.$$

Here we shall consider the analogous problem for two diagonal quadratic forms having real coefficients. Let

$$F(\boldsymbol{x}) = \sum_{i=1}^{9} \lambda_i x_i^2 \quad \text{and} \quad G(\boldsymbol{x}) = \sum_{i=1}^{9} \mu_i x_i^2. \tag{2}$$

The condition that at least one of the ratios ν_i/ν_j in (1) be irrational is equivalent to requiring that not all of the binary linear forms $\nu_i u + \nu_j v$ have coefficients which are linearly dependent over the rationals. We associate ternary linear forms

$$L_{ijk}(u, v, w) = \begin{vmatrix} u & v & w \\ \lambda_i & \lambda_j & \lambda_k \\ \mu_i & \mu_j & \mu_k \end{vmatrix}, \quad 1 \leqslant i < j < k \leqslant 9, \tag{3}$$

with the two forms F and G.

THEOREM. *Let $F(\boldsymbol{x})$ and $G(\boldsymbol{x})$ be diagonal quadratic forms, having real algebraic coefficients, in 9 variables. Suppose that*

(i) *Every member of the pencil $\{\mathscr{L}F + \mathscr{M}G\}$ $[(\mathscr{L}, \mathscr{M}) \neq (0, 0)]$ is an indefinite form with at least 5 non-zero coefficients; and*

(ii) *Not all of the ternary linear forms L_{ijk} associated with F and G have coefficients which are linearly dependent over the rationals.*

Then for any $\varepsilon > 0$ *there exist integers* $x_1, \ldots, x_9$, *not all zero, such that*

$$|F(\boldsymbol{x})| < \varepsilon \quad \textit{and} \quad |G(\boldsymbol{x})| < \varepsilon. \tag{4}$$

This is a partial complement to the analogous result for Diophantine equations [3]. By an appropriate application of Hua's Lemma analogous results may be obtained for R additive inequalities of degree k. We have assumed that the coefficients of F and G are algebraic in order to simplify the statement of the Theorem. It is possible to obtain results for forms having real coefficients, and we shall state these results in § 2.

This paper is essentially part of my PhD thesis, London 1971, and I am grateful to Prof. G. L. Watson for his advice and encouragement, and to the Science Research Council for a grant.

2. Preliminaries. We begin by normalizing the inequalities (4). Let

$$L_{123}(u, v, w) = pu + qv + rw,$$

then we may suppose that p, q and r are linearly independent over the rationals, and in particular $r \neq 0$. For any given $\varepsilon > 0$ we choose an integer $m > \varepsilon^{-1}$ and take

$$n = m\max(|\lambda_1| + |\lambda_2|, |\mu_1| + |\mu_2|).$$

We define the normalized forms $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ by

$$A = nr^{-1}(\mu_2 F - \lambda_2 G) \quad \text{and} \quad B = nr^{-1}(\mu_1 F - \lambda_1 G).$$

Thus $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ are diagonal quadratic forms in 9 variables such that every member of the pencil $\{\mathscr{L}A + \mathscr{M}B\}$ $[(\mathscr{L}, \mathscr{M}) \neq (0, 0)]$ is an indefinite form with at least 5 non-zero coefficients. We write

$$A(\boldsymbol{x}) = \sum_{i=1}^{9} a_i x_i^2 \quad \text{and} \quad B(\boldsymbol{x}) = \sum_{i=1}^{9} b_i x_i^2, \tag{5}$$

so that

$$a_1 = b_2 = n, \quad a_2 = b_1 = 0, \quad a_3 = -np/r \quad \text{and} \quad b_3 = nq/r. \tag{6}$$

In order to prove the Theorem it is sufficient to prove that there exist integers $x_1, \ldots, x_9$, not all zero, such that

$$|A(\boldsymbol{x})| < 1 \quad \text{and} \quad |B(\boldsymbol{x})| < 1. \tag{7}$$

DEFINITION. For any real number a, we say that the real linear form $pu + qv + rw$ *is of order* a if the inequalities

$$|pu + qv + rw| < U^{-a}, \quad 0 < \max(|u|, |v|, |w|) \leqslant U, \tag{8}$$

have an integer solution (u, v, w) for all U greater than some $U_0(a)$.

LEMMA 1. *Let p, q, r be algebraic numbers which are linearly independent over the rationals. Then for any $\delta > 0$ there are only finitely many integer points (u, v, w) with*

$$|pu + qv + rw| < \max(|u|, |v|, |w|)^{-2-\delta}. \tag{9}$$

This is a particular case of Corollary 1 of Schmidt [7].

COROLLARY. *The linear form L_{123} is of order at most 2.*

Proof. We have $L_{123} = pu + qv + rw$ where p, q, r are algebraic numbers which are linearly independent over the rationals. Thus

$$M = \min|pu + qv + rw| > 0,$$

where the minimum is taken over those integer points $(u, v, w) \neq (0, 0, 0)$ which satisfy (9). Hence if $U^{2+\delta} > M^{-1}$ there are no solutions of (8) so L_{123} is not of order $2 + \delta$. This is true for any $\delta > 0$ and so L_{123} is of order at most 2.

We shall only require that L_{123} is not of order ∞, and an analogue of the Theorem can be proved for quadratic forms F and G having real coefficients provided that not all of the associated ternary linear forms are of order ∞. It is straightforward to prove that the coefficients of the ternary linear forms of order ∞ form a set of Hausdorff dimension 2. Also, the proof of Theorem XIV of Cassels [2], p. 94, may readily be modified to show that there are ternary linear forms of order ∞ whose coefficients are linearly independent over the rationals.

For the rest of this paper we shall suppose that L_{123} is not of order ∞. We can choose a real number σ such that L_{123} is not of order σ and take $\Delta = 1/2(\sigma+2)$. We denote by δ a small positive constant chosen so that $\delta < \Delta/4$.

We recall that the coefficients of the normalized forms $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ are a_i and b_i respectively. We take

$$\gamma_i = a_i\alpha + b_i\beta \quad \text{for} \quad i = 1, \ldots, 9. \tag{10}$$

Let P be a large integer which will later be restricted to lie in a certain sequence. By $X \ll Y$ we mean $|X| < CY$ where C is independent of P. We let ε denote a small positive constant and we write $e(z)$ for $\exp(2\pi i z)$.

LEMMA 2. *Suppose that for every large integer P there exist real numbers α, β and integers A_i, Q_i, $i = 1, 2, 3$, satisfying*

$$\max(|\alpha|, |\beta|) \ll P^{\delta}, \tag{11}$$

$$\gamma_i = A_i/Q_i + O(P^{\Delta-2}), \quad i = 1, 2, 3, \tag{12}$$

$$0 \neq Q_i \ll P^{\Delta}, \quad i = 1, 2, 3 \tag{13}$$

and

$$(A_1, A_2) \neq (0, 0). \tag{14}$$

Then L_{123} is of order σ.

Proof. Suppose that for all sufficiently large integers P there exist solutions of (12)–(14). Now

$$\gamma_1 = n\alpha, \quad \gamma_2 = n\beta, \quad \text{and} \quad r\gamma_3 = n(q\beta - p\alpha)$$

so that

$$q(A_2/Q_2) - p(A_1/Q_1) = r(A_3/Q_3) + O(P^{\Delta-2}).$$

Hence

$$|pA_1Q_2Q_3 - qA_2Q_1Q_3 + rA_3Q_1Q_2| \ll Q_1Q_2Q_3P^{\Delta-2} \ll P^{4\Delta-2}.$$

Also $A_i \ll P^{\delta+\Delta} \ll P^{2\Delta}$ for $i = 1, 2, 3$, so taking

$$u = A_1Q_2Q_3, \quad v = -A_2Q_1Q_3 \quad \text{and} \quad w = A_3Q_1Q_2$$

we have

$$|pu + qv + rw| \ll P^{4\Delta-2}$$

and

$$0 < \max(|u|, |v|, |w|) \ll P^{4\Delta}.$$

Therefore for any $\varepsilon > 0$, L_{123} is of order $(2-4\Delta)/4\Delta - \varepsilon$, which gives the Lemma provided that ε is small.

COROLLARY. *We may suppose that there exists an infinite subsequence $\mathscr{P} = \mathscr{P}(\sigma)$ of the positive integers such that for all $P \in \mathscr{P}$ (11)–(14) are not all soluble.*

3. General lemmas

LEMMA 3. *The equations $A = B = 0$ have a non-singular real solution with none of the variables vanishing.*

This is essentially Lemma 2.4 of [3].

From such a solution we have a solution χ of the equations

$$a_1\chi_1 + \ldots + a_9\chi_9 = 0, \quad b_1\chi_1 + \ldots + b_9\chi_9 = 0$$

such that $\chi_i > 0$ for $i = 1, \ldots, 9$. Then, choosing a suitable linear multiple of this solution, we may suppose that $\chi_i > 1$ for $i = 1, \ldots, 9$. We now choose a constant C, independent of P, so that

$$1 < \chi_i < C^2 \quad \text{for} \quad i = 1, \ldots, 9. \tag{15}$$

For $i = 1, \ldots, 9$ we take

$$T_i = T(\gamma_i) = \sum_{x=P}^{CP} e(\gamma_i x^2), \tag{16}$$

$$J_i = J(\gamma_i) = \int_P^{CP} e(\gamma_i \xi^2)\,d\xi, \tag{17}$$

and we put

$$K(\alpha) = (\sin\pi\alpha/\pi\alpha)^2. \tag{18}$$

LEMMA 4.

$$\int_{-\infty}^{\infty} e(\eta\alpha)K(\alpha)\,d\alpha = \max(0, 1-|\eta|). \tag{19}$$

This is a Lemma 4 of Davenport and Heilbronn [6].

Let $\mathscr{B}$ be the box $\{\boldsymbol{x}: P \leqslant x_i \leqslant CP, i = 1, \ldots, 9\}$ and let $N(P)$ be the number of integer solutions of the normalized inequalities (7) in $\mathscr{B}$.

LEMMA 5.

$$N(P) \geqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{9} T(\gamma_i)K(\alpha)K(\beta)\,d\alpha\,d\beta \tag{20}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{9} J(\gamma_i)K(\alpha)K(\beta)\,d\alpha\,d\beta = \int_P^{CP}\ldots\int_P^{CP}\max\big(0, 1-|A(\xi)|\big)\max\big(0, 1-|B(\xi)|\big)\,d\xi. \tag{21}$$

This result follows from Lemma 4 on multiplying out the products and interchanging the orders of integration and summation.

4. Reduction to a finite integral.

We shall obtain a lower bound for $N(P)$ from (20), and begin by reducing the integral to a finite region.

LEMMA 6. *For any real y, z and any $\varepsilon > 0$*

$$\int_y^{y+1}\int_z^{z+1}\prod_{i=1}^{9}{}' |T(\gamma_i)|\,d\alpha\,d\beta \ll P^{4+\varepsilon} \tag{22}$$

where $'$ denotes the omission of any one factor from the product.

Proof. Since every member of the pencil $\{\mathscr{L}A + \mathscr{M}B\}$ $[(\mathscr{L}, \mathscr{M}) \neq (0, 0)]$ contains at least 5 terms explicitly, any ratio occurs at most 4 times among the a_i/b_i. Therefore the 8 factors in the product can be arranged into 4 pairs $T(\gamma_k)$, $T(\gamma_l)$ such that $a_kb_l - a_lb_k \neq 0$. Then

$$\int_y^{y+1}\int_z^{z+1}\prod_{i=1}^{9}{}' |T(\gamma_i)|\,d\alpha\,d\beta \ll \sum_{k,l}\int_y^{y+1}\int_z^{z+1} |T(\gamma_k)T(\gamma_l)|^4\,d\alpha\,d\beta, \tag{23}$$

where the sum is taken over such pairs k, l. The Lemma now follows on applying the generalization of Hua's Lemma obtained in [4].

COROLLARY. *For any real y, z and any $\varepsilon > 0$,*

$$\int_y^{y+1}\int_z^{z+1}\prod_{i=1}^{9} |T(\gamma_i)|\,d\alpha\,d\beta \ll P^{5+\varepsilon}. \tag{24}$$

From the α-β plane we now select 4 regions R_i:

$$R_1 = \{(\alpha, \beta):\ \alpha > P^{\delta}\}; \quad R_2 = \{(\alpha, \beta):\ \alpha < -P^{\delta}\};$$
$$R_3 = \{(\alpha, \beta):\ \beta > P^{\delta}\}; \quad R_4 = \{(\alpha, \beta):\ \beta < -P^{\delta}\}.$$

Here δ is the positive number chosen in § 2. We take

$$R = \bigcup_{i=1}^{4} R_i. \tag{25}$$

Lemma 7. *For any $\varepsilon > 0$*

$$\iint_R \prod_{i=1}^{9} |T(\gamma_i)| K(\alpha) K(\beta)\, d\alpha\, d\beta \ll P^{5+\varepsilon-\delta}. \tag{26}$$

Proof. It is sufficient to prove the result with each R_i in place of R. Using the estimate $K(\alpha) \ll \max(\alpha^{-2}, 1)$ we have

$$\iint_{R_1} \prod_{i=1}^{9} |T(\gamma_i)| K(\alpha) K(\beta)\, d\alpha\, d\beta = \sum_{y \geqslant P^{\delta}} \sum_{-\infty}^{\infty} \int_y^{y+1} \int_z^{z+1} \prod_{i=1}^{9} |T(\gamma_i)| K(\alpha) K(\beta)\, d\alpha\, d\beta$$
$$\ll \Big\{\sum_{P^{\delta}}^{\infty} y^{-2}\Big\}\Big\{\sum_{1}^{\infty} z^{-2}\Big\} P^{5+\varepsilon} \ll P^{5+\varepsilon-\delta},$$

and the other regions R_i are treated similarly.

With the linear form γ_i we associate the line

$$\Gamma_i:\ \gamma_i = 0 \tag{27}$$

in the α-β plane. We label the Γ_i so that the positive angle from $\beta = 0$ to Γ_i increases monotonically with i. Note that we may have $\Gamma_i = \Gamma_{i+1}$. If $\Gamma_i \neq \Gamma_{i+1}$, let B_i be the line bisecting the angle formed by the lines Γ_i and Γ_{i+1}. If j is the largest integer less than i such that $\Gamma_j \neq \Gamma_i$, we let $S'_{j+1} = \ldots = S'_i$ be the sector bounded by B_j and B_i. Thus $\Gamma_{j+1} = \ldots = \Gamma_i$ lie in the interior of S'_i.

We choose a positive constant c such that if $\max(|\alpha|, |\beta|) > c$, $|\gamma_i| < 1$ and $a_i b_j - a_j b_i \neq 0$ then $|\gamma_j| > 1$. Therefore, for any large integer P, if $\max(|\alpha|, |\beta|) > cP^{-1}$, $|\gamma_i| < P^{-1}$ and $a_i b_j - a_j b_i \neq 0$ then $|\gamma_j| > P^{-1}$. Let τ be a small positive constant, and take S_i to be the intersection of S'_i with the region

$$cP^{-\tau-3/2} < \max(|\alpha|, |\beta|) < cP^{-1}.$$

Lemma 8. *If $\gamma_i = O(P^{-1})$ and $\gamma_i \neq 0$ then*

$$T(\gamma_i) \ll |\gamma_i|^{-\frac{1}{2}-\varepsilon}. \tag{28}$$

This is Lemma 7 of Davenport and Heilbronn [6].

Lemma 9. *For each S_j,*

$$\iint_{S_j} \prod_{i=1}^{9} |T(\gamma_i)| K(\alpha) K(\beta)\, d\alpha\, d\beta = o(P^5). \tag{29}$$

Proof. We take new coordinates in the region S_j. These are r, the distance along Γ_j from the origin, and s, perpendicular to r. The region S_j lies in a region bounded by two lines, say $-mr \leqslant s \leqslant nr$. Also, we can choose positive constants c_0, c_1 and c_2, independent of P, so that

$$r > c_0 P^{-\tau-3/2} \quad \text{in } S_j,$$

and if $a_i b_j - a_j b_i \neq 0$, $c_1 r \leqslant |\gamma_i| \leqslant c_2 r$.

In S_j each γ_i is $O(P^{-1})$ and if $a_i b_j - a_j b_i \neq 0$ we have $\gamma_i \neq 0$ in S_j and so we can use Lemma 8 to estimate $T(\gamma_i)$. Since any ratio occurs at most 4 times among the a_i/b_i we can use Lemma 8 on at least 5 factors in the product, and use the trivial estimate $O(P)$ on the remaining terms. Hence

$$\iint_{S_j} \prod_{i=1}^{9} |T(\gamma_i)| K(\alpha) K(\beta)\, d\alpha\, d\beta \ll \iint_{S_j} P^4 (r^{-\frac{1}{2}-\varepsilon})^5\, dr\, ds$$
$$\ll \int_{c_0 P^{-\tau-3/2}}^{\infty} \int_{-mr}^{nr} P^4 (r^{-\frac{1}{2}-\varepsilon})^5\, dr\, ds$$
$$\ll P^{4+3/4+\tau+8\varepsilon} = o(P^5),$$

provided that τ and ε are sufficiently small.

We now take Σ_i to be the intersection of S'_i with the region

$$\max(|\alpha|, |\beta|) > cP^{-\tau-3/2}.$$

Lemma 10. *For each Σ_j,*

$$\iint_{\Sigma_j} \prod_{i=1}^{9} |J(\gamma_i)| K(\alpha) K(\beta)\, d\alpha\, d\beta = o(P^5). \tag{30}$$

Proof. If $\gamma_i \neq 0$ we have, as in Lemma 11 of Davenport and Heilbronn [6], $J(\gamma_i) = O(|\gamma_i|^{-\frac{1}{2}})$. The result now follows in the same way as Lemma 9.

5. The main term

Lemma 11. *For some positive constant D, independent of P,*

$$\int_P^{CP} \cdots \int_P^{CP} \max\big(0, 1 - |A(\xi)|\big) \max\big(0, 1 - |B(\xi)|\big)\, d\xi > DP^5, \tag{31}$$

for all sufficiently large P.

Proof. We put $\xi_i^2 = P^2\eta_i$, then $d\xi_i = P(2|\eta_i|^{1/2})^{-1}d\eta_i$. We take

$$\mathscr{E} = \{\boldsymbol{\eta}: 1 < \eta_i < C^2, \; i = 1, \ldots, 9\}$$

and

$$\mathscr{S} = \{\boldsymbol{\eta}: \max(|A_1(\boldsymbol{\eta})|, |B_1(\boldsymbol{\eta})|) < (2P^2)^{-1}\},$$

where $A_1(\boldsymbol{\eta}) = a_1\eta_1 + \ldots + a_9\eta_9$ and $B_1(\boldsymbol{\eta}) = b_1\eta_1 + \ldots + b_9\eta_9$. Then the left hand side of (31) is at least

$$2^{-11}P^9 \int\limits_{\mathscr{E}\cap\mathscr{S}}\!\!\ldots\!\int |\eta_1 \ldots \eta_9|^{-1/2} d\boldsymbol{\eta}.$$

The surfaces $A_1(\boldsymbol{\eta}) = 0$ and $B_1(\boldsymbol{\eta}) = 0$ are 8-dimensional linear subspaces meeting in a 7-dimensional linear subspace which contains the point $\boldsymbol{\chi}$ chosen by Lemma 3. Further, $\boldsymbol{\chi}$ is interior to $\mathscr{E}$. The set $\mathscr{E}\cap\mathscr{S}$ will therefore contain a box around $\boldsymbol{\chi}$ of volume D_0P^{-4} for some positive constant D_0 independent of P. Then

$$\int\limits_{\mathscr{E}\cap\mathscr{S}}\!\!\ldots\!\int |\eta_1 \ldots \eta_9|^{-1/2} d\boldsymbol{\eta} > C^{-9}D_0P^{-4}$$

and the result follows with $D = 2^{-11}C^{-9}D_0$.

LEMMA 12. *If* $|\gamma_i| = O(P^{-3/2})$ *then*

(32) $$|T(\gamma_i) - J(\gamma_i)| = O(1).$$

This is Lemma 5 of Davenport and Heilbronn [6].

We take $U(\tau) = \{(\alpha, \beta): \max(|\alpha|, |\beta|) \leqslant cP^{-3/2-\tau}\}$.

LEMMA 13.

(33) $$\iint\limits_{U(\tau)} \prod_{i=1}^{9} J(\gamma_i)K(\alpha)K(\beta)\,d\alpha\,d\beta = \iint\limits_{U(\tau)} \prod_{i=1}^{9} T(\gamma_i)K(\alpha)K(\beta)\,d\alpha\,d\beta + o(P^5).$$

Proof. In $U(\tau)$ we have each $|\gamma_i| = O(P^{-3/2})$, so, by Lemma 12,

$$\Big|\prod_{i=1}^{9} T(\gamma_i) - \prod_{i=1}^{9} J(\gamma_i)\Big| = O(P^8).$$

Thus the difference between the two integrals in (33) is

$$\ll P^8(P^{-3/2-\tau})^2 = P^{5-2\tau}.$$

Collecting together the results of Lemmas 13, 10, 5 and 11, we see that for some positive constant D, independent of P,

(34) $$\iint\limits_{U(\tau)} \prod_{i=1}^{9} T(\gamma_i)K(\alpha)K(\beta)\,d\alpha\,d\beta \geqslant DP^5 + o(P^5).$$

6. The residual integral. We now have to estimate the integral of the exponential sums $T(\gamma_i)$ over the region

(35) $$R_0 = \{(\alpha, \beta): cP^{-1} < \max(|\alpha|, |\beta|) \leqslant P^{\delta}\}.$$

LEMMA 14. *Suppose that* $|T(\gamma_i)| = P^{1-\theta}$, *where* $\theta < \frac{1}{2} - 2\delta$; *then* γ_i *has a rational approximation* A_i/Q_i *such that*

(36) $$1 \leqslant Q_i \ll P^{2\theta} \quad \textit{and} \quad |\gamma_i - A_i/Q_i| \ll P^{\theta-2}.$$

Proof. By Dirichlet's theorem on Diophantine approximation, there exists a rational approximation A/Q to γ_i such that

$$1 \leqslant Q \leqslant P^{1+\delta} \quad \text{and} \quad |\gamma_i - A/Q| \leqslant Q^{-1}P^{-1-\delta}.$$

If $Q \geqslant P^{1-\delta}$ then by Weyl's inequality (Lemma 1 of [4]),

$$|T(\gamma_i)| \ll P^{\frac{1}{2}+\delta},$$

which gives a contradiction. Thus $Q \leqslant P^{1-\delta}$ and so, from the Corollary to Lemma 9 of Birch and Davenport [1],

$$|T(\gamma_i)| \ll Q^{-1/2}\min(P, P^{-1}|\beta_i|^{-1}),$$

where $\beta_i = \gamma_i - A/Q$. Thus

$$P^{1-\theta} \ll Q^{-1/2}P$$

and

$$P^{1-\theta} \ll Q^{-1/2}P^{-1}|\beta_i|^{-1}$$

which gives (36).

We recall that for all $P \in \mathscr{P}(\sigma)$ there are no $(\alpha, \beta) \in R_0$ which satisfy (12)–(14). Thus for all $(\alpha, \beta) \in R_0$ and $P \in \mathscr{P}$ we have

(37) $$\min(|T(\gamma_1)|, |T(\gamma_2)|, |T(\gamma_3)|) \ll P^{1-\frac{1}{4}\Delta}.$$

Thus from (37) and Lemma 6,

(38) $$\iint\limits_{R_0} \prod_{i=1}^{9} |T(\gamma_i)|K(\alpha)K(\beta)\,d\alpha\,d\beta \ll P^{2\delta}P^{4+\varepsilon}P^{1-\frac{1}{4}\Delta}$$

for all $P \in \mathscr{P}$. Since $\delta < \Delta/4$, the right hand side of (38) is $o(P^5)$, provided that ε is sufficiently small.

7. Completion of the proof of the Theorem. It is sufficient to prove that the normalized inequalities (7) have a non-trivial integer solution. The number $N(P)$ of integer solutions of (7) in $\mathscr{B}$ satisfies

$$N(P) \geqslant \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \prod_{i=1}^{9} T(\gamma_i)K(\alpha)K(\beta)\,d\alpha\,d\beta.$$

From (38) and Lemmas 7 and 9 we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{9} T(\gamma_i)K(a)K(\beta)\,da\,d\beta = \iint_{U(\tau)}\prod_{i=1}^{9} T(\gamma_i)K(a)K(\beta)\,da\,d\beta + o(P^5),$$

as $P\to\infty$ through $\mathscr{P}$. Thus, by (34), for some positive constant D

$$N(P) \geqslant DP^5 + o(P^5)$$

as $P\to\infty$ through $\mathscr{P}$ which gives $N(P) > 0$, and the proof is complete.

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Halving an estimate obtained from Selberg's upper bound method

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Introduction. In many applications of Selberg's upper bound method, an unnecessary constant factor appears in the final estimate, due to the fact that we can only sieve up to approximately $\sqrt{x}$.

At present this restriction seems unavoidable, and arises from the necessity of squaring in order to obtain a non-negative sifting function, viz.

$$s^{(+)}(n) = \Big(\sum_{d|n}\lambda_d\Big)^2.$$

As an example, let K be any positive integer whose greatest prime factor does not exceed x. Following van Lint and Richert [1], we arrive at the estimate

$$\sum_{\substack{n\leqslant x\\(n,K)=1}} 1 \leqslant \frac{\varphi(K)}{K}x\left(\frac{1}{\log z}+\frac{z^2}{x}\right)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-1},$$

by a careful application of Selberg's method. Choosing z optimally, Mertens' formula gives

$$\sum_{\substack{n\leqslant x\\(n,K)=1}} 1 \leqslant 2e^{\gamma}\frac{\varphi(K)}{K}x\left(1+O\left(\frac{\log\log x}{\log x}\right)\right).$$

The factor e^{γ} really is necessary, as the Prime Number Theorem shows, but apart from the error term, the estimate becomes best possible if we strike out the factor 2 on the right. The object of this note is to obtain a general result of this kind.

THEOREM. *Let $f(n)$ be defined on the positive integers and satisfy*

$$f(1) = 1, \qquad 0 \leqslant f(n) \leqslant 1$$

and

$$f(nm) \leqslant f(n)f(m) \text{ provided } (n, m) = 1.$$