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# Remarks on countable models (1)

by

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**Abstract.** Theories with finitely manny countable models are investigated. Several situations are shown in which such theories have at least two universal models. A theorem of Vaught is improved and an example is given which shows that it cannot be generalized.

**Introduction.** The theme of the major part of this paper is described in the following

CONJECTURE. *If a complete not $\omega$-categorical theory has only finitely many countable models then it has at least two universal models.*

We thought we had a proof of the conjecture; it contained an "undergraduate mistake" but by strengthening the assumption of the conjecture the argument can be saved (see Theorem 3). We prove two other, maybe more interesting results, in the direction of the conjecture. Theorem 1 proves the existence of two universal models from a restriction on interactions between types of the theory. The concept we introduce may be useful in other investigations. Theorem 2 strengthens the hypothesis of the conjecture to: "every complete extension of $T$ by finitely many constants has finitely many countable models". Known examples of theories with finitely many models satisfy either of the last two assumptions and they, of course, agree with the conjecture. On the other hand we are rather ignorant about the power of the assumption in the conjecture. We know that the conclusion boils down to finding an extension of $T$ which omits a certain type (see proof of Theorem 3). But we have no syntactical characterization of theories with, say, 3 countable models. By Ryll-Nardzewski's theorem (see [9]) we know precisely when a theory has one countable model. It is thus surprising that no generalization of Ryll-Nardzewski's theorem has appeared. Other problems which emerged during our work on the conjecture are mentioned later.

Is the conjecture interesting? We, of course, think it is. Firstly, it, as any other conjecture, stimulates research which usually has value

(1) Some results in this paper were obtained in summers of 1971 and of 1972 while the author was at the University of California in Berkeley and was supported by NSF grant GP24352.

even in the case when the conjecture turns out to be false. Secondly, if the conjecture were true it would be a general fact which would imply Baldwin-Lachlan's theorem ([1]). In connection with this we should mention a theorem of Lachlan which says that no totally transcendental theory (in fact no superstable theory) has $n$ countable models where $1 < n < \omega$ (see [6]).

The last part of the paper is an improvement and, hopefully, also an explanation of the fact that no complete theory has exactly two countable models (see [11]). More precisely, we show that the definition of a saturated model cannot be made simpler (except in trivial cases) and that it is not complementary to the definition of a prime model.

The notation and terminology are standard unless otherwise indicated. An $n$-type of a theory is a maximal consistent set of formulas in variables $x_1, \ldots, x_n$. If we add to a structure elements from its universe (which is always denoted by the corresponding Roman capital) then the elements will name themselves, i.e. if $\varphi(x)$ is a formula of the language for $\boldsymbol{A}$, and $a \in A$ then $\varphi(a)$ is a legitimate formula (sentence in this case) of the language for $(\boldsymbol{A}, a)$. As can be noted above, when we talk about the number of countable models of a theory, we have in mind the number of isomorphism types of such models.

**1.** Throughout this section $T$ will stand for an arbitrary theory in a countable language.

DEFINITION 1. a) Let $x, y$ by finite sequences of distinct variables. Let $\Sigma(x)$ be a type of $T$. A formula $\alpha(x, y)$ is a *$\Sigma$-atom in $y$* if $\alpha(c, y)$ is an atom of $T \cup \Sigma(c)$.

b) Let $\Sigma(x)$ and $\Delta(y)$ be types of $T$. A type $\Lambda(x, y)$ is called a *link of $\Sigma$ and $\Delta$* if $\Sigma(x) \cup \Delta(y) \subseteq \Lambda(x, y)$. A link $\Lambda(x, y)$ is *$\Sigma$-non-principal* if it contains no $\Sigma$-atom in $y$.

c) *$T$ has few links* if for any two types $\Sigma(x)$ and $\Delta(y)$ there are only finitely many $\Sigma$-non-principal links of $\Sigma$ and $\Delta$.

To give the reader a feeling for the notion we present some examples. They will show that the notion is independent of number of types, models, total transcendence, etc.

EXAMPLE 1. Any $\omega$-categorical theory has few links since given any two types there is only finitely many links between them (see [9]).

EXAMPLE 2. Let $T_n$ be the theory of $(Q, \leqslant, U_1, \ldots, U_n, k)_{k\in\omega}$ where $Q$ is the set of rationals, $\leqslant$ is the natural order on it and $U_1, \ldots, U_n$ is a partition of $Q$ composed of dense subsets of $Q$. $T_n$ has $n+3$ countable models (see [11]) and has few links. E.g. take $T_0$ and two 1-types $\Sigma(x)$ and $\Delta(y)$. If they are both principal then they are determined by "$x = k$ and $y = m$" or "$x = k$ and $m < y < n$" or "$k < x < m$ and $k < y < m$" or... In the last case there are 3 links between them (namely $x = y$, $x < y$ and $y < x$). In the other cases there is just one link between them. All links in this case are $\Sigma$-principal. If $\Sigma(x)$ is principal (non-principal) and $\Delta(y)$ is non-principal (principal) there is just one link between them. If both are non-principal there are 3 links between them, those determined by $x < y$, $x = y$ and $y < x$ resp. Only the last one is $\Sigma$-non-principal.



EXAMPLE 3. Let $T$ be the theory in the language $\{U_n |\ n < \omega\} \cup \cup \{C_n^m |\ m, n < \omega\}$ whose axioms are: $U_n(C_m^n)$ for $n, m < \omega$; $C_k^n \neq C_p^m$ if $(n, k) \neq (m, p)$; $U_n \cap U_m = 0$ if $n \neq m$. $T$ has $2^\omega$ countable models but it has $\omega$ types. It is not difficult to see that it has few links.

EXAMPLE 4. Let $T$ be the theory of $(Q, \leqslant, q)_{q\in Q}$. This theory has $2^\omega$ 1-types but it still has few links.

Some $\omega_1$-categorical theories have few links some don't.

EXAMPLE 5. If $T$ is the theory of $(\omega, n)_{n<\omega}$ $T$ is $\omega_1$-categorical (and not $\omega$-categorical.) It clearly has few links. On the other hand if $T$ is the theory of $(I, S)$ where $I$ is the set of integers and $S(n) = n+1$ then this theory is $\omega_1$-categorical but it does not have few links: $\{x = x\}$ determines a complete 1-type $\Sigma(x)$. Let $\Delta(y_1, y_2)$ be the type determined by $\{S^k(y_1) \neq y_2 |\ k \in I\} \cup \{S^k(y_2) \neq y_1 |\ k \in I\}$. Then for every $n < \omega$ $\Sigma(x) \cup \cup \{S^n(x) = y_1\} \cup \Delta(y_1, y_2)$ determines a link of $\Sigma$ and $\Delta$ which is $\Sigma$-non-principal.

LEMMA 1. *If $T$ has few links and $\Gamma(z)$ is a type of $T$ ($z$ is a finite sequence of variables) then $T \cup \Gamma(c)$ has few links ($c$ is a sequence of new constants of appropriate length).*

Proof. Let $\Sigma(x, c)$ and $\Delta(y, c)$ be types of $T \cup \Gamma(c)$ and let $\Lambda(x, y, c)$ be a $\Sigma(x, c)$-non-principal link of them. Then $\Lambda(x, y, z)$ is a non-principal link of $\Sigma(x, z)$ and $\Delta'(y) = \{\exists z\delta |\ \delta \in \Delta(y, z)\}$. By assumption there are only finitely many of these.

The next lemma will be needed in the proof of Theorem 1.

LEMMA 2. *Let $\boldsymbol{A}$ be prime over $a = (a_1, \ldots, a_m)$ and let $b = (b_1, \ldots, b_n) \in A^n$. Assume that a non-principal type $\Sigma(b, y)$ of $\mathrm{Th}\big((\boldsymbol{A}, b)\big)$ is realized in $(\boldsymbol{A}, b)$ by $c \in A^k$. Then $a$ is not prime over $b$, i.e. $a$ realizes in $(\boldsymbol{A}, b)$ a non-principal type of $\mathrm{Th}\big((\boldsymbol{A}, b)\big)$.*

Proof. By way of contradiction assume that $a$ is prime over $b$. Then there is $\boldsymbol{B} \precsim \boldsymbol{A}$ such that $a, b \in B^{<\omega}$ and $(\boldsymbol{B}, b)$ is prime. Let $\alpha(a, x, y)$ be the $\mathrm{Th}\big((\boldsymbol{A}, a)\big)$-atom for $(b, c)$. If $c' \in B^k$ is such that $(\boldsymbol{B}, b) \models \alpha(v, b, y)[a, c']$ then the $\mathrm{Th}\big((\boldsymbol{B}, b)\big)$-type of $c'$ is $\Sigma(b, y)$; this contradicts primeness of $(\boldsymbol{B}, b)$.

DEFINITION. A structure $\boldsymbol{A}$ is *full* if every type of $\mathrm{Th}(\boldsymbol{A})$ is realized in $\boldsymbol{A}$. A structure $\boldsymbol{A}$ is *prime over a finite set* if for some $a_1, \ldots, a_n \in A$ $(\boldsymbol{A}, a_1, \ldots, a_n)$ is prime.

Note that every $\omega$-universal model is full. In [10] full models are called almost saturated. We reserve this term for models which are closer to saturatedness than full models.

THEOREM 1. *Let $T$ be a complete theory which has few links. Every full model of $T$ which is prime over a finite set is universal.*

Proof. Theorem 1 is certainly true for $\omega$-categorical theories so we may assume that $T$ is not $\omega$-categorical. If $A$ is a full model of $T$ prime over $b_1, \ldots, b_p$ then $A$ is countable so $T$ has countably many types and it has a countable saturated model. We will point by point embed the saturated model into $A$. The inductive step is singled out in

LEMMA 3. *If $\Sigma(x)$ is a 1-type of $T$ there is an $a \in A$ which realizes $\Sigma$ and such that $(A, a)$ is full.*

Proof of Lemma 3. Because $T$ is not $\omega$-categorical neither is $T \cup \Sigma(c)$. There is therefore a type $\Sigma_0(x, y_1, \ldots, y_{k_0})$ of $T$ which includes $\Sigma(x)$ and is $\Sigma(x)$-non-principal. Let $\Sigma_i(x, y_1, \ldots, y_{k_i})$ be an enumeration of all types of $T$ such that $\Sigma(x) \subseteq \Sigma_i(x, y_1, \ldots)$. Let $a_n \in A$ be such that $(A, a_n)$ realizes every $\Sigma_i(a_n, y_1, \ldots, y_{k_i})$ with $i \leqslant n$. Such an element exists because $A$ is full so a completion of $\Sigma_0(x, y_1^0, \ldots, y_{k_0}^0) \cup \ldots \cup \Sigma_n(x, y_1^n, \ldots, y_{k_n}^n)$ to a type of $T$ is realized in $A$ by, say, $(a_n, c_1^0, \ldots, c_{k_n}^n)$. Let $\Delta(y) = \Delta(y_1, \ldots, y_p)$ be the type of $b = (b_1, \ldots, b_p)$ and let $\Lambda_n(x, y)$ be the type of $(a_n, b)$. $\Lambda_n$ is a link of $\Sigma(x)$ and $\Delta(y)$. Because $(A, a_n)$ realizes a non-principal type of $\mathrm{Th}\big((A, a_n)\big)$ (e.g. $\Sigma_0(a_n, \ldots)$) $\Lambda$ is $\Sigma$-non-principal according to Lemma 2. Thus, because $T$ has few links, there is a link $\Lambda$ of $\Sigma$ and $\Delta$ such that $M = \{n \mid \Lambda_n = \Lambda\}$ is infinite. If $n, m \in M$ then $(A, b, a_n) \equiv (A, b, a_m)$. Moreover, $(A, b)$ being prime is homogeneous. The last two facts imply that the models $(A, a_n)$, $(A, a_m)$, $(n, m \in M)$ realize the same types (actually are isomorphic). By the definition of $a_n$ and the fact that $M$ is infinite $(A, a_n)$ is full for any $n \in M$.

Going back to the proof of Theorem 1 let $A$ be a full model of $T$ prime over $b \in A^n$. Let $B$ be the countable saturated model of $T$ and let $\{d_n \mid n < \omega\}$ be an enumeration of it. Assume that we have found $a_0, \ldots, a_m$ in $A$ such that $(A, a_0, \ldots, a_m)$ is full and

$$(A, a_0, \ldots, a_m) \equiv (B, d_0, \ldots, d_m)\,.$$

$(A, a_0, \ldots, a_m)$ is still prime over a finite set, $\mathrm{Th}\big((A, a_0, \ldots, a_m)\big)$ has few links by Lemma 1 so, by Lemma 2, there is $a_{m+1} \in A$ such that

$$(A, a_0, \ldots, a_{m+1}) \equiv (B, d_0, \ldots, d_{m+1})$$

with $(A, a_0, \ldots, a_{m+1})$ being full. The map $f(d_n) = a_n$ embeds $B$ into $A$ elementarily so $A$ is universal.



We now show that theories we are interested in have full models prime over a finite set.

DEFINITION. $\Sigma(x_1, \ldots, x_n)$ is a *powerful type of $T$* if every model of $T$ which realizes it is full.

Note that the argument in Theorem 1 establishes the following: If $T$ has a powerful type $\Delta$ and for every $\Sigma$ there are only finitely many links of $\Delta$ and $\Sigma$ which are $\Delta$-principal and $\Sigma$-non-principal then every model of $T$ which realizes $\Delta$ is universal.

PROPOSITION. *If a complete theory $T$ has finitely many countable models then it has a powerful type.*

Proof. If no type of $T$ is powerful we can form by induction $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$ types of $T$ and $A_0, A_1, \ldots$ models of $T$ such that $\Sigma_i$ is realized in $A_i$ but $\Sigma_{i+1}$ is not. Then if $i \neq j$ we have $A_i$ not isomorphic to $A_j$.

COROLLARY 1. *If $T$ is complete and has $n$ countable models where $1 < n < \omega$ then $T$ has a full not saturated model.*

Proof. By Proposition $T$ has a powerful type. If $A$ is a model prime over the powerful type of $T$ it is full and by Ryll-Nardzewski's theorem it cannot be saturated.

Remark. Corollary 1 can be also found in [8]. We proved Proposition independently of [8] though a year later.

Combining Theorem 1 and Corollary 1 we get:

COROLLARY 2. *If $T$ is a complete theory which has $n$ countable models where $1 < n < \omega$ and has few links then $T$ has a universal not saturated countable model.*

**2.** DEFINITION. Let $P$ be a property of theories. We say that *a theory $T$ insists on having the property $P$* if for every type $\Sigma(x_1, \ldots, x_n)$ $T \cup \Sigma(c_1, \ldots, c_n)$ has $P$ (i.e. every extension of $T$ by finitely many constants has $P$).

E.g. every $\omega$- (or $\omega_1$-) categorical theory insists on being $\omega$- (or $\omega_1$-) categorical. Or, reversing the definition, we might say that $\omega$-categoricity is a persistent property (i.e. is preserved by entensions by finitely many constants). As far as we know no systematic study of persistent properties has been undertaken.

Apart from $\omega$-categorical theories the theories in Example 2 insist on having finitely many models.

THEOREM 2. *If a complete theory $T$ insists on having finitely many countable models and is not $\omega$-categorical then it has a universal not saturated countable model.*

Proof. The argument uses König's lemma so we will build a tree. Let $B$ be the countable saturated model of $T$ and let $\{b_n \mid n < \omega\}$ be an enumeration of $B$. We form a tree whose $n$th level $L_n$ contains models of form $(A, a_i)_{i<n}$ and satisfies the following conditions:

(1) if $(A, a_i)_{i<n} \in L_n$ then $(A, a_i)_{i<n} \equiv (B, b_i)_{i<n}$ and $(A, a_i)_{i<n}$ is a full not saturated countable model,

(2) if $(A, a_i)_{i<n} \neq (A', a_i')_{i<n}$ and both are in $L_n$ then $(A, a_i)_{i<n} \not\simeq (A', a_i')_{i<n}$,

(3) if $(A', a_i')_{i<n}$ satisfies (1) then for some $(A, a_i)_{i<n} \in L_n$ $(A, a_i)_{i<n} \simeq (A', a_i')_{i<n}$.

Using our assumption and Corollary 1 we see that every level is non-empty and finite. If $(A, a_i)_{i<n}$ and $(A', a_i')_{i<m}$ are on $n$th and $m$th level resp. we define $(A, a_i)_{i<n} \leqslant (A', a_i')_{i<m}$ iff $(A, a_i)_{i<n} \simeq (A', a_i')_{i<n}$ and $n \leqslant m$.

This tree satisfies the hypothesis of König's lemma so it has an infinite branch say $\{(A_n, a_i^n)_{i<n} \mid n < \omega\}$. We claim that $A_0$ is the required model. It certainly is countable and not saturated by the definition of 0th level. Because $A_0 \simeq A_1$ there is $a_0 \in A_0$ such that $(A_0, a_0) \simeq (A_1, a_0^1)$. Since $(A_1, a_0^1) \simeq (A_2, a_0^2)$ there is $a_1 \in A_0$ such that $(A_0, a_0, a_1) \simeq (A_2, a_0^2, a_1^2)$. Similarly we find $a_2, a_3 \ldots$ The map $f(b_n) = a_n$ is by (1) an elementary embedding of $B$ into $A$ showing that $A$ is universal.

We now reproduce an example constructed by J. Rosenberg of a theory with 3 countable models whose every extension by a constant has more than 3 countable models. In other words the notion "$T$ has exactly 3 countable models" is not persistent.

Example 6. We start with the model $(Q, \leqslant)$ and add to it a sequence of "irrationals", i.e. add $P_n \subseteq Q$ such that $P_n$ is an initial segment of $Q$ which has no last element and $Q - P_n$ has no least element. We also stipulate that $P_n \subset P_m$ if $n < m$. Finally let $E$ be an equivalence relation on $Q$ which has 2 equivalence classes both of which are dense subsets of $Q$. Let $T$ be $\mathrm{Th}\big((Q, \leqslant, E, P_n)_{n<\omega}\big)$. Then $T$ has 3 countable models (one in which $\bigcup_{n<\omega} P_n = Q$, one in which $\bigcup_{n<\omega} P_n$ is $(-\infty, r)$ $(r \in Q)$, and one in which $Q - \bigcup_{n<\omega} P_n$ has no least element). If we add a constant to $T$ we are able to distinguish between the equivalence classes and we get 4 models: $\bigcup P_n = Q$ or $(-\infty, c)$ (depending where $c$ was added); $\bigcup P_n = (-\infty, r)$ and $r E c$; $\bigcup P_n = (-\infty, r)$ and not $r E c$; $Q - \bigcup P_n$ has no first element. Adding more equivalence relations one can find in a similiar way theories with 3 countable models which have, after adding a constant to them, arbitrarily large finite number of countable models.

The following question, asked by J. Rosenberg, is open [1]: is the notion "$T$ has finitely many countable models" absolute? If it is one will probably have to prove a non-trivial model-theoretical theorem about these theories. G. Sacks remarked that this had to be done for e.g. showing absoluteness of $\omega$- (or $\omega_1$-) categoricity.

[1] H. J. Keisler was kind to inform me that the notion is absolute. It follows from his book on Model Theory for Infinitary Logic (Corollary D on page 64) and Levy's absoluteness result.



**3.** The argument we mentioned in the introduction was supposed to prove the following: if $T$ has a full not saturated model then it has a universal not saturated model. We no not know a counterexample to this statement but it seems too strong to be true. If we assume the existence of a model better than a full model, the statement is true.

Definition. A model $A$ is almost saturated if for any $a \in A^n$, $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and any type $\Sigma(y_1, \ldots, y_m)$ of $\mathrm{Th}(A)$ if $\{\varphi(a, y_1, \ldots, y_m)\} \cup \Sigma$ is consistent with $T\big((A, a)\big)$ then it is realized in $(A, a)$.

Remark. The following example which shows that almost saturatedness is not stronger than universality or homogeneity is due to H. J. Keisler. Let $A$ be the set of all functions $f\colon \omega+1 \to \omega$ such that $f \restriction \omega$ is eventually constant and $f(\omega) \leqslant \lim_{n\to\infty} f(n)$. For each $n < \omega$ let $E_n \subseteq A \times A$ be defined by

$$f E_n g \quad \text{iff} \quad f \restriction (n+1) = g \restriction (n+1).$$

Each $E_n$ is an equivalence relation, $E_{n+1} \subseteq E_n$ and each equivalence class of $E_n$ contains infinitely many equivalence classes of $E_{n+1}$. $\mathrm{Th}\big((A, E_n)_{n<\omega}\big)$ admits elimination of quantifiers and this simplifies checking the claims. If $B$ is the set of functions $f\colon \omega+1 \to \omega$ which are eventually constant and $E_n'$ is defined on $B$ as $E_n$ is on $A$ then $(B, E_n')_{n<\omega}$ is saturated. It is clear that $(B, E_n')_{n<\omega}$ is not embeddable into $(A, E_n)_{n<\omega}$ so $(A, E_n)_{n<\omega}$ is not universal.

If $\Sigma(x_1, \ldots, x_m)$ is a type of the theory we let $f_1 \in A$ be the function identically equal to $m$. (Note that the theory has just one 1-type). $f_2$ is then defined so that it agrees or disagrees with $f_1$ at appropriate (according to $\Sigma$) places. If it is to agree with $f_1$ for $n < \omega$ but be different from it we define it as constant $m$ on $\omega$ and $f_2(\omega+1) = m-1$. $f_3, f_4, \ldots$ are found similarly. The model $(A, E_n)_{n<\omega}$ is therefore full so it cannot be homogeneous (full countable homogeneous models are saturated). If a consistent set $\{\varphi(a, y_1, \ldots, y_m)\} \cup \Sigma(y_1, \ldots, y_m)$ $(a \in A^n)$ is given note that $\varphi(a, y_1, \ldots, y_m)$ restricts the behavior of $y_1, \ldots, y_m$ only up to the highest $k$ such that $E_k$ appears in $\varphi$. From that point on one is free to realize $\Sigma$ as above so $(A, E_n)_{n<\omega}$ is almost saturated.

It will follow from the next theorem that this theory has a universal not saturated model: Let $A$ be the set of eventually constant functions from $\omega+1$ into $\omega+\omega$ such that if $\lim_{n\to\infty} f(n) < \omega$ then $f(\omega) \leqslant \lim_{n\to\infty} f(n)$.

$(B, E'_n)$ is then embeddable into this model by $f \to \lambda a[\omega + f(a)]$. The model is not homogeneous because if $f(g)$ is identically 1 (0) and $h \upharpoonright \omega = \lambda n[1]$ and $h(\omega) = 0$ then we have $(A, E_n, f) \equiv (A, E_n, g)$ but for no $k \in A$ is $(A, E_n, f, h) \equiv (A, E_n, g, k)$.

THEOREM 3. *If a complete theory $T$ has an almost saturated not saturated model then it has a universal not saturated model.*

Proof. Let $A$ be the saturated model of $T$, let $\{a_n \mid n < \omega\}$ be an enumeration of $A$ and let $\overline{T}$ be the (complete) diagram of $A$. If $B$ is an almost saturated not saturated model of $T$ there is a $b \in B$ and a type $\Gamma(b, y)$ of $\mathrm{Th}((B, b))$ which is not realized in $(B, b)$. Let $\Delta(x)$ be the type of $b$ (in $\mathrm{Th}(B)$). We want to find a link of $\overline{T}$ and $\Delta(b)$ which will omit $\Gamma(b, y)$. In general this is precisely what one has to do to find a universal not saturated model. To do this we define $C$ as

$$\{\Sigma(x) \mid \Sigma(x) \text{ is a type of } \overline{T} \text{ and } \Delta(x) \subseteq \Sigma(x)\}$$

and view $C$ as a subset of the Stone space of the Boolean algebra of 1-formulas of $\overline{T}$. $C$ is closed so it is compact and Baire Category theorem holds in it. For any $\varphi = \varphi(a_0, \ldots, a_n, x, y)$ where $a_0, \ldots, a_n \in A$ let $N_\varphi$ be

$$\{\Sigma \in C \mid \{(\exists y\varphi\} \cup \{(\forall y)[\varphi \to \gamma] \mid \gamma \in \Gamma(x, y)\} \subseteq \Sigma\}.$$

$N_\varphi$ is the set of those links which make $\varphi$ an "atom" for $\Gamma(b, y)$. $N_\varphi$ is a closed subset of $C$ (being an intersection of clopen sets). We show that it is nowhere dense. If not then for some $\psi(a_0, \ldots, a_m, x)$ (we may assume $m \geqslant n$) we have

$$\text{(1)} \qquad \{\Sigma \in C \mid \psi \in \Sigma\} \neq 0$$

and

$$\text{(2)} \qquad \{\Sigma \in C \mid \psi \in \Sigma\} \subseteq N_\varphi .$$

The set $\{\psi(x_0, \ldots, x_m, b)\} \cup \Sigma_m(x_0, \ldots, x_m)$ where $\Sigma_m$ is the type of $(a_0, \ldots, a_m)$ is, by (1), consistent with $\mathrm{Th}((B, b))$. Because $B$ is almost saturated the set $\{\psi\} \cup \Sigma_m$ is realized in $(B, b)$ by, say, $a'_0, \ldots, a'_m$. Let $\Lambda(x_0, \ldots, x_m, x)$ be the type of $(a'_0, \ldots, a'_m, b)$. $\psi(x_0, \ldots, x_m, x) \in \Lambda(x_0, \ldots, x_m, x)$ so, by (2), any extension of $\Lambda$ to a type in $C$ is in $N_\varphi$. Thus

$$(B, a'_0, \ldots, a'_m, b) \models (\exists y)\varphi(a'_0, \ldots, a'_n, y)$$

and

$$(B, a'_0, \ldots, a'_m, b) \models (\forall y)[\varphi \to \gamma(b, y)] \quad \text{for any } \gamma \in \Gamma(b, y) .$$

But this means that $\Gamma(b, y)$ is realized in $(B, b)$; a contradiction. Since there are only countably many formulas $\bigcup_\varphi N_\varphi$ is meager and by the Baire Category theorem there is $\Sigma \in C - \bigcup N_\varphi$. If $\varphi(a_0, \ldots, a_n, b, y)$ is consistent with $\overline{T} \cup \Sigma(b)$ then for some $\gamma \in \Gamma(b, y)$ $\varphi \wedge \neg\gamma$ must be consistent with $\overline{T} \cup \Sigma(b)$ by the definition of $\Sigma$. This means that $\overline{T} \cup \Sigma(b)$ has a model $C$ which omits $\Gamma(b, y)$. The reduct of $C$ to the language of $T$ is universal and not saturated.

It should be noted that neither of the theories listed in Example 2 satisfies the hypothesis of Theorem 3.

The proof of Theorem 3 is in the same general class as the proof of the existence of prime models (that one which uses omitting of types) or Barwise's compactness theorem or results in [3]. (We state nothing of degree of difficulty). All arguments listed above use only syntactical property of $T$ and information about countable models of $T$ unlike Morley's omitting of types theorem (see [7]) or Keisler's 2-cardinal theorem ([5]) which use information about uncountable models.

**4.** The theorem of Vaught which says that no complete theory has exactly 2 countable models puzzled me from the time I learned about it. Below we improve the theorem (another improvement is Corollary 1) and we think that the proof better explains the fact than Vaught's original proof.

We fix a complete theory $T$ which has no finite models and use the space of models for $T$ defined in [4]. So, let $S$ be $\{A \mid A = \omega \text{ and } A \models T\}$. If $\varphi(x_1, \ldots, x_n)$ is a formula and $k_1, \ldots, k_n < \omega$ let $[\varphi(k_1, \ldots, k_n)]$ be $\{A \in S \mid (A, k_1, \ldots, k_n) \models \varphi(k_1, \ldots, k_n)\}$. The sets $[\varphi]$ form a basis for a 0-dimensional, Hausdorff completely metrizable topology on $S$ (see [4] for details; actually only the part that the topology is completely metrizable is not obvious). We note that $[\varphi(k_1, \ldots, k_n)]$ is clopen and that $S$ is almost never compact. Given $A \in S$ we let $O(A)$ to be the orbit of $A$ under the group of permutations of $\omega$, in other words

$$O(A) = \{B \in S \mid B \simeq A\} .$$

Note that every orbit is dense in $S$. Assume now that $T$ has only countably many types. It then has a prime model $A \in S$ and a saturated model $B \in S$. [4] proves that $O(A)$ is $G_\delta$. In fact, if $\Sigma(x_1, \ldots, x_n)$ is a type of $T$, then the set of models in $S$ which omit $\Sigma$ is

$$\bigcap \{\bigcup_{\sigma \in \Sigma}[\neg\sigma(a)] \mid a \in \omega^n\}$$

which is a $G_\delta$ (if not empty it is comeager). $O(A)$ is the set of models which omit all non-principal types i.e. it is a countable intersection of $G_\delta$'s. The orbit of the saturated model has the following form:

$$O(B) = \bigcap_{\Sigma(x,y)} \bigcap_{a \in \omega^n} \Big(S - \bigcap_{\Delta \subseteq \Sigma} [\exists y \wedge \Delta] \cup \bigcup_{b \in \omega^m} \bigcap_{\sigma \in \Sigma} [\sigma(a, b)]\Big)$$

where $\Delta$ runs through finite subsets of $\Sigma$ which is a type of $T$. $O(\boldsymbol{B})$ is clearly $F_{\sigma\delta}$. But if $T$ is $\omega$-categorical $O(\boldsymbol{B})$ is actually open. Otherwise we have

THEOREM 4. *If $T$ is not $\omega$-categorical $O(\boldsymbol{B})$ is not $F_\sigma$ so it is true $F_{\sigma\delta}$.*

Proof. The proof resembles that of the Baire category theorem but because $O(\boldsymbol{B})$ is meager, the Baire's theorem cannot be applied directly. First of all we outline an efficient construction of a saturated model on $\omega$. Enumerate all expressions of the form $\Sigma(n_1, \ldots, n_k, y_1, \ldots, y_m)$ where $\Sigma(x_1, \ldots, y_m)$ is a type of $T$ and $n_1, \ldots, n_k < \omega$. Then construct, by induction, $T_m = T \cup \Delta(0, \ldots, p_m)$ where $\Delta$ is a type of $T$. If $T_m$ is constructed choose at random $\Sigma(n_1, \ldots, n_k, y_1, \ldots, y_j)$ with $n_1, \ldots, n_k \leqslant p_m$ and such that $\Sigma \cup \{y_i \neq k | \ i \leqslant j$ and $k \leqslant p_m\}$ is consistent with $T_m$. Extend it to a type $\Delta'(0, \ldots, p_m, y_1, \ldots, y_j)$ of $T_m$ and define $T_{m+1}$ as $T \cup \Delta'(0, \ldots, p_m, p_{m+1}, \ldots, p_{m+j})$.

Instead of taking $\Sigma$ at random we can take $\Sigma$ in orderly fashion but so that each $\Sigma(n_1, \ldots)$ is considered infinitely many times. $\overline{T} = \bigcup_m T_m$ will be the diagram of a saturated model. $\overline{T}$ is a diagram of a model since is $(\exists x)\varphi(n_1, \ldots, n_k, x) \in \overline{T}$ then $\varphi(n_1, \ldots, n_k, x) \in \Sigma(n_1, \ldots, n_k, x)$ which is sooner or later realized by, say $m$, so $\varphi(n_1, \ldots, n_k, m) \in \overline{T}$.

The construction of a prime model on $\omega$ for $T \cup \Delta(0, \ldots, m)$ is usually done in $\omega$ steps. We mention only that such a model is not saturated in our case ($T$ is not $\omega$-categorical).

Let us now assume that $O(\boldsymbol{B}) = \bigcup_{i<\omega} F_i$ where $F_i$ is closed; $F_i$ is of the form $\bigcap \{\varphi(n_1, \ldots, n_k) | \ \varphi \in F_i'\}$ where $F_i'$ is a set of formulas. We want to construct a saturated model on $\omega$ which is outside of every $F_i$. We start by building the prime model for $T$ i.e. we specify $\Delta_0(0), \Delta_1(0, 1), \ldots$ so that the result $\Delta_\omega(0, 1, \ldots)$ is the diagram of a prime model. Because $F_0 \subseteq O(\boldsymbol{B})$ for some $n_0 < \omega$ $\Delta_{n_0}(0, \ldots, n_0)$ will contain $\neg\varphi(n_1, \ldots, n_k)$ where $\varphi(n_1, \ldots, n_k) \in F_0'$. For couple of steps after $n_0$ we build according to the instructions above a saturated model for $T$ starting with $\Delta_{n_0}(0, \ldots, n_0)$. We get to, say, $\Delta_{n_1}'(0, \ldots, n_1)$ and go on to build a prime model for $\Delta_{n_1}'(0, \ldots, n_1)$. We come to $\Delta_{n_2}'(0, \ldots, n_2)$ which contains $\neg\varphi'(m_1, \ldots, m_k)$ with $\varphi'$ in $F_1'$. Then we make more steps toward creating the saturated model but again switch to the construction of the prime model for the last theory we constructed so as to get the resulting model out of $F_2$, etc. After $\omega$ steps we get a model $\boldsymbol{C} \in O(\boldsymbol{B}) - \bigcup_{i<\omega} F_i$.

COROLLARY 3. *No complete theory can have exactly 2 countable models.*

Proof. If $T$ has 2 countable models it means that $O(\boldsymbol{A}) \cup O(\boldsymbol{B}) = S$. Since $O(\boldsymbol{A}) \cap O(\boldsymbol{B}) = 0$ this would mean that $O(\boldsymbol{B})$ is $F_\sigma$ because $O(\boldsymbol{A})$ is $G_\sigma$. But this contradicts Theorem 4.



Remark. In terms of $S$ we can interpret Baldwin-Lachlan's result as follows: if $T$ is $\omega_1$-categorical theory every orbit of a not saturated model $\boldsymbol{A} \in S$ is included in a $G_\delta$ which is disjoint from the orbit of the saturated model. Indeed if $\boldsymbol{A} \in S$ is not saturated then $\boldsymbol{A}$ omits a type $\Sigma(x, x_1, \ldots, x_n)$ which says that $x_1, \ldots, x_n$ are independent elements of the strongly minimal formula $\varphi(x, .)$. In general if a not $\omega$-categorical theory has the above property then it has infinitely many non-isomorphic models. This follows from the property using the fact that finite unions of $G_\delta$'s is a $G_\delta$ and Theorem 4.

When we tried to prove Theorem 4 not knowing whether it was true we had to make sure that $S$ was uncountable for in the other case every subset of $S$ would be $F_\sigma$. We noticed that if $S$ is countable then $T$ is categorical in all powers. It is easy to see that if $T$ has two infinite, say, 1-atoms then $S$ will have power $2^\omega$ since we can interpret one of the atoms as any infinite subset of $\omega$ whose complement is infinite. There must be an infinite atom ($T$ must be $\omega$-categorical and has no finite models) and analogously to the last sentence we see that the infinite atom is strongly minimal. The converse of this remark is false. $\mathrm{Th}\big((\omega, E, F)\big)$ where $E$ is the set of even integers and $F(2n+1) = 2n$ is categorical in all powers but its space is uncountable since $E$ and $\omega - E$ are infinite atoms.

To conclude our discussion of Vaught's theorem we show that it is false for $\omega$-logic. For that we need an example of a closed linear order with an element not definable from other elements. Let $(A_0, \leqslant_0)$ be of order type $\omega+1+\omega^*$. Let $c_0 \in A_0$ be the middle element. Assume $(A_n, \leqslant_n)$ is defined and let $(A_{n+1}, \leqslant_{n+1})$ be of order type $A_n \cdot \omega + 1 + A_n \cdot \omega^*$. Each $(A_n, \leqslant_n)$ has a natural middle element, say $c_n$. Let $f_n\colon A_n \to A_{n+1}$ be defined so that: $f_n(c_n) = c_{n+1}$; $f_n$ maps $\{x \in A_n | \ x < c_n\}$ isomorphically into the initial segment of $(A_{n+1}, \leqslant_{n+1})$ and $f_n$ maps $\{x \in A_n | \ x > c_n\}$ isomorphically onto the terminal segment of $(A_{n+1}, \leqslant_{n+1})$, assuming that each $f_n$ is the identity on $A_n$ we defined $C$ as $\bigcup_{n<\omega} A_n$ and $\leqslant$ as $\bigcup_{n<\omega} \leqslant_n$. $(C, \leqslant)$ is a closed order and we claim that $c = c_0$ is not first-order definable from any $c_1, \ldots, c_k \in C - \{c\}$. For that we classify elements of $C$ into types as follows: $t(a) \geqslant 0$ for every $a \in C$; $t(a) \geqslant n+1$ if $a = \sup\{x \in C | \ x < a \wedge t(x) \geqslant n\}$. We say that $t(a) = n$ if $t(a) \geqslant n$ and $t(a) \not\geqslant n+1$. Note that $c$ is the only element $x$ of $C$ for which $t(x) \geqslant n$ for every $n < \omega$. Also note that if $a \in C$ and $n > 0$ then $\sup\{x | \ x < a \wedge t(x) = n\} = a$ iff $\inf\{x | \ x > a \wedge t(x) = n\} = a$. Let $F_n$ $(n < \omega)$ be the set of finite functions $\{(a_i, b_i) | \ i \leqslant k\}$, where $a_0 < \ldots < a_k$, $b_0 < \ldots < b_k$ are points of $C$, such that:

a) $t(a_i) \geqslant n$ iff $t(b_i) \geqslant n$ for $i \leqslant k$,

b) if $t(a_i) < n$ then $t(b_i) = t(a_i)$,

c) there are $\alpha$ ($\alpha \leqslant \omega$) points of type $q < n$ between $a_i$ and $a_{i+1}$ iff there are $\alpha$ points of type $q$ between $b_i$ and $b_{i+1}$,

d) $a_0 = b_0 =$ first element of $C$, $a_k = b_k =$ last element of $C$.

We will show, in a more general setting elsewhere, that the sets $F_n$ satisfy: if $f \in F_{n+1}$ and $a \in C$ then there are $g, h \in F_n$ such that $f \subseteq g$, $f \subseteq h$, $a \in$ domain of $g$ and $a \in$ range $h$. This means that if $\{(a_i, b_i)|\ i \leqslant k\} \in F_n$ then $(C, \leqslant, a_i)_{i\leqslant k}$ and $(C, \leqslant . b_i)_{i\leqslant k}$ satisfy the same sentences with at most $n$ quantifiers. Now, if $a_0, \ldots, a_k \in C-\{c\}$ and, say $a_0 < \ldots < a_i < c < a_{i+1} < \ldots < a_k$ and if $b \in C$ is such that $t(b) \geqslant n$ and $a_0 < \ldots < a_i < b < a_{i+1} < \ldots < a_k$ (there are infinitely many such $b$'s) then $\{(a_i, a_i)|\ i \leqslant k\} \cup \cup \{(b, c)\} \in F_n$ so it follows from above that any formula $\varphi(x, a_0, \ldots, a_k)$ with at most $n$ quantifiers which is satisfied by $c$ is also satisfied by $b$. If we put $B = C-\{c\}$ and let $\leqslant_B$ to be $\leqslant \restriction B$ then we have:

$$(B, \leqslant_B) \prec (C, \leqslant), \tag{1}$$

$$|C-B| = 1 . \tag{2}$$

Let $M$ be the set of all elements of $B$ which have direct $\leqslant$-predecessors or direct $\leqslant$-successors. $M$ will play the role of the natural numbers. Because $M$ is a definable subset of $B$, $c (\in C-B)$ is not in $M$ and because (1) holds we have

$$\boldsymbol{B} = (B, \leqslant, M, b)_{b\in B} \prec (C, \leqslant, M, a)_{a\in B} = \boldsymbol{C}.$$

We consider $\boldsymbol{B}$ and $\boldsymbol{C}$ as $\omega$-models. Let $N(.)$ be the predicate for $M$. Note that $M$ is a dense subset of $B$: if $M \cap [b_1, b_2] = 0$ the order of $B$ restricted to $[b_1, b_2]$ would be dense so $C$ being countable could not be closed. Because of this we have

$$(\forall z)[N(z) \to (z \leqslant x \leftrightarrow z \leqslant y)] \to x = y . \tag{3}$$

Let $\boldsymbol{A}$ be an $\omega$-model of $\mathrm{Th}(\boldsymbol{B})$. Since $\boldsymbol{B}$ contains name for every element in $B$ we see that $\boldsymbol{B} \prec \boldsymbol{A}$. So $N^{\boldsymbol{A}} = M$. We now show that $|A-B| \leqslant 1$. Let $a_0, a_1 \in A-B$ and let $D_i$ be $\{b \in M|\ b < a_i\}$ $(i = 0, 1)$. By (3) we have $a_0 \neq a_1$ iff $d_0 = \sup D_0 \neq d_1 = \sup D_1$ sup's being taken in $\boldsymbol{C}$. If $d_i \in B$ we have: for each $b \in M$, $b < d_i$ iff $b < a_i$ so by (3) $a_i = d_i$, contradiction. Thus $d_0, d_1 \in C-B$ i.e. $d_0 = d_1 = c$ and that means $a_0 = a_1$. If $|A-B| = 0$ then $\boldsymbol{A} = \boldsymbol{B}$ if $|A-B| = 1$ then $\boldsymbol{A} \simeq \boldsymbol{C}$. So $\boldsymbol{B}, \boldsymbol{C}$ are the only countable models of $\mathrm{Th}(\boldsymbol{B})$ considered as an $\omega$-theory.

Remark. The above example also shows a situation in which $\boldsymbol{B} \prec \boldsymbol{C}$ with $N^{\boldsymbol{B}} = N^{\boldsymbol{C}}$ but for no model $\boldsymbol{A} \succ \boldsymbol{C}$ we have $N^{\boldsymbol{A}} = N^{\boldsymbol{C}}$. Other examples of this situation were found by J. Gregory ([12]) and J. Knight in her thesis.



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