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Minimality in the Δ_2^1-degrees *

by

Harvey Friedman (Amherst, N. Y.)

Abstract. We give a proof in ZFC that every sequence of Δ_2^1-degrees has a minimal strict upper bound (Theorem 2), and characterize those sequences which have minimum strict upper bounds (Theorem 3). In addition, we prove that every countable stable ordinal is the first ordinal not Δ_2^1 in some $f\colon \omega\to\omega$, under the assumption that ω_1 is inaccessible in L (Theorem 7).

In connection with various notions of degree, consider the following statements:

A. There is a minimal degree.

B. Every sequence of degrees has a minimal strict upper bound. (This implies A).

C. No sequence of degrees has a minimum strict upper bound.

For Turing degrees, A was established in Spector [11], B was established in Sacks [7], and C was established in Spector [11].

For hyperdegrees, A was established in Gandy, Sacks [2], and reproved in Sacks [8], and C was refuted in Richter [6]. Sacks believes B is false, but is planning to publish some positive results on B in Sacks [9].

For degrees of constructibility (of functions on ω), A was established in Sacks [8] under the assumption $\omega_2^L < \omega_1$, and in Jensen [3] under the assumption $\omega_1^L < \omega_1$. We do not know how to decide B, even under the assumption that measurable cardinals exist, but we conjecture that both B and C can be decided under the assumption $(\forall x \subset \omega)(\omega_1^{L(x)} < \omega_1)$. We can refute C under the assumption $(\forall x \subset \omega)$ ($x^{\#}$ exists) by defining $d_0 = 0$, $d_{n+1} = (d_n)^{\#}$. The (degrees of constructibility of the) d_n have the least upper bound (the degree of constructibility of) $\{2^n 3^m\colon n \in d_m\}$.

The Δ_2^1-degrees are the equivalence classes of functions on ω under the equivalence relation (f is Δ_2^1 in g and g is Δ_2^1 in f).

For Δ_2^1-degrees, a proof of A may be found in Shoenfield [10]. We establish B here (Theorem 2). We also prove (Theorem 4) that C is equivalent to the statement $(\forall x \subset \omega)(\exists y \subset \omega)\big(y \notin L(x)\big)$. All of our arguments can be formalized in ZFC.

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The paper concludes with a proof that the countable stable ordinals are just those ordinals that are the first ordinal stable in some f, under the assumption that ω_1 is inaccessible in L (Theorem 7). We conjecture that $\omega^\omega \subset L$ is equivalent (in ZFC) to: the countable stable ordinals are just those ordinals that are the first ordinal stable in some $f \in \omega^\omega$. For proofs that the countable admissible ordinals are just those ordinals that are the first ordinal admissible in some f, see Friedman and Jensen [1], and Sacks [9]. A similar theorem for L-cardinals appears in Jensen and Solovay [4]. (The connection with Δ_2^1-degrees is that the first ordinal stable in f is always the first ordinal not Δ_2^1 in f.)

In the paper it is notationally convenient to use representative functions instead of their degrees.

DEFINITION 1. $K \subset \omega^\omega$ is Turing closed iff for all $f_1, \ldots, f_n \in K$, $(f_1, \ldots, f_n) \leqslant_T g$ for some $g \in K$. Let $K \subset \omega^\omega$ be Turing closed. Then f is *Δ_2^1-minimal over* K if and only if $(\forall g \in K)$ (g is Δ_2^1 in f and f is not Δ_2^1 in g), $(\forall h)$ (if h is Δ_2^1 in f and $(\forall g \in K)$ (g is Δ_2^1 in f and f is not Δ_2^1 in g) then f is Δ_2^1 in h). Also f is Δ_2^1-minimum over K if and only if f is Δ_2^1-minimal over K, and is Δ_2^1 in any g which is Δ_2^1-minimal over K.

It is convenient to speak of λ-degrees, for limit ordinals λ. Our λ-degrees are the equivalence classes of sets under ($x \in L_\lambda(y)$ and $y \in L_\lambda(x)$).

DEFINITION 2. Let $K \subset \omega^\omega$ be Turing closed, or of cardinality 1, λ a limit ordinal. Then f is *λ-minimal over* K if and only if

$$(\forall g \in K)\big(g \in L_\lambda(f)\ \&\ f \notin L_\lambda(g)\big),\ (\forall h)\Big(h \in L_\lambda(f)\ \&\ (\forall g \in K)\big(g \in L_\lambda(h)\ \&\ h \notin L_\lambda(g)\big)\Big) \to \big(f \in L_\lambda(h)\big).$$

If $K = \{h\}$, we will omit the brackets and write: f is λ-minimal over h.

DEFINITION 3. Let $\Delta_2^1(f)$ be the least ordinal not Δ_2^1 in f. For $K \subset \omega^\omega$, let $\Delta_2^1(K)$ be $\bigcup_{f \in K} \Delta_2^1(f)$.

LEMMA 1. *Let $f, g \in \omega^\omega$. Then f is Δ_2^1 in g iff $f \in L_\lambda(g)$, where $\lambda = \Delta_2^1(g)$.*

Proof. Left to the reader.

The next lemma follows easily from Lemma 1.

LEMMA 2. *Let $K \subset \omega^\omega$ be Turing closed. If f is $\Delta_2^1(K)$-minimal over K and $\Delta_2^1(f) = \Delta_2^1(K)$, then f is Δ_2^1-minimal over K.*

LEMMA 3. *Let $K \subset \omega^\omega$ be Turing closed, and let (λ_n) be a sequence of limit ordinals with limit $\Delta_2^1(K)$. Suppose that for each k there is an $h_k \Delta_2^1$ in some element of K such that (f, h_k) is λ_k-minimal over h_k. Suppose $\Delta_2^1(f) = \Delta_2^1(K)$, and $(\forall g \in K)$ (g is Δ_2^1 in f and f is not Δ_2^1 in g). Then f is Δ_2^1-minimal over K.*

Proof. First show that f is $\Delta_2^1(K)$-minimal over K. Then apply Lemma 2.

DEFINITION 4. Let $2^{<\omega}$ be the set of all finite sequences of 0's and 1's. A tree T is a nonempty subset of $2^{<\omega}$ closed under initial segments. T is *perfect* iff every element of T has at least two incomparable extensions in T. Write $f \in T$ for $(\forall n)(\bar{f}(n) \in T)$, where $\bar{f}(n) = (f(0), \ldots, f(n-1))$.

LEMMA 4. *Let T be a perfect tree, $\lambda > \omega$ closed under addition, and $g\colon \lambda \to \omega$ be one-one. Then there is a perfect subtree T^* of T such that $T^* \in L_{\lambda+1}(g, T)$ and $(\forall f \in T^*)$ ((f, T) is λ-minimal over T).*

Proof. By combining the methods of proof of Lemmas 1.4 and 3.1 in Sacks [8].

DEFINITION 5. Define a partial function $F(R, f)$, for 4-ary recursive predicates $R, f \in \omega^\omega$, by $F(R, f) =$ the least ordinal α such that $(\exists g)$ (the tree of unsecured sequence numbers of $(\forall h)(\exists n)\big(R(n, \bar{f}(n), \bar{g}(n), \bar{h}(n))\big)$ has ordinal α); undefined if $(\exists g)(\forall h)(\exists n)\big(R(n, \bar{f}(n), \bar{g}(n), \bar{h}(n))\big)$ is false.

LEMMA 5. *$\Delta_2^1(f)$ is the union of the range of $F(R, f)$ as a partial function of R.*

Proof. If a bound on $F(R, f)$ were obtained $< \Delta_2^1(f)$, then it is easy to see that every set Σ_2^1 in f would be Δ_2^1 in f, which is a contradiction. Conversely, each $F(R, f)$ must be $< \Delta_2^1(f)$, if defined. This is because the property of being a well-ordering isomorphic to the tree of unsecured sequence numbers of some $(\forall h)(\exists n)\big(R(n, \bar{f}(n), \bar{g}(n), \bar{h}(n))\big)$ is Σ_2^1 in f, and so must have a solution Δ_2^1 in f.

LEMMA 6. *Let T be a perfect tree, R a recursive predicate. Suppose that for some $f \in T$, $f \notin L(T)$, $F(R, f)$ is defined. Then there is a perfect $T^* \subset T$, Δ_2^1 in T, and an $\alpha < \Delta_2^1(T)$ such that $(\forall f \in T^*)$ $(F(R, f) = \alpha)$.*

Proof. We use a technique developed in Mansfield [5]. Let $f \notin L(T)$, $f \in T$, $F(R, f) = \beta$. Let γ be the first ordinal $> \beta$ admissible in T. Choose $\varrho\colon \beta \to \omega$ to be generic over $L_{\gamma+1}(T)$, and such that $f \notin L_\gamma(\varrho, T)$, (where one-one partial functions from β into ω are used as conditions). Define h to be the characteristic function of the relation $S(n, m) \leftrightarrow \varrho^{-1}(n) \in \varrho^{-1}(m)$. Note that S defines a well-ordering of type β. By the admissibility of $L_\gamma(\varrho, T)$, we have that $L_\gamma(h, T) \cap \omega^\omega$ is precisely the functions hyperarithmetic in (h, T). Hence h codes a well-ordering of type β and f is not hyperarithmetic in (h, T). Summarizing we have $(\exists h)\big(h$ codes a well-ordering of type some β and $(\exists f)(F(R, f) = \beta\ \&\ f \in T\ \&\ f$ is not hyperarithmetic in $(h, T))\big)$. Hence there must be such an h with the additional property that h is Δ_2^1 in T. Let the order type of h be α. Then $\alpha < \Delta_2^1(T)$, and $\{f\colon F(R, f) = \alpha\ \&\ f \in T\}$ contains an element not hyperarithmetic in (h, T), as well as being Σ_1^1 in (h, T). Hence $\{f\colon F(R, f) = \alpha\ \&\ f \in T\}$ contains a perfect subset T^* which is recursive in the hyperjump of (h, T). So in any case T^* is Δ_2^1 in T, and we are done.

The following definition is in analogy with the definition of recursively pointed in Sacks [8].

DEFINITION 6. A tree T is *λ-pointed* iff for all $f \in T$, $T \in L_\lambda(f)$, and T is perfect. T is *Δ^1_2-pointed* iff for all $f \in T$, T is Δ^1_2 in f, and T is perfect.

LEMMA 7. *Let T be λ-pointed, $f \in \omega^\omega$. Then there is a λ-pointed $T^* \subset T$ such that T^* and (f, T) have the same λ-degree.*

Proof. As in Proposition 3.2 in Sacks [8].

LEMMA 8. *Let T be $\Delta^1_2(T)$-pointed and Δ^1_2-pointed. Let $f \in \omega^\omega$. Then there is a $\Delta^1_2(T)$-pointed $T^* \subset T$ of the same Δ^1_2-degree as (f, T). Furthermore, any such T^* is Δ^1_2-pointed.*

Proof. By Lemma 7, there is a $\Delta^1_2(T)$-pointed $T^* \subset T$ of the same $\Delta^1_2(T)$-degree as (f, T). Then clearly T^* is Δ^1_2 in (f, T). To show the reverse, it is enough to show that $\Delta^1_2(T) \leqslant \Delta^1_2(T^*)$. Note that since T is Δ^1_2-pointed, T is Δ^1_2 in T^*, and we are done.

Suppose $T^* \subset T$ is $\Delta^1_2(T)$-pointed and of the same Δ^1_2-degree as (f, T). Since T is Δ^1_2-pointed, any $g \in T^*$ has $\Delta^1_2(T) \leqslant \Delta^1_2(g)$. Hence $(\forall g \in T^*)$ $(T^*$ is Δ^1_2 in $g)$.

LEMMA 9. *Let T be $\Delta^1_2(T)$-pointed and Δ^1_2-pointed. Let $\omega < \lambda < \Delta^1_2(T)$ be closed under addition. Then there is a $\Delta^1_2(T^*)$-pointed $T^* \subset T$ of the same Δ^1_2-degree as T, such that $(\forall f \in T^*)\big((f, T)$ is λ-minimal over $T\big)$.*

Proof. Choose one-one $g\colon \lambda \to \omega$ such that $g \in L_\alpha(T)$, where $\alpha = \Delta^1_2(T)$. By Lemma 4, let T^* be a perfect subtree of T, $T^* \in L_{\lambda+1}(g, T)$, and $(\forall f \in T^*)\big((f, T)$ is λ-minimal over $T\big)$. Clearly T^* is Δ^1_2 in T. Since T is Δ^1_2-pointed, T is Δ^1_2 in T^*. Since $T^* \in L_\alpha(T)$, clearly T^* is $\Delta^1_2(T)$-pointed. Hence T^* is $\Delta^1_2(T^*)$-pointed.

LEMMA 10. *Let T be $\Delta^1_2(T)$-pointed and Δ^1_2-pointed. Let R be a recursive predicate. Suppose that for some $f \in T$, $f \notin L(T)$, $F(R, f)$ is defined. Then there is a $\Delta^1_2(T^*)$-pointed $T^* \subset T$ of the same Δ^1_2-degree as T, and an $\beta < \Delta^1_2(T)$, such that $(\forall f \in T^*)\big(F(R, f) = \beta\big)$.*

Proof. By Lemma 6, choose a perfect $T^* \subset T$, Δ^1_2 in T, and $\beta < \Delta^1_2(T)$ such that $(\forall f \in T^*)\big(F(R, f) = \beta\big)$. Then T^* is $\Delta^1_2(T)$-pointed. Since T is Δ^1_2-pointed, T is Δ^1_2 in T^*. Hence T^* is $\Delta^1_2(T^*)$-pointed.

DEFINITION 7. Let T be a perfect tree, $s \in T$. Then T_s is $\{t \in T\colon s \subset t$ or $t \subset s\}$. For $s \in 2^{<\omega}$, define $s^*i = s \cup \{\langle \mathrm{dom}(s), i\rangle\}$.

LEMMA 11. *Let T be $\Delta^1_2(T)$-pointed and Δ^1_2-pointed, and let $s \in T$. Then T_s is $\Delta^1_2(T_s)$-pointed, and of the same Δ^1_2-degree as T.*

Proof. Left to the reader.

LEMMA 12. *Let $g, h, T, T^* \in L\big((g_n)\big)$, $L\big((g_n)\big) \models \beta < \omega_1$. Then g is Δ^1_2 in h iff $L\big((g_n)\big) \models g$ is Δ^1_2 in h; $\gamma = \Delta^1_2(g)$ iff $L\big((g_n)\big) \models \gamma = \Delta^1_2(g)$; T is β-pointed iff $L\big((g_n)\big) \models T$ is β-pointed; $(\forall f \in T^*)\big((f, T)$ is β-minimal over $T\big)$ iff $L\big((g_n)\big) \models (\forall f \in T^*)\big((f, T)$ is β-minimal over $T\big)$; $(\forall f \in T)\big(F(R, f) = \alpha\big)$ iff $L\big((g_n)\big) \models (\forall f \in T)\big(F(R, f) = \alpha\big)$.*

Proof. Use the absoluteness of Σ^1_2 assertions.

THEOREM 1. *Let (g_n) be a sequence of functions in ω^ω whose range K is Turing closed. Then there is a perfect tree $T_0 \in L\big((g_n)\big)$ such that $(\forall f \in T_0)$ $\big(f \notin \bigcup_n L(g_n) \to f$ is Δ^1_2-minimal over $\{g_n\colon n \geqslant 0\}\big)$. Furthermore, $(\forall n)(\forall f \in T_0)(g_n$ is Δ^1_2 in $f)$, and $(\forall f \in T_0)\big(f \notin \bigcup_n L(g_n) \to \Delta^1_2(f) = \Delta^1_2(K)\big)$.*

Proof. Let $K = \{g_n\colon n \geqslant 0\}$, $\Delta^1_2(K) = \alpha$, and let $(\lambda_n) \in L\big((g_n)\big)$ be a sequence of ordinals with limit α, each closed under addition, obeying $\omega < \lambda_n < \Delta^1_2(g_n)$. Let $(R_n) \in L\big((g_n)\big)$ be an enumeration of all 4-ary recursive predicates.

In this paragraph, we work entirely within $L\big((g_n)\big)$. We will define two functions $G\colon 2^{<\omega} \to 2^{<\omega}$, $H\colon 2^{<\omega} \to P(2^{<\omega})$ by recursion. In the next paragraph we will work in reality, and show that the definitions by recursion given in this paragraph totally define G, H. Set $G(\langle\ \rangle) = \langle\ \rangle$, $H(\langle\ \rangle) = 2^{<\omega}$. Suppose $G(s)$, $H(s)$ have been defined, so that $H(s)_{G(s)} = H(s)$, and $H(s)$ is a $\Delta^1_2\big(H(s)\big)$-pointed tree. Let $\mathrm{dom}(s) = n$, and $H(s) = T$. If $n = 3k$, define $T^* \subset T$ to be $\Delta^1_2(T)$-pointed and of the same Δ^1_2-degree as (g_k, T), and choose incomparable $r, t \in T^*$. Define $G(s^*0) = r$, $G(s^*1) = t$, $H(s^*0) = T^*_r$, $H(s^*1) = T^*_t$. If $n = 3k+1$, define $T^* \subset T$ to be $\Delta^1_2(T^*)$-pointed and of the same Δ^1_2-degree as T, such that $(\forall f \in T^*)\big((f, T)$ is λ_k-minimal over $T\big)$, and choose incomparable $r, t \in T^*$. Define $G(s^*0) = r$, $G(s^*1) = t$, $H(s^*0) = T^*_r$, $H(s^*1) = T^*_t$. If $n = 3k+2$, define $T^* \subset T$ to be $\Delta^1_2(T^*)$-pointed and of the same Δ^1_2-degree as T, such that for some $\alpha < \Delta^1_2(T)$, $(\forall f \in T^*)\big(F(R_k, f) = \alpha\big)$, *if there is such a* T^*; $T^* = T$ otherwise. Choose incomparable $r, t \in T^*$, and define $G(s^*0) = r$, $G(s^*1) = t$, $H(s^*0) = T^*_r$, $H(s^*1) = T^*_t$.

We now prove by induction that for each s, (i) $G(s)$, $H(s)$ have been defined so that $H(s)_{G(s)} = H(s)$, (ii) if $3k+1 \leqslant \mathrm{dom}(s) \leqslant 3k+3$ then $H(s)$ is of the same Δ^1_2-degree as $(g_0, \ldots, g_k)$, (iii) $H(s)$ is $\Delta^1_2\big(H(s)\big)$-pointed and Δ^1_2-pointed.

(i), (ii), (iii) clearly hold for $s = \langle\ \rangle$. Suppose they hold for s. Let $\mathrm{dom}(s) = n$, $H(s) = T$. Then T is $\Delta^1_2(T)$-pointed, Δ^1_2-pointed, and $T_{G(s)} = T$.

Assume $n = 3k$. Then T is of the same Δ^1_2-degree as $(g_0, \ldots, g_{k-1})$. By Lemma 8, there is a $\Delta^1_2(T^*)$-pointed $T^* \subset T$ of the same Δ^1_2-degree as (g_k, T). By Lemma 12, this assertion holds in $L\big((g_n)\big)$. Hence $G(s^*0)$, $H(s^*0)$, $G(s^*1)$, $H(s^*1)$ are defined. By Lemmas 8, 12, the T^* used in the definitions of $H(s^*0)$, $H(s^*1)$ in $L\big((g_n)\big)$ must be $\Delta^1_2(T^*)$-pointed, Δ^1_2-pointed, and of the same Δ^1_2-degree as (g_k, T). Hence by Lemma 11, $H(s^*0)$, $H(s^*1)$ obey (i), (ii), (iii).

Assume $n = 3k+1$. Then T is of the same Δ_2^1-degree as $(g_0, \ldots, g_k)$. Hence $\lambda_k < \Delta_2^1(T)$. By Lemma 9, there is a $\Delta_2^1(T^*)$-pointed $T^* \subset T$ of the same Δ_2^1-degree as T, such that $(\forall f \in T^*)\big((f, T)$ is λ_k-minimal over $T\big)$. By Lemma 12, this assertion holds in $L\big((g_n)\big)$. Hence $G(s^*0)$, $H(s^*0)$, $G(s^*1)$, $H(s^*1)$ are defined. By Lemma 12, the T^* used in the definitions of $H(s^*0)$, $H(s^*1)$ in $L\big((g_n)\big)$ must be $\Delta_2^1(T^*)$-pointed and of the same Δ_2^1-degree as T. So clearly T^* is also Δ_2^1-pointed. Hence by Lemma 11, $H(s^*0)$, $H(s^*1)$ obey (i), (ii), (iii).

Assume $n = 3k+2$. Then T is of the same Δ_2^1-degree as $(g_0, \ldots, g_k)$. It is obvious that $G(s^*0)$, $H(s^*0)$, $G(s^*1)$, $H(s^*1)$ are defined. Note that by Lemma 12, the T^* used in the definitions of $H(s^*0)$, $H(s^*1)$ in $L\big((g_n)\big)$ must be $\Delta_2^1(T^*)$-pointed and of the same Δ_2^1-degree as T. So clearly T^* is also Δ_2^1-pointed. Hence by Lemma 11, $H(s^*0)$, $H(s^*1)$ obey (i), (ii), (iii).

We have just shown by induction that the definition in $L\big((g_n)\big)$ of G, H totally define G, H. We thus speak of the functions G, H as objects in $L\big((g_n)\big)$.

Let $T_0 = \{t \in 2^{<\omega}\colon\ t \subset G(s)$ for some $s\}$. Then T_0 is a perfect tree in $L\big((g_n)\big)$. We presently show that T_0 is the desired tree for this Theorem. We use Lemma 3.

Let $f \in T_0$. Since $f \in H(s)$ for infinitely many s, by (ii), (iii) above we have that $(\forall k)(g_k$ is Δ_2^1 in $f)$. This establishes the second conclusion of Theorem 1.

Until the remainder of the proof, fix $f \in T_0$, $f \notin \bigcup_n L(g_n)$. Then obviously $(\forall k)(g_k$ is Δ_2^1 in f and f is not Δ_2^1 in $g_k)$.

For $\mathrm{Dom}(s) = 3k+2$, note that $L\big((g_n)\big) \models \big(\forall f \in H(s)\big)\big((f, H(s))$ is λ_k-minimal over $H(s)\big)$. Hence by Lemma 12, $\big(\forall f \in H(s)\big)\big((f, H(s))$ is λ_k-minimal over $H(s)\big)$. Now for each k there is an s with $\mathrm{dom}(s) = 3k+2$, $f \in H(s)$. Hence for each k, there is an s with $\big(f, H(s)\big)$ λ_k-minimal over $H(s)$. By clause (ii) above, we see that for each k, there is an h_k Δ_2^1 in some element of $\{g_n\colon n \geqslant 0\}$ such that (f, h_k) is λ_k-minimal over h_k.

Suppose $F(R_k, f)$ is defined. Let $f \in H(s)$, $\mathrm{dom}(s) = 3k+1$, $H(s) = T$. By (iii) above, T is $\Delta_2^1(T)$-pointed and Δ_2^1-pointed. By (ii) above, $f \notin L(T)$. Hence by Lemma 10, there is a $\Delta_2^1(T^*)$-pointed $T^* \subset T$ of the same Δ_2^1-degree as T, and a $\beta < \Delta_2^1(T)$ such that $(\forall f \in T^*)\big(F(R_k, f) = \beta\big)$. So by Lemma 12, this statement holds in $L\big((g_n)\big)$. So again by Lemma 12, the T^* used in the definitions of $H(s^*0)$, $H(s^*1)$ obeys $(\forall f \in T^*)\big(F(R_k, f) = \beta\big)$ for some $\beta < \Delta_2^1(T)$. Now $f \in H(s^*0)$ or $f \in H(s^*1)$. Hence $F(R_k, f) < \Delta_2^1(T)$. By (ii) above, we have $F(R_k, f) < \Delta_2^1(K)$. So the union of the range of $F(R, f)$ as a partial function of R is $\leqslant \Delta_2^1(K)$. Since each g_n is Δ_2^1 in f, we have by Lemma 5 that $\Delta_2^1(f) = \Delta_2^1(K)$.

We have now established the hypotheses of Lemma 3. We conclude that f is Δ_2^1-minimal over K.

LEMMA 13. *Let $K \subset \omega^\omega$ be countable and Turing closed. Assume that for no $g \in K$ is $K \subset L(g)$. Then there is a perfect tree all of whose paths are Δ_2^1-minimal over K.*

Proof. Let (g_n) enumerate K, and choose T_0 as in Theorem 1. Each g_n is Δ_2^1 in every $f \in T_0$. Hence $(\forall f \in T_0)\big(f \notin \bigcup_n L(g_n)\big)$. So $(\forall f \in T_0)$ (f is Δ_2^1-minimal over K).

LEMMA 14. *Let $K \subset \omega^\omega$ be countable and Turing closed. Let $g \in K$, $K \subset L(g)$. Suppose that for some α, $K \subset L_\alpha(g)$ and $L(g) \models \alpha < \omega_1$. Then there is an f that is Δ_2^1-minimal over K. Furthermore, if $\omega^\omega \not\subset L(g)$ then there is a perfect set every path of which is Δ_2^1-minimal over K.*

Proof. Let $\beta = \Delta_2^1(K)$. Then $L(g) \models \beta < \omega_1$. Choose γ least such that $\beta < \gamma$ and $\big(L(\gamma) - L(\beta)\big) \cap \omega^\omega \neq \emptyset$. Let $h \in L(\gamma) - L(\beta)$. Set $f = (g, h)$. Then $\Delta_2^1(f) > \gamma > \beta$. Furthermore, each element of K is Δ_2^1 in f. Suppose f^* is Δ_2^1 in f, each element of K is Δ_2^1 in f^*, and not vice versa. Let $f^* \in L_\delta(g)$, δ least. If $\delta < \beta$ then f^* is Δ_2^1 in g, contradicting the way f^* was chosen. So $\beta < \delta$. Hence $\gamma \leqslant \delta$. Then it is easy to see that f is Δ_2^1 in f^*, since g is Δ_2^1 in f^*. We conclude that f is Δ_2^1-minimal over K.

Suppose $\omega^\omega \not\subset L(g)$. In this case we can use Theorem 1 to good effect. Since $L(g) \models \beta < \omega_1$, there is a $(g_n) \in L(g)$ enumerating $K^* = \{h\colon h$ is Δ_2^1 in some element of $K\}$. By Theorem 1, let $T_0 \in L(g)$ be a perfect tree with $(\forall f \in T_0)\big(f \notin \bigcup_n L(g_n) \to f$ is Δ_2^1-minimal over $K\big)$. Hence $(\forall f \in T_0)\big(f \notin L(g) \to f$ is Δ_2^1-minimal over $K\big)$. Let $h \notin L(g)$. Since $T_0 \in L(g)$, it is easy to construct a perfect $T \subset T_0$ such that $(\forall f \in T)$ $\big(f \notin L(g)\big)$ with the aid of h.

LEMMA 15. *Let $K \subset \omega^\omega$ be countable and Turing closed. Let $g \in K$, $K \subset L(g)$. Suppose that for no α do both $K \subset L_\alpha(g)$ and $L(g) \models \alpha < \omega_1$ hold. Then there is a perfect tree all of whose paths are Δ_2^1-minimal over K.*

Proof. It is clear that under these hypotheses, $L(g) \cap \omega^\omega$ is countable. Let (g_n) enumerate K, and choose T_0 as in Theorem 1. There must be a perfect $T \subset T_0$ with $(\forall f \in T)\big(f \notin L(g)\big)$. For such a T, $(\forall f \in T)$ (f is Δ_2^1-minimal over K).

THEOREM 2. *Let K be countable and Turing closed. Then there is an f which is Δ_2^1-minimal over K.*

Proof. The hypotheses of Lemmas 13, 14, 15 exhaust all possibilities.

THEOREM 3. *Let K be countable and Turing closed. The following are equivalent:* (a) $\omega^\omega \subset \bigcup_K L(g)$, (b) *for some* $g \in K$, $\omega^\omega \subset L(g)$, (c) *there is* *an f that is Δ^1_2-minimum over K*, (d) *there is not a perfect tree each path of which is Δ^1_2-minimal over K.*

Proof. We leave (a)$\to$(b) to the reader. If (b) holds then the same f of the proof of Lemma 14 is seen to be Δ^1_2-minimum over K by the same argument. That (c)$\to$(d) holds is obvious. To see (d)$\to$(a), assume (d). Then by Lemma 13, let $g \in K$, $K \subset L(g)$. By Lemma 15, for some α, $K \subset L_\alpha(g)$ and $L(g) \models \alpha < \omega_1$. By Lemma 14, $\omega^\omega \subset L(g)$. Hence (a).

The following is obvious from Theorem 3.

THEOREM 4. *If $\omega^\omega \not\subset L(g)$ for all $g \in \omega^\omega$, K is Turing closed, then no f is Δ^1_2-minimum over K.*

DEFINITION 8. We say that α is f-*stable* iff $(L_\alpha(f), \epsilon)$ is a Σ-elementary substructure of $(L(f), \epsilon)$. We call α *stable* iff α is $\emptyset$-stable.

LEMMA 16. *The first f-stable ordinal is $\Delta^1_2(f)$. If α is f-stable then α is stable.*

Proof. These results are in the folk literature. A basic point is that $L_\alpha = \{x\colon (TC(\{x\}), \epsilon) \approx (\omega, R)$ for some $R\Delta^1_2$ in $f\}$, for $\alpha = \Delta^1_2(f)$.

THEOREM 5. *If $\omega^\omega \subset L$ then not every countable stable ordinal is of the form $\Delta^1_2(f)$ for $f \in \omega^\omega$. If $\omega^\omega \not\subset L$ then every constructibly countable stable ordinal is of the form $\Delta^1_2(f)$ for $f \in \omega^\omega$.*

Proof. For the first statement consider the ωth stable ordinal. For the second statement, let α be a constructibly countable stable ordinal.

Case 1. There is a $\gamma < \alpha$ such that α is the first stable ordinal after γ. Let $f \in L(\alpha) - L(\gamma)$, $f\colon \omega \to \omega$. Then $\Delta^1_2(f) = \alpha$.

Case 2. Case 1 does not apply. Let (α_n), (γ_n) be two strictly increasing sequences of stable ordinals with limit α, such that α_n is the first stable ordinal after γ_n. Choose $(g_n) \in L$ such that $g_n \in L(\alpha_n) - L(\gamma_n)$, $g_n\colon \omega \to \omega$, and (g_n) has Turing closed range. Then $\Delta^1_2(g_n) = \alpha_n$. By Theorem 1 there is a constructible perfect tree such that for every nonconstructible path f, $\Delta^1_2(f) = \alpha$. Since $\omega^\omega \not\subset L$, this tree must have a nonconstructible path. Hence for some f, $\Delta^1_2(f) = \alpha$.

LEMMA 17. *Let α be f-stable, and let $g\colon \omega \to \alpha$ be generic over $L(f)$ using the one-one finite partial functions from ω into α under inclusion as the conditions. Then for $\beta > \alpha$, β is f-stable if and only if β is (f, g)-stable, and β is a cardinal in $L(f)$ iff β is one in $L((f, g))$.*

Proof. Let $P(x_1, \ldots, x_n, g)$ be a true Σ-statement in $L((f, g))$ about $x_1, \ldots, x_n \in L_\beta((f, g))$. Let $\tau_1, \ldots, \tau_n$ be terms in the forcing language over $L(f)$ such that $\tau_i = x_i$ of rank $< \beta$ holds in $L((f, g))$. Let $q \subset g$ be such that $q \Vdash P(\tau_1, \ldots, \tau_n, g)$. Now it is easy to see that forcing on Σ-statements is Σ. Hence the statement $q \Vdash P(\tau_1, \ldots, \tau_n, g)$ must hold when the quantifiers are relativized to $L_\beta(f)$, since $(L_\beta(f), \epsilon)$ is an Σ-elementary substructure of $(L(f), \epsilon)$. After relativization, the statement is seen to become $q \Vdash^* P(\tau_1, \ldots, \tau_n, g)$, where $\Vdash^*$ indicates forcing over $L_\beta(f)$ instead of over $L(f)$. Since g is generic over $L(f)$ it is generic over $L_\beta(f)$. Hence $P(\tau_1, \ldots, \tau_n, g)$ holds in $L_\beta((f, g))$. So $P(x_1, \ldots, x_n, g)$ holds in $L_\beta((f, g))$. The final conclusion is well known.

THEOREM 6. *Let $\alpha \leqslant \beta < \omega_1$, β a cardinal in L. Then α is stable iff $\alpha = \Delta^1_2(f)$ for some $f\colon \omega \to \omega$.*

Proof. Case 1. For some $\gamma < \alpha$, α is the first stable ordinal after γ. Then α is not a cardinal in L, and so $\alpha < \beta$. Choose $g\colon \omega \to \gamma$, g generic over L_β using the one-one finite partial functions from ω onto γ under inclusion as the conditions. Since β is a cardinal in L, clearly g is generic over L. Hence by Lemma 17, α is g-stable. Let $f(n) = 1$ if $n = 2^a 3^b$ and $g(a) \in g(b)$; 0 otherwise. Then $\gamma < \Delta^1_2(f)$. By Lemma 16, $\Delta^1_2(f) \leqslant \alpha$, $\Delta^1_2(f)$ is stable. Hence $\Delta^1_2(f) = \alpha$. Case 2. Case 1 does not apply. Then let (α_n), (γ_n) be two strictly increasing sequences of stable ordinals whose limits are α, and such that α_n is the first stable ordinal after γ_n. By iterating the argument used in Case 1 infinitely often, we obtain a sequence g_n with Turing closed range such that $\Delta^1_2(g_n) = \alpha_n$, and β is a cardinal in $L(g_n)$, for each n. It is clear that $\bigcup_n L(g_n)$ is countable. Therefore by Theorem 1, for some f, $\Delta^1_2(f) = \Delta^1_2(\{g_n\colon n \geqslant 0\})$. Hence $\Delta^1_2(f) = \alpha$.

THEOREM 7. *If ω_1 is inaccessible in L, then the countable stable ordinals are exactly the ordinals of the form $\Delta^1_2(f)$ for $f \in \omega^\omega$.*

Proof. Immediate from Theorem 6.

Remark. The referee of this paper indicated that D. Guaspari has observed that $\omega^\omega \subset L$ is equivalent to the Δ^1_2-degrees being linearly ordered. This fact can also be obtained (in a strengthened form) through Theorem 1 by choosing (g_n) to be Δ^1_2.

For an application of Δ^1_2-degrees to a problem in descriptive set theory, see *PCA well orderings of the line*, to appear in the Journal of Symbolic Logic.

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On some conjectures connected with complete sentences

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Abstract. Some conjectures on the finite axiomatizability of complete $\aleph_1$-categorical theories are discussed and related to open problems in group theory. A problem of Chang and Keisler is solved.

0. Introduction. In this paper we want to discuss some questions and conjectures connected with the finite axiomatizability of complete theories. The oldest ones within this context may be found in Vaught [25] and Morley [16]. They are:

QUESTION 0.1. Is there a complete, finitely axiomatizable theory, categorical in every infinite power?

QUESTION 0.2. Is there a complete, finitely axiomatizable theory, categorical in every uncountable power?

Since most of the mathematicians who have attacked these questions tried to give a negative answer, they tried to prove stronger statements, e.g.:

STATEMENT 0.3 (*). *If T is a complete theory which has an infinite model and is finitely axiomatizable, then T is unstable* (*or T admits a definable order relation*).

This conjecture was inspired by the classical example of a complete finitely axiomatizable theory: any complete extension of the theory of dense linear orderings. Variatio delectat, so others tried:

STATEMENT 0.4. *If T is as in Statement* 0.3, *then T is not superstable.*

STATEMENT 0.5. *If T is as in Statement* 0.3, *then T is not ω-stable.*

For the appropriate notions of stability one can consult Shelah [22], Morley [16] or Sacks [21].

A similar connection between some Stone spaces associated with a complete first order theory and finite axiomatizability was formulated by B. Jonsson (cf. Ehrenfeucht and Fuhrken [8]):

(*) This is conjecture $C(b)$, p. 424 in [26].