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On some conjectures connected with complete sentences

by

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Abstract. Some conjectures on the finite axiomatizability of complete $\aleph_1$-categorical theories are discussed and related to open problems in group theory. A problem of Chang and Keisler is solved.

0. Introduction. In this paper we want to discuss some questions and conjectures connected with the finite axiomatizability of complete theories. The oldest ones within this context may be found in Vaught [25] and Morley [16]. They are:

QUESTION 0.1. Is there a complete, finitely axiomatizable theory, categorical in every infinite power?

QUESTION 0.2. Is there a complete, finitely axiomatizable theory, categorical in every uncountable power?

Since most of the mathematicians who have attacked these questions tried to give a negative answer, they tried to prove stronger statements, e.g.:

STATEMENT 0.3 (*). *If T is a complete theory which has an infinite model and is finitely axiomatizable, then T is unstable (or T admits a definable order relation).*

This conjecture was inspired by the classical example of a complete finitely axiomatizable theory: any complete extension of the theory of dense linear orderings. Variatio delectat, so others tried:

STATEMENT 0.4. *If T is as in Statement* 0.3, *then T is not superstable.*

STATEMENT 0.5. *If T is as in Statement* 0.3, *then T is not ω-stable.*

For the appropriate notions of stability one can consult Shelah [22], Morley [16] or Sacks [21].

A similar connection between some Stone spaces associated with a complete first order theory and finite axiomatizability was formulated by B. Jonsson (cf. Ehrenfeucht and Fuhrken [8]):

(*) This is conjecture $C(b)$, p. 424 in [26].

Statement 0.6. *Let T be a complete theory. If for all $n \in \omega$ $F_n(T)$ is atomless, then T is not finitely axiomatizable.*

For the notation see Shoenfield [24].

Clearly we have $(0.3) \Rightarrow (0.4) \Rightarrow (0.5) \Rightarrow$ (negative answer of 0.1 and 0.2). Ehrenfeucht and Fuhrken [8] disproved 0.6, giving a counterexample which is unstable. Since the assumptions of Statement 0.6 imply only that T is not ω-stable, one might ask how low one can get in the degree of stability (unstability) of the counterexample. Now we shall prove in § 2.

Theorem 2.15. *There is a theory which is*

(i) *complete,*

(ii) *finitely axiomatizable,*

(iii) *superstable,*

(iv) *for all $n \in \omega F_n(T)$ is atomless.*

This disproves Statement 0.4 and hence also Statement 0.3 and improves the result of Ehrenfeucht and Fuhrken [8]. Further we shall prove two theorems relating Question 0.2 with unsolved problems in group theory and the theory of division rings suggested by H. Läuchli and A. Macintyre. I would like to thank them for the permission of including these theorems in this paper. Statement 0.5 remains open, but we conjecture it to be false.

In § 3 we shall give a partial negative answer of Question 0.1 which is related to almost strongly minimal theories (cf. Baldwin [1], [2] or, for their algebraic characterizations, Makowsky [14]):

Theorem 3.3. *If T is a complete, $\aleph_0$-categorical, almost strongly minimal theory, then T is not finitely axiomatizable.*

Vaught independently found a weaker version of this theorem, with strongly minimal in place of almost strongly minimal (oral communication).

Unexplained notions and notation may be found in Bell and Slomson [4], Sacks [21] and Shoenfield [24].

In § 1 we shall discuss several methods of proving non-finite axiomatizability of complete theories. We shall also relate the finite axiomatizability of theories with some algebraic properties of their models. Theorem 3.3 was proved in the author's diploma thesis [12] written under the guidance of professor H. Läuchli (*). Since the proofs remained unpublished for some time and M. Dickmann was preparing a survey article on the subject, he included our proof of Theorem 3.3 in it [5]. Therefore we shall discuss in § 3 the original proof without going too much into the details. But we shall relate our results to the approach to non-finite axiomatizability results indicated in § 1 using the characterization of almost strongly minimal theories in Makowsky [14]. This will yield a new proof of Theorem 3.3.

(*) The result was announced in [13].

Some of the results of § 2 and the main theorem of [14] were obtained during the author's stay at Warsaw University as an exchange student under the Swiss-Polish-exchange program (*). It was then that I first met professor Mostowski who, together with his research group, turned my stay in Warsaw into a mathematical adventure. I am happy to contribute this paper at the occasion of his 60th birthday.

1. Finite axiomatizability and complete theories in $L_{\omega\omega}$. Let T be a theory in $L_{\omega\omega}$. From the literature several criteria of non finite axiomatizability of T are known.

A set S of sentences is an axiom system for the theory T if it is consistent and its deductive closure equals T. The elements of S are independent if for every $\varphi \in S$ $(S - \{\varphi\}) \cup \{\neg\varphi\}$ is consistent. By an abuse of language we will call S *independent* if its elements are independent. From Gödel's completeness theorem for $L_{\omega\omega}$ one easily gets the following

Criterion A. *A theory T is not finitely axiomatizable iff there is an infinite independent axiom system for T.*

Keisler (cf. Bell and Slomson [4]) translated this into the language of ultraproducts and proved, using G.C.H.:

Criterion B. *T is finitely axiomatizable iff* Mod(T) *and its complement (with respect to the appropriate structures) are closed under the formation of ultraproducts.*

Shelah [23] in the meantime has eliminated G.C.H. from the proof. These two criteria apply to arbitrary theories in L and many finitely axiomatizable theories are known.

From Ehrenfeucht's game theoretic characterization of elementary equivalence [6] one easily gets a criterion for complete theories:

Criterion C. *Let $\mathfrak{A}$ be an infinite structure.* Th$(\mathfrak{A})$ *is finitely axiomatizable iff there exists $n \in \omega$ such that for all similar structures $\mathfrak{B}$ with $\mathfrak{A} \equiv_n \mathfrak{B}$ we have $\mathfrak{A} \equiv \mathfrak{B}$ (assuming the language is finite).*

One could try to get non finitely axiomatizability results for complete theories using a similar criterion as for decidability results by means of interpretability (cf. Mostowski-Robinson-Tarski [17]). For example is the following true?

Criterion D. *Let T be a complete finitely axiomatizable theory having infinite models only. Let φ be a formula with one free variable and $\mathfrak{A}$ be a model of T. Let $\mathfrak{B}$ be the definable substructure $\mathfrak{A}|\varphi(\mathfrak{A})$. Then* Th$(\mathfrak{B})$ *is also finitely axiomatizable.*

(*) For the later period the author was also supported by ETM-grant No. 200752.

The example discussed in § 2 disproves Criterion D, and shows that, unlike for decidability problems, the situation for finitely axiomatizable complete theories is much more complicated.

The compactness of $L_{\omega\omega}$ plays an important rôle in questions about finite axiomatizability. For, assume a theory T is provably not finitely axiomatizable. Let us form a new language L^* which is the Boolean closure of $L_{\omega\omega} \cup \{\psi\}$ where $\mathrm{Mod}(\psi)$ equals $\mathrm{Mod}(T)$ and $\psi \in L_{\omega_1\omega}$. L^* is a proper extension of $L_{\omega\omega}$ and still satisfies the Downward-Löwenheim-Skolem Theorem, so by Lindström's celebrated theorem [9] L^* cannot be compact. All the criteria A, B, C have up to now been applied to prove non finite axiomatizability of particular theories. What we are interested in are theorems of the following type (cf. Makowsky [15]):

(*) Let $\mathfrak{A}$ be an infinite structure of finite signature, $P_1, P_2, \ldots, P_n$ some algebraic properties of $\mathfrak{A}$. If every model of $\mathrm{Th}(\mathfrak{A})$ satisfies $P_1, P_2, \ldots, P_n$ then $\mathrm{Th}(\mathfrak{A})$ is not finitely axiomatizable.

Candidates for such algebraic properties are:
(i) Saturatedness, (ii) homogeneity, (iii) local finiteness, (iv) analogues of the Steinitz theorem in the theory of fields, (v) being closed under substructures, (vi) being closed under direct products.

Theories with (v) are called *open* (*universal*) *theories*, theories with (vi), after their characterizations by Horn (cf. Shoenfield [24]), Horn theories.

We call a sentence (a theory) *complete* if its deductive closure is complete. We call a sentence (a theory) *semi-complete* if it has an infinite model and all its infinite models are elementarily equivalent.

The following theorems, which are all easy consequences of well known results, may illustrate (*).

THEOREM 1.1. *Let T be semi-complete and open. If T has an infinite locally finite model $\mathfrak{A}$, then $\mathrm{Th}(\mathfrak{A})$ is not finitely axiomatizable.*

A structure is locally finite if every substructure generated by a finite set is finite.

Proof. Take an ultraproduct of all the finite models of T.

In [18] Palyutin proved the following:

LEMMA 1.2. *If T is an open $\aleph_0$-categorical theory, then all the models of T are locally finite.*

THEOREM 1.3. *If T is an open $\aleph_0$-categorical theory, then $\mathrm{Th}(\mathfrak{A})$ is not finitely axiomatizable for each infinite model $\mathfrak{A}$ of T.*

THEOREM 1.4. *If T is an open $\aleph_1$-categorical theory with a locally finite infinite model $\mathfrak{A}$, then $\mathrm{Th}(\mathfrak{A})$ is not finitely axiomatizable.*

To prove these theorems we simply observe that every $\varkappa$-categorical theory, $\varkappa$ an infinite cardinal, is semi-complete by Vaught's test (cf. Shoenfield [23]) and apply again criterion B. Similarly we get:

THEOREM 1.5. *Let T be a semi-complete Horn theory with a finite model of power greater then 1. Then $\mathrm{Th}(\mathfrak{A})$ is not finitely axiomatizable for infinite models $\mathfrak{A}$ of T.*

As in Theorems 1.3 and 1.4 we can replace semi-completeness by $\varkappa$-categoricity for some infinite cardinal $\varkappa$.

Some more refined theorems of this type will be proved in § 3.

2. Graphs of groups with variations. In the sequel we study a very special class of theories to get counterexamples for several of the statements of § 0. I am very much indebted to H. Läuchli who pointed out the possible importance of such theories.

Thema (Air). Let G be a countable group and $R \to F \to G$ a countable presentation of G (i.e. $G \simeq F/R$, R, F free and countably generated). We construct now a structure $\mathfrak{A}_G$ in the following way: Let $(f_{2i})_{i \in I \subset \omega}$ be a set of generators for F, $(r_k)_{k \in K \subset \omega}$ for R. Let $f_{2i+1} = f_{2i}^{-1}$ and let $\tilde{f}_i$ be unary function symbols for every $i \in I$. Now let $\mathfrak{A}_G = \langle A, \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_i, i \in I\rangle$ be such that for all $a \in A$ and all $g \in G$ with $g = f_{k(1)} f_{k(2)} \cdots f_{k(m)}$ we have $\tilde{f}_{k(1)} \tilde{f}_{k(2)} \cdots \tilde{f}_{k(m)}(a) = a$ iff $g \in R$. For group theorists this structure is known as the graph of the group G (cf. Magnus-Karras-Solitar [11]).

1. Variation.

PROPOSITION 2.1. *If $\mathfrak{A}_G$ is infinite then $T_G = \mathrm{Th}(\mathfrak{A}_G)$ is $\aleph_1$-categorical.*

Proof. Clearly the graph of G is a model of T_G. Since it is generated by one element, it is prime in the sense of A. Robinson (cf. [19]). Now it is sufficient to prove that any substructure of a model of T_G generated by one element is isomorphic to it. But this is clear, since T_G ensures us, that all the inequalities of the graph must be true in any substructure of models of T_G, by the homogeneity of the graph.

COROLLARY 2.2. *T_G is a universal theory.*

Proof. Apply the Łoś-Tarski theorem (cf. Shoenfield [24]).

COROLLARY 2.3. *T_G is model complete.*

Proof. By Lindström's theorem on modelcompleteness applied to theories categorical in some infinite power (cf. Lindström [10] or Makowsky [14]).

COROLLARY 2.4. *T_G admits elimination of quantifiers.*

Proof. By Corollaries 2.2 and 2.3.

2. Variation. The following theorem was suggested by H. Läuchli.

THEOREM 2.5. *Assume G is infinite, finitely presentable and has only a finite number of conjugacy classes. Then T_G is finitely axiomatizable.*

Proof. Let $R \Rightarrow F \to G$ be a finite presentation of G. Let T_0 be the following set of axioms:

If $(r_i)_{i<k(R)}((f_i)_{i<k(F)})$ generate R (respectively F), and $r_i = f_{i(1)} f_{i(2)} \ldots \ldots f_{i(m)}$. Then $\forall x(\tilde{f}_{i(1)} \tilde{f}_{i(2)} \ldots \tilde{f}_{i(m)}(x) = x)$ is in T_0.

Fact 1. If H is an arbitrary subgroup of G, the graph naturally associated with G/H is a model of T_0.

Fact 2. By the homogeneity of the graph we have also:

$$\mathfrak{A}_G \models \forall x(\tilde{f}_{i(1)} \tilde{f}_{i(2)} \ldots \tilde{f}_{i(m)}(x) = x) \Leftrightarrow \exists x(\tilde{f}_{i(1)} \tilde{f}_{i(2)} \ldots \tilde{f}_{i(m)}(x) = x) .$$

Now let $(c_i)_{i<l}$ be representative members of the conjugacy classes. To ensure that the only model generated by one element is, up to isomorphism, the graph of G, we add to T_0 for every $i < l$ an axiom $\forall x(\tilde{f}_{i(1)} \tilde{f}_{i(2)} \ldots \tilde{f}_{i(c_i)}(x) \neq x)$ where $c_i = f_{i(1)} f_{i(2)} \ldots f_{i(c_i)}$ to get T_1.

Fact 3. If $g_1, g_2 \in G$ are conjugate elements, $g_1 = f_{i(1)} f_{i(2)} \ldots f_{i(m)}$, $g_2 = f_{j(1)} f_{j(2)} \ldots f_{j(n)}$ and $\mathfrak{A}_0$ is a model of T_1, then

$$\mathfrak{A}_0 \models \forall x(\tilde{f}_{i(1)} \tilde{f}_{i(2)} \ldots \tilde{f}_{i(m)}(x) = x) \Leftrightarrow \forall x(\tilde{f}_{j(1)} \tilde{f}_{j(2)} \ldots \tilde{f}_{j(n)}(x) = x) .$$

T_1 has only infinite models by Fact 3 and, by the same argument as in the proof of Proposition 2.1, T_1 is $\aleph_1$-categorical, hence complete by Vaught's test.

In the light of Morley's question the theorem also reads:

THEOREM 2.6. *If there is no finitely axiomatizable complete $\aleph_1$-categorical theory then the following is a theorem in group theory: (*) Every finitely presentable group with a finite number of conjugacy classes is finite.*

Specialists in group theory informed us that the truth of (*) is still an open problem.

3. Variation. Similarly we can define finite presentations of a ring:

DEFINITION. A Ring R is *finitely presented* if $R \simeq F/J$ where $F = Z[X_1, X_2, \ldots, X_n]$ in non commutative variables and J is a finitely generated ideal.

I am indebted to A. Macintyre who suggested the following

THEOREM 2.7. *If there is an infinite division ring D which is finitely presented, then there is a theory T_D which is finitely axiomatizable, complete and $\aleph_1$-categorical.*

Remark. In the commutative case no infinite such ring (i.e. a field) exists, but in general nothing is known about the existence of finitely presented division rings.

Proof. Vectorspaces over division rings behave very much the same as over fields. If the division ring D is finitely presented we can code it in a finite language as for graphs above. The models will be abelian groups with additional functions acting on them.

4. Variation. We consider now structures of the form $\langle \mathfrak{A}_G, P_1, P_2, \ldots, P_n \rangle$, where G is a countably presented infinite group, $\mathfrak{A}_G$ is as in Proposition 2.1 and $P_1, P_2, \ldots, P_n$ are unary predicates on $\mathfrak{A}_G$.

EXAMPLE. Let G be $Z \oplus Z$, then the dominos of R. Robinson [20] can be thought of as structures of the described type.

PROPOSITION 2.8. $\mathrm{Th}(\langle \mathfrak{A}_G, P_1, P_2, \ldots, P_n \rangle)$ *is model complete.*

Proof. Look at the reducts to models of T_G.

PROPOSITION 2.9. $\mathrm{Th}(\langle \mathfrak{A}_G, P_1, P_2, \ldots, P_n \rangle) = T$ *is superstable.*

Proof. Call a submodel (of a model of T) which is generated by one element a component of a model of T. Every component is countable and every model of T splits into disjoint components. There are at most continuum many non-isomorphic components of models of T, hence the result.

PROPOSITION 2.10. *If T from Proposition 2.9 is open (universal), then the following are equivalent:*

(i) *T is $\aleph_1$-categorical,*

(ii) *T is ω-stable (= totally transcendental).*

Proof. (i) $\Rightarrow$ (ii) holds generally by Morley's theorem [16]. To prove (ii) $\Rightarrow$ (i) we observe first that all components are locally isomorphic (i.e. every finite part of it may be isomorphically imbedded in any other component). This follows from Ehrenfeucht's characterization of elementary equivalence [6] and the fact that every substructure (= component) of a model of an open theory is itself a model.

Now assume T is not $\aleph_1$-categorical. Then there are two components $\mathfrak{A}$, $\mathfrak{B}$ in some model of T which are not isomorphic. But by Proposition 2.8 $\mathfrak{A}$ and $\mathfrak{B}$ are elementary submodels, and since they are generated by one element, they are minimal. So T has no prime model, which contradicts ω-stability.

COROLLARY 2.11. *T has either continuum many isomorphism types of countable models or T is $\aleph_1$-categorical.*

Proof. By Ehrenfeucht's theorem [7] and the fact T has no prime model unless it is $\aleph_1$-categorical.

COROLLARY 2.12. *T has either only one or continuum many isomorphism types of components.*

Proof. By the same theorem of Ehrenfeucht, if T has no prime model then for some $n \in \omega$, T has continuum many n-types, hence continuum many non-isomorphic components.

COROLLARY 2.13. *In T no linear order is definable.*

Proof. By a Theorem of Shelah [22], if T admits a definable linear ordering, T is unstable, which contradicts Proposition 2.9.

Application. We now take $G = Z \oplus Z$ and interpret it as a domino in the sense of R. Robinson [20]. Robinson proved (private communication) the following theorem, which relies on a domino set similar to the domino in [20]:

THEOREM 2.14. *There is a finite set of domino axioms A such that*

(i) *A has only non periodic solutions (= components),*

(ii) *two solutions are locally isomorphic and*

(iii) *there are continuum many non isomorphic solutions.*

THEOREM 2.15. *There is a complete theory T such that*

(i) *T is finitely axiomatizable,*

(ii) *T is super-stable and*

(iii) *for every $n \in \omega$ $F_n(T)$ is atomless.*

Proof. Take any solution of the domino set in Theorem 2.14 and interpret it as a structure as in Proposition 2.8. Let us denote this structure with $\mathfrak{A}_{\mathrm{Rob}}$, and $T = \mathrm{Th}(\mathfrak{A}_{\mathrm{Rob}})$. T is superstable by Proposition 2.9. Let T_0 consist of the axioms asserting, that $\tilde{f}, \tilde{f}^{-1}, \tilde{g}, \tilde{g}^{-1}$ are functions and relating these functions with the domino predicates as prescribed through the domino set. Clearly T_0 is finite. Using Ehrenfeucht's characterization of elementary equivalence T_0 is complete by Theorem 2.14 (ii) and the fact that T_0 has no finite models by 2.14 (i). Hence T is finitely axiomatizable. Since T_0 is open, by 2.14 (ii) and (iii), T has continuum many minimal models, hence for all $n \in \omega$ $F_n(T)$ is atomless.

Remark. In Theorem 2.14, (iii) already follows from (i) and (ii) if the domino set has, up to isomorphism, more then one solution, just by Corollary 2.12.

If we restrict a minimal model of T to one of the domino predicates, say P_1, one sees easily, by Criterion C, that the complete theory of the resulting structure is not finitely axiomatizable. So we have

COROLLARY 2.16. *T of Theorem 2.15 is a counterexample to criterion* D.

3. Categoricity and non finite axiomatizability.

DEFINITIONS. A complete theory T is *almost strongly minimal* if there is a principal extension T' of T with a strongly minimal formula φ such that for every model $\mathfrak{A}$ of T' we have $\mathfrak{A} = \mathrm{cl}(\varphi(\mathfrak{A}))$. A theory T (not necessarily complete) has the *weak intersection property* (w.i.p.) if for every model $\mathfrak{A}$ of T and for every two submodels $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathfrak{A}$, $\mathfrak{B} \cap \mathfrak{C}$ is either finite or a model of T.

Almost strongly minimal theories have been introduced by Baldwin [1]. They form a special class of $\aleph_1$-categorical theories. In [14] we proved the following:

THEOREM 3.1. *Let T be a complete $\aleph_1$-categorical theory. Then the following are equivalent:*

(i) *T is almost strongly minimal;*

(ii) *some principal extension $T^{*\prime}$ of T^* has the* w.i.p.

T^* denotes the full expansion of T (cf. Bell-Slomson [4]).

Let T be a complete almost strongly minimal $\aleph_0$-categorical theory and let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be models of T. From the results of Makowsky [12] (cf. Dickmann [5]) there is a function $f\colon \omega \to \omega$ such that (i) f is monotone almost everywhere on ω, (ii) for $\mathfrak{A} < \mathfrak{C}$, $\mathfrak{B} < \mathfrak{C}$ and $|\mathfrak{A} \cap \mathfrak{B}| = n$ then $\mathfrak{A} \cap \mathfrak{B} \equiv_{f(n)} \mathfrak{C}$. So using Criterion C of § 1 we get:

THEOREM 3.2. *If T is complete, categorical in every infinite power and has* w.i.p., *then T is not finitely axiomatizable.*

or

THEOREM 3.3. *If T is complete, almost strongly minimal and $\aleph_0$-categorical, then T is not finitely axiomatizable.*

By a result of Baldwin (private communication) not every $\aleph_1$-categorical theory is almost strongly minimal, even if T is $\aleph_0$-categorical, but there is no obvious way of generalizing the methods of [12]. Analyzing the proof, one sees easily that it focuses around algebraic properties of the models of almost strongly minimal theories, from which we do not know if all models of theories categorical in every infinite power share them.

DEFINITION. T has the *unbounded weak intersection property* (u.w.i.p.) if T has the w.i.p. and for every $n \in \omega$ there are models $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ of T such that $\mathfrak{A} \subset \mathfrak{C}$, $\mathfrak{B} \subset \mathfrak{C}$ and $\mathfrak{A} \cap \mathfrak{B}$ is finite but of cardinality $> n$.

THEOREM 3.4. *If T is semi-complete and has the* u.w.i.p. *then* $\mathrm{Th}(\mathfrak{A})$ *is not finitely axiomatizable for infinite models $\mathfrak{A}$ of T.*

Proof. Choose $\mathfrak{A}_n, \mathfrak{B}_n \subset \mathfrak{C}_n$, all models of T, such that $f(n) = \mathrm{card}(\mathfrak{A}_n \cap \mathfrak{B}_n)$ is finite and f is an increasing function of n. (This is possible because T has u.w.i.p.). Let F be an ultrafilter on ω which is not principal. Now set $\mathfrak{A} = \Pi\mathfrak{A}_n/F$, $\mathfrak{B} = \Pi\mathfrak{B}_n/F$ and $\mathfrak{C} = \Pi\mathfrak{C}_n/F$. We have $\mathfrak{A} \cap \mathfrak{B} = \Pi\mathfrak{A}_n \cap \mathfrak{B}_n/F$ and $\mathfrak{A} \cap \mathfrak{B}$ is infinite, so $\mathfrak{A} \cap \mathfrak{B} \equiv \mathfrak{C}$, since T is semi-complete. Now the result follows by Criterion B.

LEMMA 3.5. *If T is categorical in every infinite power and has the* w.i.p. *then T has the* u.w.i.p.

Proof. Let $(x_n)_{n\in\omega}$ be a basis for the strongly minimal set of some countable model $\mathfrak{A}$ of T. Put $X_n = \{x_k \mid k < n\}$. From the proof of Theorem 3.2 of [14] we know that $\mathrm{cl}(X_n) = \mathrm{cl}(X_n \cup \{x_{2k} \mid k > n\}) \cap \mathrm{cl}(X_n \cup \{x_{2k+1} \mid k > n\})$ where $\mathrm{card}(\mathrm{cl}(X_n)) > n$ but finite by Ryll-Nardzewski's theorem (cf. Shoenfield [24]).

Lemma 3.5 gives us new proofs of Theorems 3.2 and 3.3. For Theorem 3.3 one has to consider the reducts of the models of the full expansion of T to the original similarity type.

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The boundedness principle in ordinal recursion

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Abstract. An application of Spector's boundedness principle to ordinal recursion yields, for α recursively regular and $\beta < \alpha$: (1) Any α-recursive functional total on $\beta \times P(\beta)$ can be defined without the search operator. Let $|\alpha/\beta|$ be the closure ordinal of the class of α-recursive inductive operators over β. For example, an operator over ω is ω_1-recursive iff it is Δ_1^1. A new proof of the fact that $|\Delta_1^1|$ is recursively singular follows from the more general result (2) $|\alpha/\beta| > \alpha$ iff $|\alpha/\beta|$ is singular. Characterizations of closure ordinals are obtained in terms of projectibility. For example, (3) $|\alpha/\beta| \geqslant \alpha$ iff α is absolutely projectible to β.

1. Introduction. The boundedness principle, due to Spector [6], is basically that any Σ_1^1 set of well-orderings is bounded below ω_1. We apply this principle to ordinal recursion to obtain several results regarding functionals and specifically inductive definitions on sets of ordinals.

We assume that the reader is familiar with the concepts of inductive definitions and of ordinal recursion as outlined in [1].

Briefly, an inductive operator Γ over a set X is a map from $P(X)$ to $P(X)$ such that for all A, $A \subseteq \Gamma(A)$. Γ determines a transfinite sequence $\{\Gamma^\xi\colon\ \xi \in \mathrm{ORD}\,(\text{ordinals})\}$, where $\Gamma^\xi = U\{\Gamma^\sigma\colon\ \sigma < \xi\}$ for $\xi = 0$ or ξ a limit ordinal and $\Gamma^{\xi+1} = \Gamma(\Gamma^\xi)$. The closure ordinal $|\Gamma|$ of Γ is the least ordinal ξ such that $\Gamma^{\xi+1} = \Gamma^\xi$. The closure $\overline{\Gamma}$ of Γ is $\Gamma^{|\Gamma|}$, the set inductively defined by Γ.

The definition of the α-recursive functionals and the primitive ordinal recursive (p.o.r.) functionals is a natural generalization of standard recursion over the natural numbers. We list in § 3 some basic facts about ordinal recursion essential to this paper.

In this paper we consider the notion or ordinal recursive inductive operators on sets of ordinals. Given recursively regular ordinals α and β, $\beta \leqslant \alpha$, let $|\alpha/\beta|$ be the closure ordinal of the class of inductive operators over β which are α-recursive in parameters less than β; let $|(\alpha/\beta)|$ be the closure ordinal of inductive operators over β which are α-recursive in parameters less than α. (The latter are called weakly α-recursive.) For example, $|\omega/\omega| = |(\omega/\omega)| = \omega$. In this paper we consider only countable α and β.