Proof. (i) For $n \in \omega$, set

$$b_n = \{\langle n\rangle\} \cup \{p_n \mid p \text{ has the property stated in the definition of } \varphi \text{ with respect to } x\}.$$

Then each b_n has points at every level $< \omega_1^L$, so it remains to prove that b_n is linearly ordered. Obviously, $x_n \leqslant_n y$ for $n \in \omega$ and $y \in b_n$.

So assume that p and p' have the properties given in φ with respect to x, and suppose that $p_n | p'_n$ for some $n \in \omega$. We seek a contradiction.

First we prove that $p_m | p'_m$ for $m \geqslant n$. If not, let $m \geqslant n$ be such that $p_m | p'_m$ but $p_{m+1} \leqslant_{m+1} p'_{m+1}$. Let z be the largest $z \leqslant_m p_m, p'_m$, and let $z' <_{m+1} p_{m+1}$, $|z'| = |z|$. (Notice that $|z| \geqslant 1$.) By our assumptions about p and p', $f(z') \leqslant_m p_m, p'_m$. But this is impossible, since $f(z') >_m z$.

So let z_m be the largest $z \leqslant_m p_m, p'_m$ for $m \geqslant n$. By the same argument we must have

$$|z_n| > |z_{n+1}| > |z_{n+2}| > \ldots,$$

which is impossible.

(ii) If $x \neq y$, then $x_n \neq y_n$ for some n. But then, by (i), T_n will contain two different ω_1^L-branches, which is impossible by Claim 4. Q.E.D.

Now, (a) in the theorem is trivially satisfied by φ. From (i) in Claim 9 we obtain $\exists x \varphi(x) \to V \neq L$, which is equivalent to (b). (c) is exactly (ii) in Claim 9. In $M[a]$, $(*)$ holds, so (d) is clear. (GCH is implied by $V = L^a$.) (e) is clear from the absoluteness in the construction (or simply by Shoenfield's absoluteness theorem). The proof is complete.

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Banach spaces and large cardinals

by

Jussi Ketonen (Berkeley, Cal.)

Abstract. The purpose of this paper is to introduce a new type of a basis-notion; sets of indiscernibles, for Banach spaces. A structural theory for Banach spaces generated by sets of indiscernibles is developed. It is shown that any Banach space of the cardinality of a Ramsey cardinal has a set of indiscernibles of the same cardinality and that consequently it has a big subspace admitting non-trivial projections. The behaviour of linear operators on spaces of large cardinality is also studied.

0. Introduction and notation. Our intent is to study the applications of the theory of large cardinals to Banach spaces. The cardinals we choose to work with, Ramsey cardinals, are of a fairly high order. It is shown that the notion of sets of indiscernibles, which usually arises in the theory of Ramsey cardinals, has a natural interpretation in the context of Banach spaces. Chapter 1 is devoted to the study of the structural theory of Banach spaces generated by sets of indiscernibles. No large cardinality assumptions are needed here except that we do require the density character of the spaces in question to be uncountable. It seems from the many counterexamples one can construct that the countable case has very little coherence. In the remainder of this paper we then invoke large cardinality assumptions in order to get sets of indiscernibles; the general idea behind all of our proofs being that every big enough Banach space has a big, fairly homogeneous, subspace.

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The notation and terminology conforms to that used in [1] and [2]. For example, cardinals are initial ordinals. Ordinals are denoted by small Greek letters $\alpha, \beta, \ldots$ The cardinality of the set X is denoted by $|X|$. The finite linear span of the set X (if it makes sense) is denoted by $[X]$. Operator always means bounded linear operator.

0.1. DEFINITION. A cardinal $\varkappa$ is *Ramsey* if

(1) $\varkappa$ is regular,

(2) if $[\varkappa]^{<\omega}$ denotes the set of finite subsets of $\varkappa$, and $f\colon [\varkappa]^{<\omega}\to\lambda$, where $\lambda<\varkappa$, then the function f has a homogeneous set of cardinality $\varkappa$; i.e., if $S_1, S_2\subseteq X$ and S_1, S_2 have the same finite ordertype, then $f(S_1)=f(S_2)$.

It is well-known that every Ramsey cardinal is inaccessible and that every measurable cardinal is Ramsey. For more, see J. Silver [1].

0.2. DEFINITION. Let B be a Banach space. A set $X\subseteq B$ is a SOI (*set of indiscernibles*) in B if $X=\{x_\alpha|\ \alpha<\varkappa\}$ (indexed by a cardinal $\varkappa$) is a linearly independent family so that for every finite set $F\subseteq\varkappa$, order-preserving map $\pi\colon F\to\varkappa$, scalars a_α $(\alpha\in F)$

$$\Big\|\sum_{\alpha\in F}a_\alpha x_\alpha\Big\|=\Big\|\sum_{\alpha\in F}a_\alpha x_{\pi(\alpha)}\Big\|$$

and

$$\|x\|_\alpha=1\qquad(\alpha<\varkappa)\,.$$

X is a GSOI (*generating set of indiscernibles*) if $[X]$ is dense in B.

0.3. DEFINITION. Let B be a Banach space. A family $X=\{x_\alpha|\ \alpha<\varkappa\}$ is an *unconditional basis-set* for B if there is a constant C so that

$$\Big\|\sum\varepsilon_\alpha c_\alpha x_\alpha\Big\|\leqslant C\Big\|\sum c_\alpha x_\alpha\Big\|$$

for all (ε_α) with $\varepsilon_\alpha=\pm1$ and all (c_α) where c_α is a scalar with $c_\alpha\neq0$ for only finitely many α's.

The following simple proposition gives our fundamental observation.

0.4. PROPOSITION. *If $\varkappa$ is a Ramsey cardinal, B a Banach space of cardinality $\varkappa$, then B contains a SOI of cardinality $\varkappa$. As a matter of fact, every set of norm-1 elements of B of power $\varkappa$ has a subset of power $\varkappa$ which is a SOI for B.*

Proof. Let $\{b_\alpha|\ \alpha<\varkappa\}$ be a collection of distinct, norm-1 elements of B. For each set $F\in[\varkappa]^{<\omega}$ define a norm $\|\cdot\|_F$ on the Euclidean space $S^{|F|}$ (S denotes the field of scalars) by setting

$$\|\langle a_1\dots a_n\rangle\|_F=\Big\|\sum_{i\leqslant n}a_{\alpha_i}b_{\alpha_i}\Big\|,$$

where $F=\{\alpha_1\dots\alpha_n\}$ and $\alpha_1<\alpha_2<\dots<\alpha_n$. Define a function $G\colon[\varkappa]^{<\omega}\to\bigcup_{n<\omega}S^{S^n}$ by $G(F)=\|\cdot\|_F$. Let X be a homogeneous set of cardinality $\varkappa$ for G. Then $\{b_\alpha|\ \alpha\in X\}$ is the required SOI for B. ∎

1. Sets of indiscernibles in Banach spaces. In the following, fix a Banach space B, a set of indiscernibles $X=\{x_\alpha|\ \alpha<\varkappa\}$ for B, where $\varkappa$ is a regular cardinal $\geqslant\omega_1$, and set $D=[X]$. We can then define linear functions $k_\alpha(x)=\hat{x}(\alpha)$ $(\alpha<\varkappa)$ for $x\in D$ so that

$$x=\sum_{\alpha<\varkappa}\hat{x}(\alpha)\cdot x_\alpha\qquad(x\in D)\,.$$

Define the support of x for $x\in D$ to be the set

$$\operatorname{Supp}(x)=\{\alpha|\ \hat{x}(\alpha)\neq0\}\,.$$

Given an order-preserving map $\pi\colon\operatorname{Supp}(x)\to\varkappa$, define

$$\pi^*(x)=\sum_{\alpha<\varkappa}\hat{x}(\alpha)\cdot x_{\pi(\alpha)}.$$

Thus, if $\pi\colon\varkappa\to\varkappa$ is order-preserving, π^* is an isometry and hence can be extended in an unique fashion to $\overline{[X]}$. Define projections p_A for subsets $A\subseteq\varkappa$ by

$$p_A(x)=\sum_{\alpha\in A}\hat{x}(\alpha)\cdot x\,.$$

The above notation will remain fixed throughout this paper. Also, let $q_\alpha x=x-p_\alpha x$ and

$$C_A=\overline{[\{x_\alpha|\ \alpha\in A\}]}\qquad(A\subseteq\varkappa)\,.$$

1.1. PROPOSITION. *If X is a GSOI for B, then no proper subset of X generates B; for any $\alpha<\varkappa$*

$$x_\alpha\notin C_{\varkappa-\{\alpha\}}\,.$$

Proof. By way of contradiction, assume that $\alpha<\varkappa$ and $x_\alpha\in C_{\varkappa-\{\alpha\}}$. Pick $\beta>\alpha$, $\varepsilon>0$. Let $x\in C_{\varkappa-\{\alpha\}}\cap D$ so that

$$\|x-x_\alpha\|<\varepsilon\,.$$

Let $\pi_i\colon\operatorname{Supp}(x)\cup\{\alpha\}\to\varkappa$ $(i=1,2)$ be order-preserving maps so that

$$\pi_1=\begin{cases}\text{id} & \text{on } \alpha\,,\\ \beta & \text{on } \{\alpha\}\end{cases}$$

and

$$\pi_2=\begin{cases}\text{id} & \text{on } \alpha\cup\{\alpha\}\,,\\ \pi_1 & \text{otherwise.}\end{cases}$$

Then

$$\|\pi_1(x)-x_\beta\|=\|\pi_1(x-x_\alpha)\|<\varepsilon$$

and

$$\|\pi_1(x)-x_\alpha\|=\|\pi_2(x-x_\alpha)\|<\varepsilon\,.$$

Hence, for any $\varepsilon > 0$

$$\|x_\alpha - x_\beta\| < 2\varepsilon$$

i.e. $x_\alpha = x_\beta$; a contradiction. ∎

As an easy corollary, we obtain:

1.2. PROPOSITION. *Each k_α $(\alpha < \varkappa)$ is a bounded linear functional and hence can be extended in a unique fashion to $\bar{D}$. The family*

$$\{\langle x_\gamma, k_\gamma\rangle \mid \gamma < \varkappa\}$$

forms a bounded biorthogonal system.

Hence the definition of the support of an element naturally extends to $\bar{D}$. We can also define the support of a linear functional T:

$$\mathrm{Supp}(T) = \{\alpha \mid Tx_\alpha \neq 0\}.$$

This definition leads us to the key technical concepts of this paper:

1.3. DEFINITION (1) A linear functional T is *restricted with respect to X* if there is a $\gamma < \varkappa$ so that

$$\mathrm{Supp}(T) \subseteq \gamma.$$

(2) A linear functional T is *finitely restricted with respect to X* if for every $\delta > 0$ the set

$$\{\alpha \mid |Tx_\alpha| \geq \delta\}$$

is finite.

Thus every finitely restricted functional has countable support.

1.4. PROPOSITION. (1) *If T is restricted and $x \in D$, $\gamma < \varkappa$, then*

$$|T(p_\gamma(x))| \leq \|T\| \cdot \|x\|.$$

(2) *If T is finitely restricted, $Y \subseteq \varkappa$ a set so that*

$$\alpha, \beta \in Y, \quad \alpha < \beta \to |(\alpha, \beta)| = \omega$$

then for any finite $F \subseteq Y$, $x \in C_Y$

$$|T(p_F(x))| \leq \|T\| \cdot \|x\|.$$

Proof. (1) Given the T, let $\eta < \varkappa$ so that $\mathrm{Supp}(T) \subseteq \eta$. Let π: $\varkappa \to \varkappa$ be an order-preserving map so that $\pi = \mathrm{id}$ on γ and π: $[\gamma, \varkappa) \to [\eta+1, \varkappa)$. Then for any $x \in D$

$$|T(p_\gamma(x))| = |T(\pi^*(x))| \leq \|T\| \cdot \|\pi^*(x)\| = \|T\| \cdot \|x\|.$$

(2) Fix T, F and $x \in C_Y \cap D$. Let $n = |\mathrm{Supp}(x)|$ and $\varepsilon > 0$. Since T is finitely restricted, we can find an order-preserving map π: $\mathrm{Supp}(x) \to \varkappa$ so that $\pi = \mathrm{id}$ on $F \cap \mathrm{Supp}(x)$ and for any $\alpha \in \mathrm{Supp}(x) \cap (-F)$

$$|Tx_{\pi(\alpha)}| < \varepsilon.$$

Then

$$|T(p_F(x))| \leq |T(\pi^*(x))| + \varepsilon \cdot n \leq \|T\| \cdot \|x\| + \varepsilon \cdot n.$$

Letting $\varepsilon \to 0$ we get the claim. ∎

It turns out to be useful to subdivide SOI's into two different types: singular and non-singular sets of indiscernibles according to whether or not there are non-restricted linear functionals. The importance of this distinction is borne out by the following result:

1.5. THEOREM. *There is a non-restricted linear functional if and only if there is a non-finitely restricted linear functional if and only if there is a $M < \infty$ so that for any $x \in D$*

$$\Big|\sum_{\alpha<\varkappa} \hat{x}(\alpha)\Big| \leq M \cdot \|x\|.$$

Thus, the functional

$$\Pi(x) = \sum_{\alpha<\varkappa} \hat{x}(\alpha)$$

can be in this case extended from D to a continuous linear functional with support $= \varkappa$.

Proof. Suppose that the functional T is not finitely restricted. Then there is a scalar $t \neq 0$ so that for any $\varepsilon > 0$ the set

$$A_\varepsilon = \{\alpha \mid |Tx_\alpha - t| < \varepsilon\}$$

is infinite. Given any $x \in D$, $\varepsilon > 0$, let π: $\mathrm{Supp}(x) \to A_\varepsilon$ be a order-preserving map. Then

$$\|x\| \cdot \|T\| = \|\pi^*(x)\| \cdot \|T\| \geq |T(\pi^*(x))| \geq \Big|t \cdot \Big(\sum_{\alpha<\varkappa} \hat{x}(\alpha)\Big)\Big| - \varepsilon \cdot n,$$

where $n = |\mathrm{Supp}(x)|$. Letting $\varepsilon \to 0$ we get the claim with $M = \|T\|/|t|$. ∎

The singular case can be subdivided further: From now on till the end of this chapter we will assume that X is a GSOI for B.

1.6. THEOREM. *Suppose that X is a singular* SOI *(i.e. there are non-restricted linear functionals). Then either $B \cong l^1(\varkappa)$ (i.e. there is a continuous one-to-one onto linear operator from B onto $l^1(\varkappa)$) or for every $T \in B^*$ there is a (unique) scalar λ_T so that $T - \lambda_T \Pi$ is restricted.*

It is easy to see that the map $T \mapsto \lambda_T$ is a continuous linear functional of norm 1 on B^*.

Proof. For assume that there exist scalars $u \neq t$, $t \neq 0$ so that for every $\varepsilon > 0$ the sets

$$P_\varepsilon = \{\alpha |\ |Tx_\alpha - t| < \varepsilon\} \quad \text{and} \quad Q_\varepsilon = \{\alpha |\ |Tx_\alpha - u| < \varepsilon\}$$

have cardinality $\varkappa$. Given $x \in D$, $\varepsilon > 0$, and a finite subset $F \subseteq \varkappa$, let $H = \mathrm{Supp}(x)$ and $\pi\colon H \to \varkappa$ order-preserving so that

$$\pi\colon H \cap F \to P_\varepsilon, \qquad \pi\colon H - F \to Q_\varepsilon .$$

Then, if $N = |H|$,

$$\|T\| \cdot \|x\| = \|T\| \cdot \|\pi^*(x)\|$$
$$\geqslant \Big| \Big(\sum_{\alpha \in F} \hat{x}(\alpha)\Big) + \Big(\sum_{\alpha \notin F} \hat{x}(\alpha)\Big) \cdot u \Big| - \varepsilon \cdot N .$$

Letting $\varepsilon \to 0$ we find: There are constants $\lambda \neq 1$, $M < \infty$ so that for any $x \in D$, $F \subseteq \varkappa$-finite

$$\Big| \Big(\sum_{\alpha \in F} \hat{x}(\alpha)\Big) + \Big(\sum_{\alpha \notin F} \hat{x}(\alpha)\Big) \cdot \lambda \Big| \leqslant M \cdot \|x\|$$

and

$$\Big| \sum_{\alpha < \varkappa} \hat{x}(\alpha) \Big| \leqslant M \cdot \|x\| .$$

Applying these two inequalities with $F = \{\alpha |\ \hat{x}(\alpha) > 0\}$ we get: There is a constant C so that for any $x \in D$

$$\sum_{\alpha < \varkappa} |\hat{x}(\alpha)| \leqslant C \cdot \|x\| .$$

Hence $B \cong l^1(\varkappa)$. ∎

By Proposition 1.4 every projection p_α is norm-1 continuous on B in the non-singular case.

1.7. Theorem. *If X is a non-singular SOI, X is an unconditional basis-set: For all scalars a_α $(\alpha < \varkappa)$, finite sets $F \subseteq G$*

$$\Big\| \sum_{\alpha \in F} a_\alpha x_\alpha \Big\| \leqslant \Big\| \sum_{\alpha \in G} a_\alpha x_\alpha \Big\| .$$

Proof. If X is non-singular, every linear functional is finitely restricted by Theorem 1.5. Given $x \in D$ and a finite $F \subseteq \varkappa$, select an order-preserving map $\pi\colon \mathrm{Supp}(x) \to \varkappa$ so that for $\alpha, \beta \in \mathrm{Supp}(x)$

$$\alpha < \beta \to \big|(\pi(\alpha), \pi(\beta))\big| = \omega .$$

Let $T \in B^*$ so that $\|T\| = 1$ and

$$\big|T\big(p_H(\pi^*(x))\big)\big| = \big\|p_H(\pi^*(x))\big\|$$

where $H = \pi(F)$. Then, by Proposition 1.4

$$\|p_F(x)\| = \big\|p_H(\pi^*(x))\big\| = \Big| T\big(p_H(\pi^*(x))\big) \Big|$$
$$\leqslant \|\pi^*(x)\| = \|x\| . \ ∎$$

Thus, in this case every $x \in B$ is uniquely determined by the sequence $\langle k_\alpha x |\ \alpha < \varkappa\rangle$ and for every $A \subseteq \varkappa$ the projection p_A is continuous. Consequently $C_A \cap C_B = \{0\}$ for $A, B \subseteq \varkappa$ so that $A \cap B = 0$. We shall now show that the above-mentioned facts almost hold in the singular case.

1.8. Proposition. *If X is a GSOI for B and every p_α $(\alpha < \varkappa)$ is continuous, then every element $x \in B$ is uniquely determined by the sequence $\langle k_\alpha x |\ \alpha < \varkappa\rangle$.*

Proof. First of all, since X is a SOI, there is a constant M so that $\|p_\alpha\| \leqslant M$ for every $\alpha < \varkappa$. Given $x \neq 0$, let α be the least ordinal so that $p_\alpha x \neq 0$. Then α must be a successor-ordinal: For if α is a limit, select $y \in D$ so that $\|y - x\| < \varepsilon$, where

$$\varepsilon = \|p_\alpha x\|/2M .$$

Then there is a $\beta < \alpha$ so that $p_\alpha y = p_\beta y$. Therefore

$$\|p_\alpha x\| = \|p_\alpha x - p_\beta x\| \leqslant \|p_\alpha x - p_\alpha y\| + \|p_\beta y - p_\beta x\| < 2M\varepsilon ,$$

a contradiction. Therefore there is a $\gamma < \alpha$ so that $\alpha = \gamma + 1$ and $p_\gamma x = 0$. Hence $k_\gamma x \neq 0$. ∎

1.9. Proposition. *If X is a GSOI for B with every p_α $(\alpha < \varkappa)$ discontinuous, then for every $T \in B^*$ there is a scalar λ_T so that $T - \lambda_T \Pi$ is finitely restricted.*

Proof. We can argue as in the proof of Theorem 1.5. If the claim were not true, then we can find a $T \in B^*$ of norm 1 so that there exist $\gamma < \varkappa$ and a scalar $t \neq 0$ so that every $\varepsilon > 0$ the set

$$A_\varepsilon = \{\alpha |\ |Tx_\alpha - t| < \varepsilon\}$$

is infinite and $\mathrm{Supp}(T) \subseteq \gamma$. Given any $x \in D$, $\varepsilon > 0$ we can therefore find an order-preserving map π on $\mathrm{Supp}(x)$ so that $\mathrm{Supp}(x) \cap \gamma$ is mapped into A_ε and π is otherwise the identity. It follows that the functional

$$\Pi_\gamma x = \sum_{\alpha < \gamma} \hat{x}(\alpha)$$

is a continuous linear functional on B. But this implies that p_γ is continuous: For if $x \in D$ and $T \in B^*$ has norm 1 so that $|T(p_\gamma(x))| = \|p_\gamma(x)\|$, by Proposition 1.4 (1),

$$\|p_\gamma(x)\| \leqslant |\lambda_T \Pi(p_\gamma(x))| + |(T-\lambda_T\Pi)(p_\gamma(x))| \leqslant |\Pi_\gamma x| + (1+\|\Pi\|)\cdot\|x\| . \blacksquare$$

1.10. THEOREM. *Let X be a GSOI for B.*

(1) *There is a $b \in B$ so that the following statements are equivalent for any $x \in B$*

(a) *there is a λ s.t. $x = \lambda b$,*

(b) *there are $A, B \subseteq \varkappa$, $A \cap B = 0$ s.t. $x \in C_A \cap C_B$,*

(c) *for every $\alpha < \varkappa$, $k_\alpha x = 0$,*

(d) *for every restricted T, $Tx = 0$,*

(e) *for every finitely restricted T, $Tx = 0$,*

(f) *if $\pi\colon \varkappa \to \varkappa$ is order-preserving, $\pi^*(x) = x$.*

(2) *If any of* (a)-(f) *hold for a $x \neq 0$, there is a λ so that the family $\{x_\alpha - \lambda x|\ \alpha < \varkappa\}$ (when normalized to 1) is a non-singular* SOI *in B whose span has codimension one.*

1.1. COROLLARY. *If B is reflexive, then $\{k_\alpha x|\ \alpha < \varkappa\}$ (or $\{k_\alpha x|\ \alpha < \varkappa\} \cup \{\Pi\}$ in the singular case) generates B^*. If X is a* SOI *in B, then there is a $b \in B$ so that $X - b = \{x - b|\ x \in X\}$ is a non-singular* SOI.

Proof of Corollary 1.1. Given X, there is a $b \in B$ and a sequence i_k $(k < \omega)$ of natural numbers so that $x_{i_k} \to b$ weakly. Therefore $k_\alpha b = 0$ for any $\alpha < \varkappa$. Theorem 1.9 now implies the claim. $\blacksquare$

Proof of Theorem 1.10. (1) First of all, it is easy to see that the all of the statements (b)-(e) imply (c) and that (f)$\to$(b). For example, suppose that (b) holds. Let $x \in C_A \cap C_B$ and $A \cap B = 0$. If f. ex. $\alpha \in A$, then $k_\alpha = 0$ on C_B and therefore $k_\alpha x = 0$.

We can (by Proposition 1.8) without a loss of generality assume that X has the properties stated in Proposition 1.9.

(e)$\to$(f): Let $M = \{x|$ If $T \in B^*$ is finitely restricted, $Tx = 0\}$. By Proposition 1.9 this space is at most one-dimensional. If $\pi\colon \varkappa \to \varkappa$ is order-preserving and $T \in B^*$ is finitely restricted, so is $T \circ \pi^*$. Therefore $\pi^*(x) \in M$ for every $x \in M$. Since π is an isometry, $\pi^*(x) = x$ for every $x \in M$.

(c)$\to$(e). Suppose that for every $\alpha < \varkappa$ $k_\alpha x = 0$. Let $\pi\colon \varkappa \to \varkappa$ be order-preserving so that for $\alpha < \beta$ $|(\pi(\alpha), \pi(\beta))| = \omega$. Let $y = \pi^*(x)$. Then $k_\alpha y = 0$ for every $\alpha < \varkappa$. Let T be finitely restricted and $\varepsilon > 0$. Pick $z \in C_F$ where $F \subset \pi(\varkappa)$ is finite so that $\|z - y\| < \varepsilon$. Then, by Proposition 1.4 we have:

$$|Tz| = |T(p_F(z) - p_F(y))| \leqslant \|T\|\cdot\|z - y\| < \|T\|\cdot\varepsilon .$$

Letting $\varepsilon \to 0$ we find $Ty = 0$. Therefore $\pi^*(x) \in M$. If $x \neq \pi^*(x)$, then $\pi^*(x) \neq \pi^*(\pi^*(x))$. But this is impossible by the preceding. Hence $x = \pi^*(x) \in M$. $\blacksquare$

(2) By part (f) of (1), $\pi^*(x) = x$ for every order-preserving π. Therefore $\{x_\alpha - x|\ \alpha < \varkappa\}$ is a SOI for B. Select x so that $\Pi x = 1$. Then we are back in the non-singular case, for otherwise there would exist a constant M so that for any $y \in D$

$$|\Pi y| \leqslant M\cdot\|y - (\Pi y)x\|$$

letting $y \to x$ we get a contradiction. $\blacksquare$

The following result shows that we can always transform a SOI into an unconditional basis-set.

1.12. THEOREM. *Suppose X is a* SOI *with $B \not\cong l^1(\varkappa)$. Let $\pi\colon \varkappa \to \varkappa$ be an order-preserving map so that $\pi(\alpha) > \alpha$ for every $\alpha < \varkappa$. Then $X_\pi = \{x_\alpha - x_{\pi(\alpha)}|\ \alpha < \varkappa\}$ (when normalized to 1) is a non-singular* SOI.

Proof. For it is obvious in this case that for every $T \in B^*$ there is $\alpha < \varkappa$ so that

$$\beta > \alpha \to Tx_\beta = Tx_\alpha \to T(x_\beta - x_{\pi(\beta)}) = 0 .$$

2. Some applications and examples. For the duration of this chapter, let B denote a Banach space of the cardinality of a Ramsey cardinal $\varkappa$. The following theorem is a direct consequence of the results of chapter 1.

2.1. THEOREM. (1) *B contains an unconditional basis-set of cardinality $\varkappa$.*

(2) *If B has no subspaces isomorphic to $l^1(\varkappa)$, then B contains an unconditional basis-set $\{x_\alpha|\ \alpha < \varkappa\}$ with the associated projections*

$$p_F\Big(\sum a_\alpha x_\alpha\Big) = \sum_{\alpha \in F} a_\alpha x_\alpha$$

of norm 1.

(3) *If B is uniformly convex, then every subset of B of cardinality $\varkappa$ contains a subset of cardinality $\varkappa$ which is an unconditional basis-set of the type described in part* (2).

Next we shall study linear operators in spaces of high cardinality.

2.2. THEOREM. *If T is a continuous linear operator on B so that $|T(B)| = \varkappa$, then there is a subspace $M \subseteq B$ of cardinality $\varkappa$ so that for some $\delta > 0$*

$$\|Tx\| \geqslant \delta\|x\| \quad \text{on } M .$$

Proof. Let $X = \{x_\alpha|\ \alpha < \varkappa\}$ be a SOI for B so that X is an unconditional basis-set and for each $\alpha < \varkappa$ $Tx_\alpha \neq 0$. For $\alpha < \varkappa$,

$$A_\alpha = \{\gamma|\ k_\gamma(Tx_\alpha) \neq 0\} .$$

Then we can find $Y \subseteq \varkappa$, $A \subseteq \varkappa$, $f\colon A \to R$, $\delta > 0$ so that $|Y| = \varkappa$ and for every $\alpha, \beta \in Y$, $\alpha < \beta$,

$$\|Tx_\alpha\| = \delta$$

and

$$k_\gamma(Tx_\alpha) = f(\gamma) \quad \text{for} \quad \gamma \in A\,.$$

and

$$A_\alpha \cap A_\beta = A\,.$$

If $A \neq 0$, let $\gamma \in A$. Then, for any finite $F \subseteq Y$, scalars a_α $(\alpha < \varkappa)$

$$\|T\| \cdot \Big\| \sum_{\alpha \in F} a_\alpha x_\alpha \Big\| \geqslant \Big| k_\gamma\Big(\sum_{\alpha \in F} a_\alpha T x_\alpha \Big) \Big| = |f(\gamma)| \cdot \Big| \sum_{\alpha \in F} a_\alpha \Big|\,,$$

a contradiction. Hence $A = 0$.

Thus, without a loss of generality we can assume that there is a $\delta_0 > 0$ so that for any $\alpha \in Y$ we can find a $\gamma_\alpha \in A_\alpha$ so that

$$k_{\gamma_\alpha}(Tx_\alpha) = \delta_0\,.$$

Let $S = \{\gamma_\alpha |\ \alpha < \varkappa\}$. Then for any $F \subseteq Y$ finite, scalars a_α

$$\Big\| \sum_{\alpha \in F} a_\alpha T x_\alpha \Big\| \geqslant \Big\| p_S\Big(\sum_{\alpha \in F} a_\alpha T x_\alpha \Big) \Big\| = \delta_0 \Big\| \sum_{\alpha \in F} a_\alpha x_\alpha \Big\|\,. \ ■$$

In the particular case of a Hilbert space this theorem reads:

2.3. THEOREM. *If $\varkappa$ is Ramsey and T a continuous linear operator on $l^2(\varkappa)$ so that $|T(l^2(\varkappa))| = \varkappa$, then there is a $\delta > 0$ and a set $Y \subset \varkappa$ of cardinality $\varkappa$ so that*

$$\|Tx\| = \delta \|x\| \quad \text{on } l^2(Y)\,.$$

Proof. For in this case the A's are disjoint and for finite $F \subseteq \varkappa$, scalars a_α

$$\Big\| \sum_{\alpha \in F} a_\alpha T x_\alpha \Big\|^2 = \sum_\gamma \Big(\sum_\alpha a_\alpha (Tx_\alpha)\gamma \Big)^2 = \sum_{\alpha \in F} a_\alpha^2 \sum_{\gamma \in A_\alpha} (Tx_\alpha)^2(\gamma)$$
$$= \delta^2 \Big(\sum_{\alpha \in F} a_\alpha^2 \Big)\,. \ ■$$

Actually, the above method of proof extends to show that Theorem 2.3 is true for every regular cardinal $\geqslant (2^\omega)^+$. It is trivially false for 2^ω.

An interesting situation arises when $B = L^1(\mu)$ where μ is a positive bounded measure and $|B| \geqslant \varkappa$ where $\varkappa$ is Ramsey. We can then show that every subspace of B of cardinality $\varkappa$ has an unconditional basis-set of cardinality $\varkappa$ with the corresponding projections of norm 1:

2.4. THEOREM. *If $\{f_\alpha |\ \alpha < \varkappa\}$ SOI in $L^1(\mu)$, then there is a $f \in L^1(\mu)$ s.t. $\{f_\alpha - f |\ \alpha < \varkappa\}$ (when normalized to one) is a non-singular* SOI.

Proof. For suppose that $\{f_\alpha |\ \alpha < \varkappa\}$ is a singular SOI. Therefore there is a $\varphi \in L^\infty(\mu)$ so that for every $\alpha < \varkappa$

$$\int f_\alpha \varphi \, d\mu = 1\,.$$

It is a result of Rosenthal ([1], p. 214, Rem. 2) that $L^1(\mu)$ cannot contain a subspace isomorphic to $l^1(\varkappa)$. Therefore for every $\psi \in L^\infty(\mu)$ there is a scalar λ_ψ so that there is a $\alpha_0 < \varkappa$ s.t.

$$\int f_\alpha \psi \, d\mu = \lambda_\psi \quad (\alpha > \alpha_0)\,.$$

The map $\psi \to \lambda_\psi$ is a linear functional on $L^\infty(\mu)$ of norm 1. By the Radon-Nikodymin theorem, we can find a function $f \in L^1(\mu)$ so that for every $\psi \in L^\infty(\mu)$

$$\lambda_\psi = \int f \psi \, d\mu\,.$$

Therefore for every $\psi \in L^\infty(\mu)$ there is a α_0 s.t.

$$\int (f_\alpha - f) \psi \, d\mu = 0 \quad (\alpha > \alpha_0)\,.$$

Hence, by Theorem 1.10 $\{f_\alpha - f |\ \alpha < \varkappa\}$ is a non-singular SOI for B. ■

It is perhaps worth noting that not every singular SOI in a Banach space is translatable into a non-singular one in the above fashion. For example, if we define a norm on a space generated by the sequence $\{x_\alpha |\ \alpha < \varkappa\}$ by

$$\Big\| \sum a_\alpha x_\alpha \Big\| = \sup_{\gamma < \varkappa} \Big| \sum_{\alpha < \gamma} a_\alpha \Big|$$

then every projection p_α is continuous.

3. Measurable cardinals. In the following, let $\varkappa$ denote a measurable cardinal and D a normal ultrafilter over $\varkappa$. (For definitions, and all the relevant facts and notation, see J. Silver [2]). Given a Banach space B, the ultrapower $\Pi_D B$ can be easily endowed with a Banach space structure by setting for $f\colon \varkappa \to B$

$$\|[f]_D\| = \lim_D f(\alpha) = \text{the real } r$$

so that

$$\{\alpha |\ f(\alpha) = r\} \in D\,.$$

For $T \in B^*$, let $\tilde{T}$ denote the canonical extension of T into $\Pi_D B$: For $f\colon \varkappa \to B$, define

$$\tilde{T}([f]_D) = \lim_D T f(\alpha)\,.$$

Note that by a theorem of Rowbottom, for any one-to-one function $f\colon \varkappa \to B$ there is a $X \in D$ so that $\{f(\alpha) |\ \alpha \in X\}$ is a SOI. Now let C denote the set of all constant-elements of $\Pi_D B$ and $NS(S)$ denote the set of all non-constant elements of $\Pi_D B$ which yield a non-singular (singular) SOI. Then

$$\Pi_D B = C \cup NS \cup S .$$

The following result is then obvious:

3.1. Proposition. (1) $NS = \{x \in \Pi_D B|\ \textit{for every}\ T \in B^*,\ \tilde{T}x = 0\}$.

(2) *If B is a Banach space of cardinality $\geqslant \varkappa$ so that $|B^*| < 2^\varkappa$, then* $\dim(NS) = 2^\varkappa$.

An interesting situation arises when $\{x_a|\ a < \varkappa\}$ is a GSOI for the Banach space B. We can construct a subspace M of the space $L(B)$ of all continuous linear operators by

$$M = \{T \in L(B)|\ \{a|\ Tx_a = 0\} \in D\} .$$

The resulting quotient space

$$B_D \overset{\text{def}}{=} L(B)/M$$

is then a Banach space. In the case of a Hilbert space, the arguments of Theorem 2.3 imply that

$$l^2(\varkappa)_D \cong l^2(2^\varkappa) .$$

In the general case we can define an embedding into the ultrapower

$$\varphi\colon B_D \xrightarrow{1-1} \Pi_D B .$$

by setting

$$\varphi([T]) = [(Tx_a|\ a < \varkappa)]_D .$$

Obviously, $\|\varphi\| \leqslant 1$. It is also easy to see that $\varphi''(B_D) \subseteq L$ where

$$L = \{[f]_D|\ \exists X \in D \text{ s.t. } \operatorname{supp}(f(x_a)) \text{ disjointed } (a \in X)\} .$$

Actually, in the non-singular case it is easy to see that

$$\Pi_D B = C \oplus L$$

where the associated projections have norm 1.

3.2. Proposition. (1) *$\varphi''(B_D)$ is dense in L.*

(2) *The following are equivalent:*

(a) *$\varphi''(B_D)$ is closed,*

(b) *$\exists \delta > 0$ so that for any $T \in L(B)$*

$$\lim_D \|Tx_a\| \geqslant \delta \|T\| .$$

(c) $\varphi''(B_D) = L$.

(d) *There is a $\delta > 0$ so that if $x \in C_{\omega_1}$ and $\pi_a\colon \omega_1 \to \varkappa$ $(a < \varkappa)$ are order-preserving maps with disjoint ranges,*

$$\delta \Big\| \sum a_a \pi_a x \Big\| \leqslant \Big\| \sum a_a x_a \Big\| \leqslant \frac{1}{\delta} \Big\| \sum a_a \pi_a x \Big\| .$$

References

[1] H. Rosenthal, *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. 124 (1970).

[2] J. Silver, *Some applications of model theory in set theory*, Annals of Math. Logic 3 (1), pp. 45–110.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA AT BERKELEY

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