Infinitary equivalence of abelian groups

by

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Abstract. Necessary and sufficient conditions are given for an abelian group to be $L_{\infty\varkappa}$ equivalent to a free group or, more generally, to a direct sum of cyclic groups. As applications, examples are given of groups of cardinality ω_1 which are $L_{\infty\omega_1}$ equivalent but not isomorphic. Also, generalizations are proved of some theorems of abelian group theory about free groups and direct sums of cyclic groups.

0. Introduction. In this paper we use infinitary logic to generalize and put in perspective some theorems of abelian group theory viz., Pontryagin's criterion (1.3), and theorems of Baer (1.6), Prufer (2.5) and Kurosh (2.7); and, conversely, we make use of abelian group theory to exhibit some natural counterexamples in infinitary logic viz., counterexamples to a generalization of Scott's theorem to $L_{\infty\omega_1}$ (1.9) and (2.10).

Our main theorem from which the above results follow as corollaries is a necessary and sufficient condition for an abelian group to be $L_{\infty\varkappa}$-equivalent to a direct sum of cyclic groups (1.1) and (2.2). The principal tool of the proof is the "back-and-forth criterion" for $L_{\infty\varkappa}$-equivalence which was developed by Ehrenfeucht [6] and Fraissé [8] for finitary languages, by Karp [15] for $L_{\infty\omega}$ and by Benda [3] and Calais [4] for $L_{\infty\varkappa}$ for $\varkappa > \omega$. (Recall that $L_{\infty\varkappa}$ is a language whose formulas are those constructed from function and relation symbols by taking conjunctions over arbitrary sets of formulas; applying the negation symbol; and quantifying over sets of variables of cardinality $<\varkappa$. See, for example, [7], for more details.)

In order to state the back-and-forth criterion, let us first define a *partial isomorphism* from a structure $\mathfrak{A}$ to a structure $\mathfrak{B}$ to be an isomorphism $f\colon \mathfrak{A}' \to \mathfrak{B}'$ where $\mathfrak{A}'$ (resp. $\mathfrak{B}'$) is a substructure of $\mathfrak{A}$ (resp. $\mathfrak{B}$). A set $\mathfrak{J}$ of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is called a *$\varkappa$-extendable system* if whenever $f\colon \mathfrak{A}' \to \mathfrak{B}'$ is a member of $\mathfrak{J}$ and X (resp. Y) is a subset of the universe A of $\mathfrak{A}$ (resp. of $\mathfrak{B}$) then there exists $f'\colon \mathfrak{A}'' \to \mathfrak{B}''$ such that $f \subseteq f'$ and $X \subseteq A''$ (resp. $Y \subseteq B''$). Now we can state the back-and-forth criterion:

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$\mathfrak{A}$ is $L_{\infty\varkappa}$-equivalent to $\mathfrak{B}$ if and only if there is a $\varkappa$-extendable system of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

Then Scott's Theorem (in a weak form) says:

If $\mathfrak{A}$ and $\mathfrak{B}$ are countable and $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ [19].

1. Groups equivalent to free groups. Throughout this paper "group" will mean abelian group. A direct sum of α copies of a group A will be denoted $A^{(\alpha)}$. $Z(p^n)$ denotes the cyclic group of order p^n and Z denotes the infinite cyclic group. A group A is a direct sum of cyclic groups, denoted Σ-cyclic, if it is isomorphic to

$$\bigoplus_{p,n} Z(p^n)^{(\alpha_{p,n})} \oplus Z^{(\beta)}$$

for some cardinals $\alpha_{p,n}$ and β.

The symbol $\varkappa$ will always denote an infinite cardinal. A is $\varkappa$-generated if it is generated by fewer than $\varkappa$ elements. A is *$\varkappa$-Σ-cyclic* (resp. *$\varkappa$-free*) if every $\varkappa$-generated subgroup of A is Σ-cyclic (resp. free). Thus A is $\varkappa$-free if and only if it is $\varkappa$-Σ-cyclic and torsion-free. Note also that for uncountable $\varkappa$, A is $\varkappa$-generated if and only if A is of cardinality $<\varkappa$; thus for uncountable $\varkappa$, our definition agrees with the usual one (cf. [10], p. 94). For $\varkappa = \omega$, A is $\varkappa$-free if and only if A is torsion-free.

A subgroup S of A is called *$\varkappa$-pure* if for every subgroup G of A such that $S \subseteq G \subseteq A$ and G/S is $\varkappa$-generated, S is a direct summand of G (see [9], § 27). An ω-pure subgroup is called, simply, a *pure* subgroup; S is a pure subgroup of A if and only if for every $n < \omega$, $nS = (nA) \cap S$. Obviously, a direct summand of a $\varkappa$-pure subgroup is $\varkappa$-pure, and S is a $\varkappa$-pure subgroup of A for all $\varkappa$ if and only if S is a direct summand of A.

We are going to give a criterion for a group to be $L_{\infty\varkappa}$-equivalent to a Σ-cyclic group. We take up first the simpler case of a torsion-free group.

1.1. Theorem. *A is $L_{\infty\varkappa}$-equivalent to a free group if and only if every $\varkappa$-generated subgroup of A is contained in a free, $\varkappa$-pure subgroup of A.*

Proof. The necessity of the condition follows from the fact that the condition is expressible in $L_{\infty\varkappa}$ and true in a free group. (Alternately, one may use the fact that there is a $\varkappa$-extendable system $\mathfrak{I}$ of partial isomorphisms from A to a free group F; thus for any $\varkappa$-generated subgroup S of A there is an $f \in \mathfrak{I}$ which is an isomorphism between an extension S' of S and a $\varkappa$-generated direct summand of F; the $\varkappa$-extendability of $\mathfrak{I}$ insures that S' is $\varkappa$-pure in A). To prove sufficiency, notice first that if A is $\varkappa$-generated, then A is free; hence any A satisfying the hypothesis is $\varkappa$-free. We may then assume that A is not $\varkappa$-generated; let $F =$ the free group of rank $\varkappa$. Let $\mathfrak{I} =$ the set of all partial isomorphisms $f\colon S \to T$ where S and T are $\varkappa$-generated, $\varkappa$-pure subgroups of A and F respectively. Given a subset X of A of cardinality $<\varkappa$, there is by hypothesis a free, $\varkappa$-pure extension S' of S containing S and X; S' can be taken to be $\varkappa$-generated. Then $S' = S \oplus S''$ for some free S'' because S is $\varkappa$-pure in A. Since F has rank $\geqslant \varkappa$, we can find a T'' isomorphic to S'' and such that $T \cap T'' = \{0\}$. Thus we can extend f to $f'\colon S \oplus S'' \to T \oplus T''$. Since we have used only properties common to A and F, we can also prove that we can extend f to include any given subset $Y \subseteq F$ of cardinality $<\varkappa$ in its range. Therefore $\mathfrak{I}$ is $\varkappa$-extendable and the theorem is proved.

When we take $\varkappa = \omega$ in the above theorem, we obtain the following three corollaries, all of which were previously known:

1.2. Corollary. *A group is $L_{\infty\omega}$-equivalent to a free group if and only if every subgroup of finite rank is free.*

Proof. For any ω-generated subgroup S of A, $\{a \in A\colon na \in S$, some $n < \omega\}$ is a pure subgroup of finite rank, hence free. Thus by the theorem, A is $L_{\infty\omega}$-equivalent to a free group. Conversely, if A is $L_{\infty\omega}$-equivalent to a free group and S is of finite rank let $\{a_1, \ldots, a_n\}$ be a maximal independent set of elements in S and let B be the subgroup generated by $\{a_1, \ldots, a_n\}$. By the theorem B is contained in a pure, free subgroup B'; then S is also contained in B', so S is free.

1.3. Corollary. (Pontryagin [17]). *A countable group is free if and only if every subgroup of finite rank is free.*

Proof. Apply Scott's theorem to Corollary 1.2.

The logical significance of Pontryagin's criterion has been observed in different forms by a number of persons, among them J. Barwise, E. Fisher, and G. Sabbagh.

1.4. Corollary (Kueker). *A group is $L_{\infty\omega}$-equivalent to a free group if and only if it is ω_1-free.*

Proof. If A is ω_1-free, then every subgroup of finite rank, being countable, is free; so A is $L_{\infty\omega}$-equivalent to a free group by 1.2. Conversely, if A is $L_{\infty\omega}$-equivalent to a free group, by 1.2 every subgroup of finite rank is free, so by 1.3 every countable subgroup is free.

Applying the theorem in the case $\varkappa = \omega_1$, to the Specker group Z^ω we obtain the following (Z^ω denotes the product of ω copies of Z).

1.5. Corollary. *Z^ω is not $L_{\infty\omega_1}$-equivalent to a free group.*

Proof. $Z^{(\omega)}$ is an ω_1-generated subgroup of Z^ω. If G is any countable extension of $Z^{(\omega)}$ in Z, there exists $a \in \prod_n (n!Z) \subseteq Z^\omega$ such that $a \notin G$. If H is a countable extension of G containing a and such that H/G is pure in Z^ω/G, then G is not a direct summand of H, because $0 \neq a + G$ is divisible by all integers in H/G; but Z^ω has no non-zero elements divisible by all integers. Hence H/G is not isomorphic to a subgroup of Z^{ω}. Thus we have proved that $Z^{(\omega)}$ has no ω_1-generated (i.e. countable) ω_1-pure extension, and the corollary is proved.

1.6. Corollary (Baer [1]). *Z^{ω} is not free.*

In order to apply the theorem to give non-trivial examples of groups which are $L_{\infty\varkappa}$-equivalent to free groups, we make use of known results in abelian group theory. For example, it is known that Z^{ω} is ω_1-free (Specker [20]; see also [10] Thm. 19.2). Hence by Corollaries 1.4 and 1.6 we have:

1.7. Corollary (Keisler-Kueker). *Z^{ω} is $L_{\infty\omega}$-equivalent to a free group. The class of free groups is not definable in $L_{\infty\omega}$.*

A is a *Fuchs* 5*-group* if every infinite subgroup of A is contained in a direct summand of A of the same cardinality (cf. [9], p. 96, Problem 5).

A direct sum of countable groups is a Fuchs 5-group. It was proved by Hill [13] that there is a p-group which is a Fuchs 5-group and is not a direct sum of countable groups. Then Griffith ([11]; see also [12], Thm 147) used the idea of Hill's proof to construct an example of a non-trivial torsion-free Fuchs 5-group; Griffith's proof shows the following:

1.8. Theorem (Griffith [11]). *There is a torsion-free group G of cardinality ω_1 which is a Fuchs 5-group and which is ω_1-free but not free.*

Now a Fuchs 5-group G obviously has the property that for any uncountable $\varkappa$, any $\varkappa$-generated subgroup of G is contained in a $\varkappa$-generated, $\varkappa$-pure (even a direct summand) subgroup of G. Hence as a corollary of 1.8 and 1.1 we have the following:

1.9. Corollary. *There is a group G of cardinality ω_1 which is $L_{\infty\omega_1}$-equivalent, but not isomorphic, to a free group of cardinality ω_1. The class of free groups is not definable in $L_{\infty\omega_1}$* ([1]).

The following corollary was suggested by E. Fisher.

1.10. Corollary. *If A is $L_{\infty\varkappa}$-equivalent to a free group and A is a $\varkappa$-pure subgroup of B such that $A \equiv_{\infty\varkappa} B$ then $A <_{\infty\varkappa} B$.*

Proof. We may assume Card$(A) \geqslant \varkappa$. It suffices to prove that for any $\varkappa$-generated subgroup A' of A, A' is contained in some $A'' \subseteq A$ such that the identity map $i_{A''}: A'' \to A''$ belongs to $\mathfrak{I}$, the system of partial isomorphisms defined in the proof of 1.1. But given A' we can take A'' to be any $\varkappa$-generated, $\varkappa$-pure subgroup of A containing A'; $i_{A''}$ is in $\mathfrak{I}$ because A'' is also $\varkappa$-pure in B ([9], p. 89).

(1) Recently Hill (*New criteria for freeness in abelian groups*, preprint) has shown that there is a non-free group A which is $L_{\infty\omega_2}$ equivalent to a free group. This proof does not determine whether A has cardinality $\aleph_2$ or $\aleph_3$.

2. Groups equivalent to Σ-cyclic groups. We turn now to the more general question of when an arbitrary abelian group A is $L_{\infty\varkappa}$-equivalent to a Σ-cyclic; this matter is more complicated in that there are very many different direct sums of cyclic groups to which A could be equivalent.

2.1. Lemma. (1) *If A is isomorphic to $\bigoplus_{p,n} Z(p^n)^{(\alpha_{p,n})} \oplus Z^{(\beta)}$ then the cardinals $\alpha_{p,n}$ and β are uniquely determined by A. In fact, β (resp. $\alpha_{p,n}$) $\geqslant \lambda$ if and only if A contains a subgroup (resp. pure subgroup) isomorphic to $Z^{(\lambda)}$ (resp. $Z(p^n)^{(\lambda)}$).*

(2) *Let A be any group and C one of the indecomposable cyclic groups $Z(p^n)$ or Z. Let $\lambda = \sup\{\gamma$: A contains a pure subgroup isomorphic to $C^{(\gamma)}\}$. If S is a pure subgroup of A, there is an extension $S' = S \oplus C^{(\gamma)}$, for some $\gamma \leqslant \lambda$, which is pure in A and which has no extension of the form $S' \oplus C$ which is pure in A.*

Proof. (1) follows from the fact that $\beta =$ the maximal number of independent torsion-free elements in A and $\alpha_{p,n} =$ the $(n-1)$st Ulm invariant of $A = \dim p^{n-1}A[p]/p^nA[p]$ (where $p^kA[p] = \{a \in A\colon pa = 0, a = p^kb$, some $b \in A\})$.

(2) If S does not have the property desired of S', then there exists an extension $S_1 = S \oplus C$ which is pure in A; we continue by induction: $S_{\nu+1} =$ a pure subgroup of A of the form $S_\nu \oplus C$ if there is one; otherwise $S_{\nu+1} = S_\nu = S'$. If σ is a limit ordinal, $S_\sigma = \bigcup_{\nu<\sigma} S_\nu$, which is pure in A ([10], p. 114 (f)). Thus S_ν is isomorphic to $S \oplus \bigoplus_{\mu<\nu} C$ and is pure in A. By the definition of λ we must have $S_{\nu+1} = S_\nu$ for some $\nu < \lambda^+$.

2.2. Theorem. *Let $\varkappa$ be an infinite cardinal. An abelian group A is $L_{\infty\varkappa}$-equivalent to a direct sum of cyclic groups if and only if every $\varkappa$-generated subgroup of A is contained in a $\varkappa$-pure subgroup of A which is Σ-cyclic.*

Proof. Note that the condition is equivalent to: A is $\varkappa$-Σ-cyclic and every $\varkappa$-generated subgroup of A is contained in a $\varkappa$-generated, $\varkappa$-pure subgroup of A. The necessity of the condition — as in 1.1 — follows from the fact that it is expressible in $L_{\infty\varkappa}$ and true in a Σ-cyclic, or from the existence of a $\varkappa$-extendable system $\mathfrak{I}$ from A to a Σ-cyclic. For sufficiency, let $\{C_n\colon n<\omega\}$ enumerate the indecomposable cyclics; let $L_n(A) = \{\gamma$: A contains a pure subgroup isomorphic to $C_n^{(\gamma)}\}$ and $\lambda_n(A) = \sup L_n(A)$. Let $\mathcal{E} = \{n < \omega\colon \lambda_n(A) < \varkappa\}$ and let

$$A' = \bigoplus_{n\in\mathcal{E}} C_n^{(\lambda_n(A))} \oplus \bigoplus_{n\notin\mathcal{E}} C_n^{(\varkappa)}.$$

Notice that A' satisfies the conditions stated in the theorem and that for each $n < \omega$ either $\lambda_n(A) = \lambda_n(A') = \lambda_n$ or $\lambda_n(A) \geqslant \varkappa$ and $\lambda_n(A') \geqslant \varkappa$. We shall use only these facts about A and A' to prove that $A \equiv_{\infty\varkappa} A'$. There are two cases.

Case 1. Cofinality of $\varkappa > \omega$. Then $\sum_{n\in\mathcal{E}} \lambda_n(A) < \varkappa$ and

$$B' \stackrel{\text{def}}{=} \sum_{n\in\mathcal{E}} C_n^{(\lambda_n(A))}$$

is $\varkappa$-generated. It follows from Lemma 2.1 (1) and (2) and the hypotheses of the theorem that A contains a subgroup B isomorphic to B' which is $\varkappa$-pure in A and has the property that for every $n \in \mathcal{E}$ there is no pure subgroup of A of the form $B \oplus C_n$. Now fix an isomorphism $g\colon B \to B'$ and let $\mathfrak{I}$ be the set of all partial isomorphisms $f\colon S \to S'$, extending g and such that S and S' are $\varkappa$-generated, $\varkappa$-pure subgroups of A and A' respectively. If X is a subset of A of cardinality $<\varkappa$, let T be a $\varkappa$-generated, $\varkappa$-pure subgroup of A which contains S and X. By the definition of B, T is of the form $S \oplus \bigoplus_{n\notin\mathcal{E}} C_n^{(\tau_n)}$ where $\tau_n < \varkappa$. One may prove that if S' is any $\varkappa$-generated subgroup of A' and $n \notin \mathcal{E}$, that there is an extension $S' \oplus C_n^{(\tau_n)}$ which is pure in A'. (We use the fact that $\lambda_n(A') \geqslant \varkappa > \tau_n$ and apply Lemma 2.3, following the proof.) Thus there is a pure subgroup of A' of the form $S' \oplus \bigoplus_{n\notin\mathcal{E}} C_n^{(\tau_n)}$. Let T'' be a $\varkappa$-generated $\varkappa$-pure subgroup of A' containing a subgroup of the form $S' \oplus \bigoplus_{n\notin\mathcal{E}} C_n^{(\tau_n)}$ which is pure in A'. Since T''/S' contains a pure subgroup isomorphic to $\bigoplus_{n\notin\mathcal{E}} C_n^{(\tau_n)}$ we see, using Lemma 2.1 (1), that T'' has a direct summand of the form $S' \oplus \bigoplus_{n\notin\mathcal{E}} C_n^{(\tau_n)}$. Hence $\mathfrak{I}$ is $\varkappa$-extendable.

Case 2. Cofinality of $\varkappa = \omega$. In this case we have to modify the argument if $\sum_{n\in\mathcal{E}}\lambda_n(A) = \varkappa$. We let $\mathfrak{I}$ = the set of all partial isomorphisms $f\colon S \to S'$ where S and S' are $\varkappa$-generated, $\varkappa$-pure subgroups of A and A' respectively such that for each $n \in \mathcal{E}$,

(*) if S contains a direct summand isomorphic to $C_n^{(\lambda_n)}$ then S (resp. S') has no extension of the form $S \oplus C_n$ (resp. $S' \oplus C_n$) which is pure in A (resp. A').

Let $f\colon S \to T$ be a member of $\mathfrak{I}$. If X is a subset of A of cardinality $<\varkappa$, let S_0 be a $\varkappa$-generated, $\varkappa$-pure subgroup of A which contains S and X; say S_0 is ϱ^+-generated where $\varrho < \varkappa$. Let $\mathcal{E}' = \{n \in \mathcal{E}\colon S_0$ contains a direct summand isomorphic to $C_n^{(\lambda_n)}\}$. Using Lemma 2.1 (2) we see that there is an extension of S_0 of the form $\widetilde{S}_0 = S_0 \oplus \bigoplus_{n\in\mathcal{E}'} C_n^{(\gamma_n)}$ which is ϱ^+-generated, pure in A and has no extension of the form $S_0 \oplus C_n$, $n \in \mathcal{E}'$, which is pure in A. By hypothesis there is an extension S_1 of $\widetilde{S}_0$ which is ϱ^+-generated, Σ-cyclic and $\varkappa$-pure in A. Moreover we may assume S_1 has property (*). (If necessary i.e. if for some $n \in \mathcal{E} - \mathcal{E}'$, $C_n^{(\lambda_n)}$ is a direct summand of S_1, repeat the construction above; after at most a countable number of repetitions one must obtain a ϱ^+-generated extension with the desired properties). Since S is $\varkappa$-pure we can write $S_1 = S \oplus S_2$, where S_2 is Σ-cyclic and such that for any summand C_n of S_2, $C_n^{(\lambda_n)}$ is not a direct summand of S. In order to show that f can be extended to S_1 we must prove that there exists $S_1' = S' \oplus S_2'$ which is $\varkappa$-pure in A', satisfies (*), and $S_2' \cong S_2$. Let

$$S' = \bigoplus_n C_n^{(\sigma_n)}, \qquad S_2 = \bigoplus_n C_n^{(\tau_n)}.$$

If $\tau_n > 0$ then $\sigma_n < \lambda_n$ by (*). For any n such that $\tau_n > 0$, let $\delta_n = \max\{\sigma_n + \tau_n, \sigma_n^+\}$. Then $\delta_n < \varkappa$, $\delta_n \leqslant \lambda_n$ so there exists a $\varkappa$-generated pure subgroup of A' isomorphic to $\bigoplus_{\tau_n\neq 0} C_n^{(\delta_n)}$. If S_1'' is a $\varkappa$-pure, $\varkappa$-generated subgroup of A' containing S' and $\bigoplus_{\tau_n\neq 0} C_n^{(\delta_n)}$, then by a cardinality argument, using Lemma 2.1 (1) S_1'' has a direct summand of the form

$$S_1' = S_1' \oplus \bigoplus_n C_n^{(\tau_n)}.$$

Therefore $\mathfrak{I}$ is $\varkappa$-extendable and the proof of the theorem is complete.

2.3. Lemma. *Let $\varkappa > \omega$ and let B be a $\varkappa$-generated pure subgroup of A and $C = \bigoplus_{\nu<\varkappa} C_\nu$ a pure subgroup of A. There exists $S \subseteq \varkappa$ of cardinality $\varkappa$ such that $B \cap \bigoplus_{\nu\in S} C_\nu = \{0\}$ and $B \oplus \bigoplus_{\nu\in S} C_\nu$ is pure in A.*

Proof. Let us fix $n \geqslant 0$ and $b \in B$ and consider $c \in C$ such that n divides $b+c$ in A but $n \nmid b$ and $n \nmid c$. Then $n \nmid c(\nu_0)$ for some $\nu_0 < \varkappa$. If c' is another element of C such that $n \mid b+c'$ then $n \mid c-c'$ so $n \nmid c'(\nu_0)$. Hence there exists $\nu_{n,b} \in \varkappa$ such that if $T = \varkappa - \{\nu_{n,b}\}$, then for $c \in \bigoplus_{\nu\in T} C_\nu$, $n \mid b+c \Leftrightarrow n \mid b$ and $n \mid c$. So we may take $S = \varkappa - \{\nu_{n,b}\colon n < \omega;\ b \in B\}$.

Theorem 1.11 has a series of corollaries parallel to those of Theorem 1.1.

2.4. Corollary (Barwise-Eklof [2]). *A p-group A is $L_{\infty\omega}$-equivalent to a direct sum of cyclic groups if and only if it has no (non-zero) elements of infinite height.*

Proof. In order to prove that a p-primary group without elements of infinite height is $L_{\infty\omega}$-equivalent to a Σ-cyclic, it suffices, by the theorem, to prove that every p-group of finite rank without elements of infinite height is finitely-generated, hence Σ-cyclic. To prove this first note that for any p-group G, if for each k we choose $\{x_{i,k}\colon i \in I_k\}$ such that $\{p^k x_{i,k}\}$ generate $p^kG[p]$, then by an induction on the order p^t of $g \in G$ one can prove that g is in the group generated by the $x_{i,k}$, $k < t$; therefore G is generated by $\bigcup_{k<\omega} \{x_{i,k}\colon i \in I_k\}$. If G is of finite rank, then clearly for each $k < \omega$, $p^kG[p]$ is finite, and so we may take I_k to be finite.

Moreover, since $p^kG[p] \supseteq p^{k+1}G[p]$ there is an m such that $p^mG[p] = p^{m+1}G[p]$ and since G has no elements of infinite height, $p^mG[p] = 0$. Therefore G is finitely-generated, in fact generated by $\bigcup_{k<m} \{x_{i,k}: i \in I_k\}$.

2.5. COROLLARY (Prufer [18]). *A countable p-group is a direct sum of cyclic groups if and only if it has no (non-zero) elements of infinite height.*

Proof. Apply Scott's Theorem to 2.4.

Remark. One may check that the proof of 2.2 is simpler for the case $\varkappa = \omega$ and uses only the fundamental theorem of (finitely-generated) abelian groups. Thus the proof of 2.5 depends only on this result.

Let T be the maximal torsion subgroup of $\prod_n Z(p^n)$. T is an uncountable group without elements of infinite height. In a manner similar to the proof of 1.5 we can prove:

2.6. COROLLARY. *T is not $L_{\infty\omega_1}$-equivalent to a direct sum of cyclic groups.*

2.7. COROLLARY. (Kurosh [16]). *T is not ad irect sum of cyclic groups.*

2.8. COROLLARY. *The class of direct sums of cyclic p-groups is not definable in $L_{\infty\omega}$.*

As we mentioned above, Hill proved that there was a non-trivial p-primary Fuchs 5-group. In fact he proved the following:

2.9. THEOREM. (Hill [13]). *There is a p-group G of cardinality ω_1 which is a Fuchs 5-group and which is ω_1-Σ-cyclic but is not Σ-cyclic.*

2.10. COROLLARY. *There is a p-group G of cardinality ω_1 which is $L_{\infty\omega_1}$-equivalent, but not isomorphic to a direct sum of cyclic groups of cardinality ω_1. The class of direct sums of cyclic p-groups is not definable in $L_{\infty\omega_1}$.*

We also have the following analogue of Corollary 1.10.

2.11. COROLLARY. *If A is $L_{\infty\omega}$-equivalent to a Σ-cyclic and A is a pure subgroup of A' such that $A \equiv_{\infty\omega} A'$ then $A <_{\infty\omega} A'$.*

Proof. It suffices to prove that any finitely-generated subgroup of A is contained in a pure finitely-generated subgroup S of A which satisfies (*) (see proof of 2.2) both with respect to A and with respect to A'. But in fact, for $\varkappa = \omega$, if S is any pure Σ-cyclic subgroup of A (resp. A') satisfies (*) with respect to A (resp. A').

Remarks. (1) Corollary 2.11 is false for $\varkappa > \omega$ as may be seen by the example $A = Z^{(\varkappa)} \oplus Z(p)^{(\omega)}$, $B = A \oplus Z(p)$.

(2) We do not know if for cardinals $\varkappa > \omega_1$ there are groups A of cardinality $\varkappa$ which satisfy the hypotheses of Theorem 2.2 but are not Σ-cyclic. (Of course, a necessary condition on $\varkappa$ is that it have cofinality $> \omega$, cf. [5], Prop. 5).

(3) The first counterexamples to Scott's Theorem for uncountable cardinals were given by Morley (unpublished, but see [5], p. 45).

(4) It may be argued that 2.10 (resp. 1.9) is evidence that there is no complete system of invariants for uncountable p-groups (resp. torsion-free groups) analogous to the Ulm invariants since one would expect such invariants for groups of cardinality ω_1 to be definable in $L_{\infty\omega_1}$. We hope to make this more precise in a later paper.

Added in proof. Since this paper was written the following developments have occurred. The author and P. Hill have proved, independently, that for each $n < \omega$ there is a non-free group of cardinality ω_n which is $L_{\infty\omega_n}$-equivalent to a free group. A. Mekler has proved that if $\varkappa$ is strongly compact then the class of free groups is definable in $L_{\infty\varkappa}$. J. Gregory has proved that the axiom of constructibility ($V = L$) implies that for every regular $\varkappa$ which is not weakly compact, there is a non-free group of cardinality $\varkappa$ which is $L_{\infty\varkappa}$-equivalent to a free group. (For references and more information see the author's *On the existence of $\varkappa$-free abelian groups* to appear in Proc. Amer. Math. Soc.).

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Forcing in a general setting (¹)

by

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Abstract. Abstract topological notions of forcing and generic set are presented. These notions are independent of the general notions of language and structure. Most particular notions of forcing in the literature are subsumed under this notion. The abstract notion is used to construct notions of forcing for languages containing the equi-cardinality quantifier, infinitary languages containing dependent quantifiers, and second-order languages.

The method of *forcing* was first invented by Cohen [Coh 1], [Coh 2] to solve questions regarding the logical independence of the axiom of choice and the continuum hypothesis with regard to the axioms of Zermelo-Fraenkel set theory. Subsequently Feferman [Fe] transferred the method to the settings of number theory and analysis and Robinson [Ro 1], [Ro 2] extended if the setting of general first order model theory.

Takenti realized that the existence of generic sets in set-theoretic forcing could be derived from the Baire Category Theorem and developed this point of view in [Ta] and lectures at the University of Illinois during 1965-66. This point of view was further developed in [Bo 3] and its extension to first order model theory was announced in [Bo 1]. The extension to second order logic was presented in [Bo 2].

In this paper we develop extremely abstract topological notions of forcing and generic objects which are entirely independent of the notions of language and structure. This development is presented in § 2. That it apparently subsums a great many of the forcing notions already extant in the literature is sketched in § 3. The extension of the notion of forcing to languages involving the equicardinality quantifier Q and to infinitary languages involving dependent quantifiers in the sense of [Ma] is presented in § 4.

The formulation of abstract forcing as given in § 2 is more general than necessary in that in §§ 3-5 we always take the sets X and X_0 to be $X = \{0, 1\}$ and $X_0 = \{0\}$. We hope to use this generality to extend the forcing concept to continuous model theory in the sense of $[C/K]$ in a future publication.

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