able but the negationless interpretation of A may be different from that A^G. Thus the Dialectica interpretation does not succeed in showing that every negationless acceptable statement which is provable using non-acceptable formulae has a derivation which only uses negationless acceptable formulae.

References

S. Feferman

[1962] *Classifications of recursive functions by means of hierarchies*, Trans. American Math. Soc. 104 (1962), pp. 101–122.

G. F. C. Griss

[1946–1951] *Negationless intuitionistic mathematics*, and ibid., II, III, IVa, IVb, Kon. Ned. Akad. Wet. (Amsterdam), 49 (1946), pp. 1127–1133, 53 (1950), pp. 456–463, 54 (1951), pp. 193–199, 452–471.

[1955] *La mathématique intuitioniste sans negation*, Nieuv Archief voor Wiskunde 3 (3) (1955), pp. 134–142.

S. C. Kleene

[1945] *On the interpretation of intuitionistic number theory*, J. Symbolic Logic 10 (1945), pp. 109–124.

K. Minichiello

[1968] *Negationless intuitionistic mathematics*, Dissertation, The George Washington University, D.C., 1968.

D. N. Nelson

[1966] *Non-Null implication*, J. Symbolic Logic 31 (1966), pp. 562–572.

D. Prawitz

[1965] *Natural Deduction, a proof theoretic study*, Upsala 1965.

[1971] *Ideas and results in proof theory*, in: Proceedings of the Second Scandinavian Logic Symposium, Amsterdam (1971), pp. 237–309.

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On nonmonotone inductive definability

by

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Abstract. The paper studies the class of relations on a set A which are defined inductively by nonmonotone operators in some collection $\mathfrak{F}$ satisfying certain minimal structural conditions. There are several concrete applications, including the construction of some interesting admissible sets.

The purpose of this paper is to apply the methods of EIAS (2) to the study of nonmonotone inductions.

In the first three sections we have attempted to codify the most basic properties of nonmonotone induction. These are general versions of tricks and methods which have been well known to the researchers in this field for some time. Many of them were formulated in similar abstract forms independently by P. Aczel, see Aczel [1973].

After the basics, we apply the theory of *Spector classes* of Chapter 9 of EIAS in Sections 4, 5 to obtain a characterization of the class of inductive relations relative to a "typical, nonmonotone class of operators." This is Theorem 15, the main result of the paper.

In Section 6 we consider in some detail the important examples of inductive definability in the higher order language over a structure — i.e. Σ_k^m- and Π_k^m-inductive definability, $m = 0$ and $k \geqslant 2$ or $m, k \geqslant 1$. The significant but "atypical" case of Π_1^0-induction is discussed briefly in Section 8.

Finally, in Section 7 we apply the theory of *companions of Spector classes* of Chapter 9 of EIAS to characterize various nonmonotone inductions in terms of admissible sets with related, interesting properties. The main result here is Theorem 21. There are also some applications

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(2) By EIAS we will refer to *Elementary Induction on Abstract Structures*, Amsterdam 1973.

to the theory of admissible sets; for example, we construct the *next Π_k-reflecting set* ($k \geqslant 3$) and the *next admissible set $\mathcal{M}$ which is $\mathcal{M}^+$-stable.*

The starting points for this work were the fundamental papers Grilliot [1971] and Aczel–Richter [1971]. The more recent Aczel–Richter [1973] was not available to me until the manuscript of this paper was almost finished, but I have attempted to introduce references to this rather than the earlier Aczel–Richter [1971], whenever possible. In addition to various references to these papers in the text, there is a brief discussion of the relevant results of Aczel and Richter at the end of Section 7.

1. The basic notions. Let A be a fixed infinite set, the domain on which we will study inductive definability. A *second order relation* on A is a relation with arguments elements and relations on A, say

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_k),$$

where x_i $(1 \leqslant i \leqslant n)$ ranges over A and Y_j $(1 \leqslant j \leqslant k)$ ranges over the r_j-ary relations on A. The sequence

$$\nu = (n, r_1, \ldots, r_k)$$

is the *signature* of the second order relation φ. If ν is of the form (n, n), so that φ has n individual arguments and one n-ary relation argument, then φ defines naturally an *operator* on the n-ary relations on A,

$$\Phi(S) = \{\bar{x}\colon \varphi(\bar{x}, S)\}.$$

We call a relation φ of this type *operative* and we define the ξth *iterate* of (the operator associated with) φ in the usual way, by the transfinite recursion

$$I_\varphi^\xi = \bigcup_{\eta<\xi} I_\varphi^\eta \cup \{\bar{x}\colon \varphi(\bar{x}, \bigcup_{\eta<\xi} I_\varphi^\eta)\}.$$

Putting

$$I_\varphi^{<\xi} = \bigcup_{\eta<\xi} I_\varphi^\eta,$$

the basic equation above becomes

$$I_\varphi^\xi = I_\varphi^{<\xi} \cup \{\bar{x}\colon \varphi(\bar{x}, I_\varphi^{<\xi})\}.$$

The *fixed point* of φ or the *set built up by* φ is

$$I_\varphi = \bigcup_\xi I_\varphi^\xi$$

and the *closure ordinal* of φ is the least stage at which we throw no new elements in I_φ,

$$\|\varphi\| = \textit{least } \xi \textit{ such that } I_\varphi^\xi = I_\varphi^{<\xi}.$$

An n-ary relation R is *defined inductively* by the operative relation φ with signature $(k+n, k+n)$, if there are constants $\bar{a} = a_1, \ldots, a_k$ in A such that for all $\bar{x}$,

$$R(\bar{x}) \Leftrightarrow (\bar{a}, \bar{x}) \in I_\varphi.$$

The main aim of the theory of inductive definability is to deduce properties of the fixed point I_φ, the closure ordinal $\|\varphi\|$ and the relations defined inductively by an operative relation φ from some assumed (simple) form of definition of φ. For example, we may take the case when φ is elementarily definable on some structure

$$\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$$

with domain A, or when φ is Σ_1^1 on $\mathfrak{A}$, i.e. definable by a prenex second order formula on $\mathfrak{A}$ with only existential relation quantifiers. In EIAS we studied the situation when $\varphi(\bar{x}, S)$ is definable by an elementary formula of some structure $\mathfrak{A}$ with only positive occurrences of S, so that φ was *monotone*, i.e.

$$S \subseteq T \;\&\; \varphi(\bar{x}, S) \Rightarrow \varphi(\bar{x}, T).$$

Here we are concerned with some nonmonotone examples which we will list at the end of this section. For the purpose of developing a theory of fairly wide applicability, it is best to assume that we are given a collection $\mathfrak{F}$ of second order relations on A satisfying some minimal conditions and then derive general properties of the relations defined inductively by the operative relations in $\mathfrak{F}$.

It will be convenient to call a collection of second order relations on A a *class of operators*. There is a slight abuse of language here, since nonoperative relations do not correspond directly to operators, but allowing relations of all signatures in classes of operators will make the axiomatic approach we prefer much easier. (It is simpler to describe the closure properties of the class of Σ_1^1 second order relations than those of the subclass of Σ_1^1 operative relations.)

If $\mathfrak{F}$ is a class of operators on A, the $\mathfrak{F}$-*fixed points* are the relations I_φ with φ an operative relation in $\mathfrak{F}$. We call $R \subseteq A^n$ $\mathfrak{F}$-*inductive* if there is an $\mathfrak{F}$-fixed point I_φ and constants $\bar{a} = a_1, \ldots, a_k$ in A such that

$$R(\bar{x}) \Leftrightarrow (\bar{a}, \bar{x}) \in I_\varphi,$$

we call R $\mathfrak{F}$-*coinductive* if $A^n - R$ is $\mathfrak{F}$-inductive and we call R $\mathfrak{F}$-*bi-inductive* or $\mathfrak{F}$-*hyperdefinable* if R is both $\mathfrak{F}$-inductive and $\mathfrak{F}$-coinductive. We denote these classes of relations on A by

$$\mathfrak{F}\text{-IND},\ \mathfrak{F}\text{-COIND or } \neg\mathfrak{F}\text{-IND},\ \mathfrak{F}\text{-HYP}$$

respectively. The *closure ordinal* of $\mathfrak{F}$ is the supremum of the closure ordinals of the operative relations in $\mathfrak{F}$,

$$\|\mathfrak{F}\| = \mathit{supremum}\{\|\varphi\|\colon \varphi \in \mathfrak{F},\ \varphi \ \mathit{operative}\}.$$

Let us now describe the main examples that we intend the present theory to cover, the classes of Σ_k^m- and Π_k^m-inductive relations over a structure $\mathfrak{A}$.

The *set of types* is defined by the induction

$$0 \ \mathit{is\ a\ type},$$

$$\mathit{if}\ \tau_1, \ldots, \tau_n\ \mathit{are\ types,\ then\ the\ tuple}\ (\tau_1, \ldots, \tau_n)\ \mathit{is\ a\ type}.$$

For a fixed set A, the set $T^\tau(A)$ of *objects* (or relations) *of type* τ *over* A is defined by the corresponding inductions

$$T^0(A) = A,$$

$$\mathit{if}\ \tau = (\tau_1, \ldots, \tau_n),\ \mathit{then}\ R \in T^\tau(A) \Leftrightarrow R \subseteq T^{\tau_1}(A) \times \ldots \times T^{\tau_n}(A).$$

For example, the objects of type $(0, 0, 0)$ are the ternary relations on A, the objects of type $(0, 0, (0, 0))$ are the second order relations of signature (2,2) over A, etc.

The full *higher order* (or *higher type*) language for a structure $\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$ has relation constants for $=$ (identity on A) and $R_1, \ldots, R_l$ and an infinite sequence of variables of type τ for each type τ. *Terms* of type 0 are the constants from A and the variables of type 0 and terms of type $\tau \neq 0$ are just the variables of type τ. The *prime formulas* are of the form

$$t_1 = t_2, \quad R_i(t_1, \ldots, t_{n_i}), \quad Y^\tau(Y_1^{\tau_1}, \ldots, Y_n^{\tau_n}),$$

where the t_j are terms of type 0 and $Y^\tau, Y_1^{\tau_1}, \ldots, Y_n^{\tau_n}$ are terms of types $\tau = (\tau_1, \ldots, \tau_n), \tau_1, \ldots, \tau_n$ respectively. *Formulas* are constructed from these using $\neg$, &, $\vee$, $\rightarrow$ and both sorts of quantifiers of every type.

We define truth (or satisfaction) for formulas of the higher order language for $\mathfrak{A}$ in the obvious, standard manner.

The *order of a type* τ is defined by the induction

$$\mathit{order}(0) = 0,$$

$$\mathit{order}(\tau_1, \ldots, \tau_n) = \mathit{supremum}\{\mathit{order}(\tau_1), \ldots, \mathit{order}(\tau_n)\} + 1$$

and the *order of an object* of type τ is the order of τ. (Clearly relations of order 2 are exactly what we have been calling second order relations.) Similarly, the *order of a variable* of type τ is the order of τ.

A relation R of any type is *m-th order definable over* $\mathfrak{A}$ if there is a formula $\varphi \equiv \varphi(Y_1, \ldots, Y_n)$ with the appropriate type free variables and all bound variables of order $< m$ *which defines* R, i.e.

$$R(Y_1, \ldots, Y_n) \Leftrightarrow \mathfrak{A} \models \varphi(Y_1, \ldots, Y_n).$$

To define the finer classifications of Σ_k^m and Π_k^m relations we must take cases on $m \geqslant 1$ or $m = 0$.

For $m \geqslant 1$ and $k \geqslant 1$, a relation R of any type is Σ_k^m *over* $\mathfrak{A}$ if there is a formula φ which defines R and which is of the form

$$(1)\quad \underbrace{(\exists Z_1^1)(\exists Z_2^1)\ldots(\exists Z_{n_1}^1)}\underbrace{(\forall Z_1^2)(\forall Z_2^2)\ldots(\forall Z_{n_2}^2)}\ldots\underbrace{(\mathrm{Q}Z_1^k)(\mathrm{Q}Z_2^k)\ldots(\mathrm{Q}Z_{n_k}^k)}\psi,$$

where the Z_j^i are variables of order m and ψ has no bound variables of order $\geqslant m$: i.e. φ starts with k alternating blocks of mth order quantifiers beginning with $\exists$ and continues with a formula having no quantifiers of order $\geqslant m$.

A relation R is Π_k^m $(m \geqslant 1, k \geqslant 1)$ if $\neg R$ is Σ_k^m, i.e. if R can be defined by a formula φ of form dual to (1).

A relation R is $\exists_k$ over $\mathfrak{A}$ if there is a formula φ which defines R and which is of the form (1) with the Z_j^i of type 0 and ψ quantifier free. These relations are interesting from the model theoretic point of view, but to study inductive definability, we must also look at another, indirectly defined class of relations.

Recall from Section 5A of EIAS the definition of a coding scheme $\mathfrak{C} = \langle N^{\mathfrak{C}}, \leqslant^{\mathfrak{C}}, \langle\ \rangle^{\mathfrak{C}}\rangle$ on A with decoding relations and functions $\mathrm{Seq}^{\mathfrak{C}}$, $lh^{\mathfrak{C}}$, $q^{\mathfrak{C}}$. A relation P of any type is *restricted on* $\mathfrak{A}$ *relative to* $\mathfrak{C}$ if it can be defined by a formula in the language of the expanded structure

$$(\mathfrak{A}, \mathfrak{C}) = \langle A, R_1, \ldots, R_l, \leqslant^{\mathfrak{C}}, \mathrm{Seq}^{\mathfrak{C}}, lh^{\mathfrak{C}}, q^{\mathfrak{C}}\rangle$$

in which all quantifiers are of type 0 and occur in one of the following two forms:

$$(\exists x)[x \leqslant^{\mathfrak{C}} y \ \&\ \ldots], \quad (\forall x)[x \leqslant^{\mathfrak{C}} y \rightarrow \ldots].$$

Notice that the expanded structure $(\mathfrak{A}, \mathfrak{C})$ has functions as well as relations, so complicated terms like $lh(q(x, i))$ occur in prime formulas.

We say that R is Σ_k^0 *on* $\mathfrak{A}$ *relative to* $\mathfrak{C}$ or $\Sigma_k^0(\mathfrak{C})$ *on* $\mathfrak{A}$ if R satisfies an equivalence

$$(2)\quad R(Y_1, \ldots, Y_n)$$
$$\Leftrightarrow \underbrace{(\exists z_1^1)\ldots(\exists z_{n_1}^1)}\underbrace{(\forall z_1^2)\ldots(\forall z_{n_2}^2)}\ldots\underbrace{(\mathrm{Q}z_1^k)\ldots(\mathrm{Q}z_{n_k}^k)}P(\ldots, Y_1, \ldots, Y_n),$$

where the z_j^i are variables of type 0 and P is *restricted relative to* $\mathfrak{C}$: i.e. R can be defined by applying a string of k alternating blocks of 0-th order quantifiers beginning with $\exists$ to a restricted relation.

As above, R *is* Π^0_k *on* $\mathfrak{A}$ *relative to* $\mathcal{C}$ or $\Pi^0_k(\mathcal{C})$ *on* $\mathfrak{A}$ if $\neg R$ is $\Sigma^0_k(\mathcal{C})$ over $\mathfrak{A}$, i.e. if R can be defined by an expression dual to (2).

The main examples we intend to study are Σ^m_k and Π^m_k inductions with $m, k \geq 1$ and $\Sigma^0_k(\mathcal{C})$, $\Pi^0_k(\mathcal{C})$ inductions with $k \geq 2$ and $\mathcal{C}$ a hyperelementary coding scheme in the structure $\mathfrak{A}$ (we will show that the theory of these classes of operators does not depend on the particular hyperelementary $\mathcal{C}$ we choose). However the general theory we will develop has many other applications and we will discuss some of them in section 7. Many of the results will also apply to $\Pi^0_1(\mathcal{C})$ — this is a very interesting class of operators and we will consider it briefly in section 8.

2. Elementary results. Let us consider some reasonable conditions on a class of operators $\mathfrak{F}$ which will insure that the theory of $\mathfrak{F}$-induction is not trivial. Throughout, $\mathfrak{F}$ is a collection of operators (second order relations) on an infinite set A.

CONDITION (A). *$\mathfrak{F}$ contains all second order relations on A which are definable by (first order) universal formulas of the trivial structure $\langle A\rangle$.*

By a universal formula of $\langle A\rangle$ we mean one of the form

$$(\forall x_1)\ldots(\forall x_n)\psi,$$

where ψ is quantifier free, i.e. built up from prime formulas of the form

$$t_1 = t_2, \qquad Y(t_1, \ldots, t_n)$$

(the t_i constants from A or individual variables, Y an n-ary relation variable) by the propositional connectives $\neg$, &, $\vee$, $\rightarrow$.

This first condition already puts nonmonotone operative relations in $\mathfrak{F}$ and excludes the case $\Sigma^0_1(\mathcal{C})$. The other two conditions we need now are minimal closure properties.

CONDITION (B). *$\mathfrak{F}$ is closed under & and $\vee$, i.e. if $\varphi(\bar{x}, \bar{Y})$ and $\psi(\bar{x}, \bar{Y})$ are in $\mathfrak{F}$ with the same signature, then the relations*

$$\varphi(\bar{x}, \bar{Y})\,\&\,\psi(\bar{x}, \bar{Y}), \qquad \varphi(\bar{x}, \bar{Y})\vee\psi(\bar{x}, \bar{Y})$$

are also in $\mathfrak{F}$.

To state Condition (C) we need the notion of a trivial, combinatorial function with individual or relation values, see EIAS 9A. If $f(\bar{x}, \bar{Y})$ is a function with values in A, we call it *trivial, combinatorial* if the relation

$$\varphi(\bar{x}, \bar{Y}, z) \Leftrightarrow z = f(\bar{x}, \bar{Y})$$

is definable by a quantifier free formula of the trivial structure $\langle A\rangle$, as we defined these above. For example, the function

$$f(x, Y) = z \Leftrightarrow [Y(x)\,\&\,z = c_0]\vee[\neg Y(x)\,\&\,z = x]$$

is trivial combinatorial. Similarly, if $f(\bar{x}, \bar{Y})$ is a function with m-ary relations as values, we call it *trivial, combinatorial* if the relation

$$\varphi(\bar{x}, \bar{Y}, z_1, \ldots, z_m) \Leftrightarrow (z_1, \ldots, z_m) \in f(\bar{x}, \bar{Y})$$

is definable by a quantifier free formula of $\langle A\rangle$. For example, the function

$$f(x, Y, Z) = \{(c_0, u, v):\ Y(u, x)\vee Z(v, x)\}$$

is trivial, combinatorial, since

$$(t, u, v) \in f(x, Y, Z) \Leftrightarrow t = c_0\,\&\,[Y(u, x)\vee Z(v, x)].$$

CONDITION (C). *$\mathfrak{F}$ is closed under trivial, combinatorial substitutions, i.e. if $\psi(z_1, \ldots, z_m, W_1, \ldots, W_k)$ is in $\mathfrak{F}$ and $f_1(\bar{x}, \bar{Y}), \ldots, f_m(\bar{x}, \bar{Y})$, $g_1(\bar{x}, \bar{Y}), \ldots, g_k(\bar{x}, \bar{Y})$ are trivial, combinatorial with the appropriate sorts of values, then the relation $\varphi(\bar{x}, \bar{Y})$ defined by*

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow \psi\big(f_1(\bar{x}, \bar{Y}), \ldots, f_m(\bar{x}, \bar{Y}), g_1(\bar{x}, \bar{Y}), \ldots, g_k(\bar{x}, \bar{Y})\big)$$

is also in $\mathfrak{F}$.

It will be convenient to call a class of operators $\mathfrak{F}$ which satisfies conditions (A), (B), (C) *reasonable, nonmonotone.* It is clear that the classes Σ^m_k, Π^m_k $(m \geq 1, k \geq 1)$ and $\Sigma^0_k(\mathcal{C})$ $(k \geq 2)$, $\Pi^0_k(\mathcal{C})$ $(k \geq 1)$ on a structure $\mathfrak{A}$ are reasonable, nonmonotone. There are many other obvious examples that come to mind, e.g. the class of $\exists_k$ operators on a structure as we defined them in Section 1 $(k \geq 2)$, the similarly defined class of $\forall_k$ operators $(k \geq 1)$ or even the class of all elementarily definable operators on a structure.

THEOREM 1. *Let $\mathfrak{F}$ be a reasonable, nonmonotone class of operators on A, suppose $\varphi_1(\bar{x}_1, S_1, \ldots, S_n), \varphi_2(\bar{x}_2, S_1, \ldots, S_n), \ldots, \varphi_n(\bar{x}_n, S_1, \ldots, S_n)$ are relations in $\mathfrak{F}$, where $\bar{x}_i$ is an r_i-tuple of variables and S_i ranges over r_i-ary relations and define the sets $J^\xi_1, \ldots, J^\xi_n$ by the simultaneous recursion*

$$\bar{x}_1 \in J^\xi_1 \Leftrightarrow \bar{x}_1 \in J^{<\xi}_1 \vee \varphi_1(\bar{x}_1, J^{<\xi}_1, \ldots, J^{<\xi}_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\bar{x}_n \in J^\xi_n \Leftrightarrow \bar{x}_n \in J^{<\xi}_n \vee \varphi_n(\bar{x}_n, J^{<\xi}_1, \ldots, J^{<\xi}_n),$$

where

$$J^{<\xi}_1 = \bigcup_{\eta<\xi} J^\eta_1, \ldots, J^{<\eta}_n = \bigcup_{\eta<\xi} J^\eta_n.$$

Let also

$$J_1 = \bigcup_\xi J^\xi_1, \ldots, J_n = \bigcup_\xi J^\xi_n.$$

Then $J_1, \ldots, J_n$ are all $\mathfrak{F}$-inductive. In fact there is an operative relation φ^* in $\mathfrak{F}$ and sequences of constants $\bar{c}_1, \ldots, \bar{c}_n$ such that for all ξ,

$$\bar{x}_1 \in J_1^\xi \Leftrightarrow (\bar{c}_1, \bar{x}_1) \in I_{\varphi^*}^\xi,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\bar{x}_n \in J_1^\xi \Leftrightarrow (\bar{c}_n, \bar{x}_n) \in I_{\varphi^*}^\xi.$$

(Definition by simultaneous induction.)

Proof. For simplicity take $n = 2$ and assume that

$$\bar{x}_1 = y, \qquad \bar{x}_2 = z_1, z_2,$$

i.e. $\bar{x}_1$ ranges over elements of A and $\bar{x}_2$ ranges over pairs from A. Put

$$\begin{aligned}&\varphi^*(t, u, v, S)\\ &\quad\Leftrightarrow [t = c_0 \,\&\, u = c_0 \,\&\, \varphi_1(v, \{v' \colon S(c_0, c_0, v')\}, \{(u', v') \colon S(c_1, u', v')\})] \vee\\ &\quad\vee [t = c_1 \,\&\, \varphi_2(u, v, \{v' \colon S(c_0, c_0, v')\}, \{(u', v') \colon S(c_1, u', v')\})],\end{aligned}$$

where c_0, c_1 are distinct members of A. Clearly φ^* is an operative relation in $\mathfrak{F}$ and an easy transfinite induction on ξ shows that

$$y \in J_1^\xi \Leftrightarrow (c_0, c_0, y) \in I_{\varphi^*}^\xi,$$

$$(z_1, z_2) \in J_2^\xi \Leftrightarrow (c_1, z_1, z_2) \in I_{\varphi^*}^\xi. \qquad \blacksquare$$

Definition by simultaneous induction is the basic tool for constructing complicated inductive definitions. We put it to use at once to prove the elementary transitivity and closure properties of the classes $\mathfrak{F}$-IND and $\mathfrak{F}$-HYP.

Recall that a second order relation $\varphi(\bar{x}, Y_1, \ldots, Y_m)$ is *monotone* if

$$\varphi(\bar{x}, Y_1, \ldots, Y_m) \,\&\, Y_1 \subseteq Z_1 \,\&\, \ldots \,\&\, Y_m \subseteq Z_m \Rightarrow \varphi(\bar{x}, Z_1, \ldots, Z_m).$$

THEOREM 2. *Let $\mathfrak{F}$ be reasonable, nonmonotone on A, let $\psi(\bar{x}, Y_1, \ldots, Y_m, S)$ be a monotone relation in $\mathfrak{F}$, let $R_1, \ldots, R_m$ be $\mathfrak{F}$-inductive relations and define φ by*

$$\varphi(\bar{x}, S) \Leftrightarrow \psi(\bar{x}, R_1, \ldots, R_m, S);$$

then I_φ is $\mathfrak{F}$-inductive and $\|\varphi\| \leqslant \|\mathfrak{F}\|$. (The Monotone Transitivity Theorem.)

Proof. By hypothesis, there are operative relations $\psi_1, \ldots, \psi_m$ in $\mathfrak{F}$ and sequences of constants $\bar{a}_1, \ldots, \bar{a}_m$ in A such that

$$R_1(\bar{x}_1) \Leftrightarrow (\bar{a}_1, \bar{x}_1) \in I_{\psi_1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$R_m(\bar{x}_m) \Leftrightarrow (\bar{a}_m, \bar{x}_m) \in I_{\psi_m}.$$

Consider the system of simultaneous induction

$$\varphi_1(\bar{u}_1, \bar{x}_1, S_1, \ldots, S_m, S_{m+1}) \Leftrightarrow \psi_1(\bar{u}_1, \bar{x}_1, S_1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\varphi_m(\bar{u}_m, \bar{x}_m, S_1, \ldots, S_m, S_{m+1}) \Leftrightarrow \psi_m(\bar{u}_m, \bar{x}_m, S_m),$$

$$\begin{aligned}\varphi_{m+1}(\bar{x}, S_1, \ldots, S_m, S_{m+1}) \Leftrightarrow \psi(\bar{x}, \{\bar{x}_1 \colon S_1(\bar{a}_1, \bar{x}_1)\}, \ldots\\ \ldots, \{\bar{x}_m \colon S_m(\bar{a}_m, \bar{x}_m)\}, S_{m+1}).\end{aligned}$$

Clearly all the φ_i's are in $\mathfrak{F}$, hence in the notation of Theorem 1, each J_i is $\mathfrak{F}$-inductive. It is immediate from the definition that

$$J_i^\xi = I_{\psi_i}^\xi, \qquad i = 1, \ldots, m,$$

so that

$$R_1 = \{\bar{x}_1 \colon (\bar{a}_1, \bar{x}_1) \in J_1\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$R_m = \{\bar{x}_m \colon (\bar{a}_m, \bar{x}_m) \in J_m\}.$$

First we verify by induction on ξ that

$$\bar{x} \in J_{m+1}^\xi \Rightarrow \bar{x} \in I_\varphi^\xi; \tag{1}$$

because

$$\begin{aligned}\bar{x} \in J_{m+1}^\xi &\Rightarrow \bar{x} \in J_{m+1}^{<\xi} \vee \psi(\bar{x}, \{\bar{x}_1 \colon (\bar{a}_1, \bar{x}_1) \in J_1^{<\xi}\}, \ldots, J_{m+1}^{<\xi})\\ &\Rightarrow \bar{x} \in I_\varphi^{<\xi} \vee \psi(\bar{x}, R_1, \ldots, R_m, I_\varphi^{<\xi}) \quad \text{(ind. hyp. \& monotonicity)}\\ &\Rightarrow \bar{x} \in I_\varphi^{<\xi} \vee \varphi(\bar{x}, I_\varphi^{<\xi})\\ &\Rightarrow \bar{x} \in I_\varphi^\xi.\end{aligned}$$

Also by induction on ξ,

$$\bar{x} \in I_\varphi^\xi \Rightarrow \bar{x} \in J_{m+1}; \tag{2}$$

because

$$\begin{aligned}\bar{x} \in I_\varphi^\xi &\Rightarrow \bar{x} \in I_\varphi^{<\xi} \vee \psi(\bar{x}, R_1, \ldots, R_m, I_\varphi^{<\xi})\\ &\Rightarrow \bar{x} \in J_{m+1}^{<\lambda} \vee \psi(\bar{x}, \{\bar{x}_1 \colon (\bar{a}_1, \bar{x}_1) \in J_1^{<\lambda}\}, \ldots, J_{m+1}^{<\lambda})\end{aligned}$$

(choosing λ large enough and using the induction hypothesis and monotonicity)

$$\Rightarrow \bar{x} \in J_{m+1}^\lambda.$$

Now (1) and (2) yield $I_\varphi = J_{m+1}$, so I_φ is $\mathfrak{F}$-inductive by Theorem 1. Also if φ^* is the operative relation which combines the system $\varphi_1, \ldots, \varphi_{m+1}$ as in Theorem 1, then

$$\begin{aligned} \bar{x} \in I_\varphi \Rightarrow \bar{x} \in J^{\xi}_{m+1}\,, &\quad \text{for some } \xi < \|\varphi^*\| \text{ by (2)}\,, \\ \Rightarrow \bar{x} \in I^{\xi}_{\varphi}\,, &\quad \text{for some } \xi < \|\varphi^*\| \text{ by (1)}\,, \end{aligned}$$

so that $\|\varphi\| \leqslant \|\varphi^*\| \leqslant \|\mathfrak{F}\|$. ∎

As an immediate corollary of Theorem 2 we obtain the closure properties of the class $\mathfrak{F}$-IND.

THEOREM 3. *Let $\mathfrak{F}$ be reasonable, nonmonotone A, suppose $\psi(\bar{x}, Y_1, \ldots, Y_m)$ is a monotone relation in $\mathfrak{F}$, $R_1, \ldots, R_m$ are $\mathfrak{F}$-inductive and*

$$P(\bar{x}) \Leftrightarrow \psi(\bar{x}, R_1, \ldots, R_m)\,;$$

then P is $\mathfrak{F}$-inductive. In particular $\mathfrak{F}$-IND contains all the first order relations in $\mathfrak{F}$ and is closed under &, $\vee$, $\forall$ and trivial combinatorial substitutions. (Closure properties of $\mathfrak{F}$-IND.)

Proof is immediate by taking

$$\begin{aligned} \varphi(\bar{x}, Y_1, \ldots, Y_m, S) &\Leftrightarrow \psi(\bar{x}, Y_1, \ldots, Y_m)\,, \\ \varphi(\bar{x}, S) &\Leftrightarrow \psi(\bar{x}, R_1, \ldots, R_m, S) \end{aligned}$$

and noticing that $I_\varphi = P$. The special cases mentioned are trivial, e.g. taking

$$\psi(\bar{x}, Y_1, Y_2) \Leftrightarrow Y_1(\bar{x})\ \&\ Y_2(\bar{x})$$

to show closure under &. ∎

The restriction to monotone ψ in Theorem 3 is necessary, since we do not expect $\mathfrak{F}$-IND to be closed under $\neg$. We can remove this restriction when we relativize to $\mathfrak{F}$-hyperdefinable relations, as in the next two results.

THEOREM 4. *Let $\mathfrak{F}$ be reasonable, nonmonotone on A and for each sequence*

$$\bar{Q} = Q_1, \ldots, Q_m$$

of first order relations on A, let $\mathrm{Rel}(\mathfrak{F}; \bar{Q})$ be the class of all second order relations $\varphi(\bar{x}, \bar{Y})$ such that for some $\psi(Z_1, \ldots, Z_m, \bar{x}, \bar{Y})$ in $\mathfrak{F}$ we have

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow \psi(\bar{Q}, \bar{x}, \bar{Y})\,.$$

Then $\mathrm{Rel}(\mathfrak{F}; Q)$ is a reasonable, nonmonotone class and if $Q_1, \ldots, Q_m$ are all $\mathfrak{F}$-hyperdefinable, then

$$\begin{gathered} \mathfrak{F}\text{-IND} = \mathrm{Rel}(\mathfrak{F}; \bar{Q})\text{-IND}\,, \\ \|\mathfrak{F}\| = \|\mathrm{Rel}(\mathfrak{F}; \bar{Q})\|\,. \end{gathered}$$

(The Nonmonotone Transitivity Theorem.)

Proof. That $\mathrm{Rel}(\mathfrak{F}; \bar{Q})$ is reasonable, nonmonotone is trivial.

Since $\mathfrak{F} \subseteq \mathrm{Rel}(\mathfrak{F}; \bar{Q})$, clearly $\mathfrak{F}$-IND $\subseteq \mathrm{Rel}(\mathfrak{F}; \bar{Q})$-IND and $\|\mathfrak{F}\| \leqslant \|\mathrm{Rel}(\mathfrak{F}; \bar{Q})\|$.

By hypothesis, there are operative relations $\psi_1, \ldots, \psi_{2m}$ in $\mathfrak{F}$ and constants $\bar{a}_1, \ldots, \bar{a}_{2m}$ such that

$$\begin{gathered} Q_1(\bar{x}_1) \Leftrightarrow (\bar{a}_1, \bar{x}_1) \in I_{\psi_1}, \ldots, Q_m(\bar{x}_m) \Leftrightarrow (\bar{a}_m, \bar{x}_m) \in I_{\psi_m}\,, \\ \neg Q_1(\bar{x}_1) \Leftrightarrow (\bar{a}_{m+1}, \bar{x}_1) \in I_{\psi_{m+1}}, \ldots, \neg Q_m(\bar{x}_m) \Leftrightarrow (\bar{a}_{2m}, \bar{x}_m) \in I_{\psi_{2m}}\,. \end{gathered}$$

If $\varphi(\bar{x}, S)$ is any operative relation in $\mathrm{Rel}(\mathfrak{F}; \bar{Q})$, choose some $\psi(Z_1, \ldots, Z_m, \bar{x}, S)$ in $\mathfrak{F}$ such that

$$\varphi(\bar{x}, S) \Leftrightarrow \psi(Q_1, \ldots, Q_m, \bar{x}, S)$$

and consider the following system of simultaneous inductions:

$$\begin{aligned} &\varphi_1(\bar{u}_1, \bar{x}_1, S_1, \ldots, S_{2m}) \Leftrightarrow \psi_1(\bar{u}_1, \bar{x}_1, S_1)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\varphi_{2m}(\bar{u}_{2m}, \bar{x}_m, S_1, \ldots, S_{2m}, S) \Leftrightarrow \psi_{2m}(\bar{u}_{2m}, \bar{x}_m, S_{2m})\,, \\ &\varphi_{2m+1}(\bar{x}, S_1, \ldots, S_{2m}, S) \Leftrightarrow (\forall \bar{x}_1)[S_1(\bar{a}_1, \bar{x}_1) \vee S_{m+1}(\bar{a}_{m+1}, \bar{x}_1)] \\ &\qquad\qquad \&\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad\qquad \&\ (\forall \bar{x}_m)[S_m(\bar{a}_m, \bar{x}_m) \vee S_{2m}(\bar{a}_{2m}, \bar{x}_m)] \\ &\qquad\qquad \&\ \psi(\{\bar{x}_1 : S_1(\bar{a}_1, \bar{x}_1)\}, \ldots, \{\bar{x}_m : S_m(\bar{a}_m, \bar{x}_m)\}, \bar{x}, S)\,. \end{aligned}$$

These relations are all in $\mathfrak{F}$, and in the notation of Theorem 1 we have immediately

$$J_1 = I_{\psi_1}, \ldots, J_{2m} = I_{\psi_{2m}}\,,$$

so that

$$\begin{gathered} Q_1 = \{\bar{x}_1 : (\bar{a}_1, \bar{x}_1) \in J_1\}, \ldots, Q_m = \{\bar{x}_m : (\bar{a}_m, \bar{x}_m) \in J_m\}\,, \\ \neg Q_1 = \{\bar{x}_1 : (\bar{a}_{m+1}, \bar{x}_1) \in J_{m+1}\}, \ldots, \neg Q_m = \{\bar{x}_m : (\bar{a}_m, \bar{x}_m) \in J_{2m}\}\,. \end{gathered}$$

It follows that for each $i = 1, \ldots, m$ and all sufficiently large ξ,

$$(3) \qquad (\forall \bar{x}_i)[(\bar{a}_i, \bar{x}_i) \in J_i^{<\xi} \vee (\bar{a}_{m+1}, \bar{x}_i) \in J_{m+i}^{<\xi}]$$

and that if *ξ is good for Q_i* in the sense that (3) holds, then

$$Q_i = \{\bar{x}_i : (\bar{a}_i, \bar{x}_i) \in J_i^{<\xi}\}\,.$$

Let

$$\lambda = \textit{least } \xi \textit{ which is good for } Q_1, \ldots, Q_m\,;$$

we have easily that

$$J^{\xi}_{2m+1} = \emptyset \quad \text{for} \quad \xi < \lambda,$$

$$(4) \qquad J^{\lambda+\xi}_{2m+1} = I^{\xi}_{\varphi},$$

so that

$$J_{2m+1} = I_{\varphi}$$

and I_{φ} is $\mathfrak{F}$-inductive by Theorem 1.

This argument shows that every $\mathrm{Rel}(\mathfrak{F}; \bar{Q})$-fixed point is $\mathfrak{F}$-inductive, so that every $\mathrm{Rel}(\mathfrak{F}; \bar{Q})$-inductive relation is $\mathfrak{F}$-inductive. The second assertion of Theorem 1 together with (4) tells us that the $\mathfrak{F}$-induction which combines the simultaneous inductions for $J_1, \ldots, J_{2m+1}$ has closure ordinal $\geqslant \|\varphi\|$, so that $\|\mathrm{Rel}(\mathfrak{F}; \bar{Q})\| \leqslant \|\mathfrak{F}\|$. ■

This yields again immediately the closure properties of the class $\mathfrak{F}$-HYP.

THEOREM 5. *Let $\mathfrak{F}$ be reasonable, nonmonotone on A, suppose $\psi(\bar{x}, Y_1, \ldots, Y_m)$ is a relation in $\mathfrak{F}$, $R_1, \ldots, R_m$ are $\mathfrak{F}$-hyperdefinable and*

$$P(\bar{x}) \Leftrightarrow \psi(\bar{x}, R_1, \ldots, R_m);$$

then P is $\mathfrak{F}$-hyperdefinable.

In particular $\mathfrak{F}$-HYP contains all the first order relations in $\mathfrak{F}$ and is closed under all the elementary operations $\neg$, &, $\vee$, $\rightarrow$, $\forall$, $\exists$. (Closure properties of $\mathfrak{F}$-HYP.)

Proof. Let us first observe that if P is a first order relation in $\mathfrak{F}$, then P is $\mathfrak{F}$-hyperdefinable; that P is $\mathfrak{F}$-inductive is trivial and that $\neg P$ is $\mathfrak{F}$-inductive follows by considering the system.

$$\varphi_1(t, S_1, S_2, S_3) \Leftrightarrow t = a,$$

$$\varphi_2(\bar{x}, S_1, S_2, S_3) \Leftrightarrow P(\bar{x}),$$

$$\varphi_3(\bar{x}, S_1, S_2, S_3) \Leftrightarrow S_1(a) \;\&\; \neg S_2(\bar{x})$$

and verifying that $J_2 = J_2^0 = P$, $J_3 = J_3^1 = \neg P$.

Now, given $\psi(\bar{x}, Y_1, \ldots, Y_m)$ and $R_1, \ldots, R_m$ as in the hypothesis, apply this observation to $\mathfrak{F}^* = \mathrm{Rel}(\mathfrak{F}; R_1, \ldots, R_m)$ to infer that P is $\mathfrak{F}^*$-hyperdefinable and hence $\mathfrak{F}$-hyperdefinable by Theorem 4. ■

3. Structure theory. We go now to those results about $\mathfrak{F}$-induction which deal with the stages I^{ξ}_{φ} of an inductive definition. As in EIAS 2A, if φ, ψ are operative relations, put

$$\bar{x} \leqslant^*_{\varphi,\psi} \bar{y} \Leftrightarrow \bar{x} \in I_{\varphi} \;\&\; [\bar{y} \notin I_{\psi} \vee |\bar{x}|_{\varphi} \leqslant |\bar{y}|_{\psi}],$$

$$\bar{x} <^*_{\varphi,\psi} \bar{y} \Leftrightarrow \bar{x} \in I_{\varphi} \;\&\; [\bar{y} \notin I_{\psi} \vee |\bar{x}|_{\varphi} < |\bar{y}|_{\psi}],$$

where, of course

$$|\bar{x}|_{\varphi} = \textit{least } \xi \textit{ such that } \bar{x} \in I^{\xi}_{\varphi}$$

and similarly for $|\bar{y}|_{\psi}$. A very slight weakening of the Stage Comparison Theorem of EIAS holds for nonmonotone inductions with an entirely different and easier proof.

THEOREM 6. *Let $\mathfrak{F}$ be a reasonable, nonmonotone class of operators on A, and suppose that φ, ψ are operative relations in $\mathfrak{F}$; then both $\leqslant^*_{\varphi,\psi}$, $<^*_{\varphi,\psi}$ are $\mathfrak{F}$-inductive.* (The Stage Comparison Theorem.)

Proof. Consider the system of inductions

$$\varphi_1(\bar{x}, S_1, S_2, S_3) \Leftrightarrow \varphi(\bar{x}, S_1),$$

$$\varphi_2(\bar{y}, S_1, S_2, S_3) \Leftrightarrow \psi(\bar{y}, S_2),$$

$$\varphi_3(\bar{x}, \bar{y}, S_1, S_2, S_3) \Leftrightarrow \varphi(\bar{x}, S_1) \;\&\; \neg S_2(\bar{y}).$$

These relations are obviously in $\mathfrak{F}$ and a trivial induction shows, in the notation of Theorem 1,

$$J^{\xi}_1 = I^{\xi}_{\varphi}, \qquad J^{\xi}_2 = J^{\xi}_{\psi};$$

hence

$$(\bar{x}, \bar{y}) \in J^{\xi}_3 \Leftrightarrow (\bar{x}, \bar{y}) \in J^{<\xi}_3 \vee [\varphi(\bar{x}, J^{<\xi}_1) \;\&\; \neg(\bar{y} \in J^{<\xi}_2)]$$
$$\Leftrightarrow (\bar{x}, \bar{y}) \in J^{<\xi}_3 \vee [\bar{x} \in I^{\xi}_{\varphi} \;\&\; \bar{y} \notin I^{<\xi}_{\psi}],$$

which implies by a trivial induction that

$$\leqslant^*_{\varphi,\psi} = J_3.$$

The proof for $<^*_{\varphi,\psi}$ is a slight variation. ■

Using the Stage comparison theorem, we can obtain appropriate versions of all the results in Chapter 2 of EIAS by slight variations of the proofs there, so we shall be brief.

THEOREM 7. *Let $\mathfrak{F}$ be reasonable, nonmonotone and let φ be an operative relation in $\mathfrak{F}$. Then for each $\xi < \|\mathfrak{F}\|$ the set $I^{<\xi}_{\varphi}$ is $\mathfrak{F}$-hyperdefinable, and the fixed point I_{φ} is $\mathfrak{F}$-hyperdefinable if and only if $\|\varphi\| < \|\mathfrak{F}\|$.* (The Closure Theorem.)

Proof. If $\xi = |\bar{y}_0|_{\psi}$ for some operative ψ in $\mathfrak{F}$ and some $\bar{y}_0 \in I_{\psi}$, then

$$\bar{x} \in I^{<\xi}_{\varphi} \Leftrightarrow \bar{x} <^*_{\varphi,\psi} \bar{y}_0 \Leftrightarrow \neg(\bar{y}_0 \leqslant^*_{\psi,\varphi} \bar{x}),$$

so that $I^{<\xi}_{\varphi}$ is $\mathfrak{F}$-hyperdefinable. In particular, if $\|\varphi\| = \xi < \|\mathfrak{F}\|$, then $I_{\varphi} = I^{<\xi}_{\varphi}$ is $\mathfrak{F}$-hyperdefinable.

Assuming now that I_φ is $\mathfrak{F}$-hyperdefinable, consider the system

$$\varphi_1(\bar{x}, S_1, S_2) \Leftrightarrow \varphi(\bar{x}, S_1)\,,$$
$$\varphi_2(t, S_1, S_2) \Leftrightarrow (\forall \bar{x})\{\bar{x} \in I_\varphi \Rightarrow S_1(\bar{x})\}\,,$$

where φ_1, φ_2 are obviously in the class of operators $\mathrm{Rel}(\mathfrak{F}; I_\varphi)$. It is easy to verify that in the notation of Theorem 1,

$$J_1^\xi = I_\varphi^\xi\,, \quad \text{for all } \xi\,,$$

$$J_2^\xi = \begin{cases} \emptyset & \text{if} \quad \xi < \|\varphi\|\,, \\ A & \text{if} \quad \xi \geqslant \|\varphi\|\,, \end{cases}$$

so that if φ^* is the operative relation which combines the simultaneous induction by Theorem 1, then $\|\varphi^*\| = \|\varphi\|+1 > \|\varphi\|$. But φ^* is in $\mathrm{Rel}(\mathfrak{F}; I_\varphi)$, hence using Theorem 4,

$$\|\mathfrak{F}\| = \|\mathrm{Rel}(\mathfrak{F}; I_\varphi)\| \geqslant \|\varphi^*\| > \|\varphi\|\,. \quad \blacksquare$$

Recall that if $\precsim$ is a wellfounded relation, then $\mathrm{rank}(\precsim)$ is the smallest ordinal on which we can map the field of $\precsim$ and carry $\precsim$ to the usual ordering on ordinals.

THEOREM 8. *Let $\mathfrak{F}$ be reasonable, nonmonotone on A. Then*

$$\begin{aligned}\|\mathfrak{F}\| &= \textit{supremum } \{\mathrm{rank}(\precsim) : \precsim \textit{ is an } \mathfrak{F}\textit{-hyperdefinable prewellordering on some } A^n\}\,,\\ &= \textit{supremum } \{\mathrm{rank}(\precsim) : \precsim \textit{ is an } \mathfrak{F}\textit{-coinductive wellfounded relation on some } A^n\}.\end{aligned}$$

Moreover, none of these suprema are attained.

Proof is exactly like that of Theorem (2B.5) in EIAS and we omit it. $\blacksquare$

Recall from Section 9A of EIAS that if Γ is a collection of first order relations (of all numbers of arguments) on A, $P \subseteq A^n$ and $\sigma: P \to \lambda$ is a *norm* on P, we call σ a Γ-*norm* if there are relations $J_\sigma, \check{J}_\sigma$ in Γ such that

$$\bar{y} \in P \Rightarrow (\forall \bar{x})\{[\bar{x} \in P \;\&\; \sigma(\bar{x}) \leqslant \sigma(\bar{y})] \Leftrightarrow J_\sigma(\bar{x}, \bar{y}) \Leftrightarrow \check{J}_\sigma(\bar{x}, \bar{y})\}\,.$$

For Γ closed under &, $\vee$ and trivial combinatorial substitutions, this is equivalent to assuming that both relations

$$\bar{x} \leqslant_\sigma^* \bar{y} \Leftrightarrow \bar{x} \in P \;\&\; \neg[\bar{y} \in P \;\&\; \sigma(\bar{y}) < \sigma(\bar{x})]\,,$$
$$\bar{x} <_\sigma^* \bar{y} \Leftrightarrow \bar{x} \in P \;\&\; \neg[\bar{y} \in P \;\&\; \sigma(\bar{y}) \leqslant \sigma(\bar{x})]$$

are in Γ, by the argument given in 3A of EIAS.

THEOREM 9. *If $\mathfrak{F}$ is reasonable, nonmonotone on A, then every $\mathfrak{F}$-inductive relation admits an $\mathfrak{F}$-inductive norm.* (The Prewellordering theorem.)

Proof is exactly that of Theorem (3A.3) in EIAS and we omit it. $\blacksquare$

From the Prewellordering theorem we get immediately the Reduction property for $\mathfrak{F}$-IND and the Separation property for the dual class $\neg\mathfrak{F}$-IND, exactly as in (3A.4) and (3A.5) of EIAS. Some versions of the remaining structure results in 3B, 3C of EIAS go through in this situation too, but not all, since their proofs often use closure of $\mathfrak{F}$-IND under E.

4. Typical, nonmonotone classes of operators. Here we strengthen the hypotheses on $\mathfrak{F}$ so that we can show that $\mathfrak{F}$-IND is a Spector class in the sense of EIAS 9A. A class of operators $\mathfrak{F}$ on A is *typical, nonmonotone* if it satisfies Conditions (A), (B), (C) of Section 2 and also the following additional three conditions.

CONDITION (D). *$\mathfrak{F}$ contains all second order relations definable by existential formulas of the trivial structure $\langle A\rangle$.*

CONDITION (E). *There is an ordering $\leqslant \subseteq A \times A$, isomorphic to the ordering on ω and a one-to-one function $f: A\times A \to A$ which are $\mathfrak{F}$-hyperdefinable.*

CONDITION (F). *For each $n \geqslant 1$, there is an $\mathfrak{F}$-inductive $(n+1)$-ary relation U^n which parametrizes the n-ary $\mathfrak{F}$-inductive relations, i.e. for $R \subseteq A^n$,*

$$R \in \mathfrak{F}\text{-IND} \Leftrightarrow \textit{for some } a \in A,\ R = U_a^n = \{\bar{x} : (a, \bar{x}) \in U^n\}\,.$$

It is not hard to verify that Condition (F) is implied by the following

CONDITION (F′). *For each signature $\nu = (n, r_1, \ldots, r_k)$ there is a relation $\psi(a, \bar{x}, \bar{Y})$ in $\mathfrak{F}$ of signature $(n+1, r_1, \ldots, r_k)$ which parametrizes the ν-ary relations in $\mathfrak{F}$, i.e. a ν-ary φ is in $\mathfrak{F}$ if and only if there is some $a \in A$ such that*

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow \psi(a, \bar{x}, \bar{Y})\,.$$

This is perhaps a more natural condition than (F) and in most cases we will verify directly that (F′) holds. There are interesting examples where we prove (F) by showing that (F′) holds for some class $\mathfrak{F}'$ satisfying $\mathfrak{F}'$-IND $= \mathfrak{F}$-IND.

These conditions leave out the case of $\Pi_1^0(\mathrm{C})$-inductive definability, but they include all the other examples of Section 1 on "nice" structures. There are grounds for thinking of $\Pi_1^0(\mathrm{C})$ as an "atypical" case.

In the remainder of this paper we will use heavily the material on Spector classes developed in Chapter 9 of EIAS. Recall, in particular, the Parametrization theorem for the self-dual part $\Delta = \Gamma \cap \neg\Gamma$ of a Spector class Γ, 9C.8 of EIAS: for each $n \geqslant 1$, there is a set I^n in $\Gamma - \Delta$ and $(n+1)$-ary relations H^n, $\check{H}^n$ in Γ and $\neg\Gamma$ respectively, such that

(i) if $R \subseteq A^n$, then R is in Δ if and only if there is some $a \in I^n$ such that $R = H^n_a = \{\bar{x}\colon (a, \bar{x}) \in H^n\}$,

(ii) if $a \in I^n$, then $H^n_a = \check{H}^n_a$.

A second order relation φ of signature $(n, r_1, \ldots, r_k)$ is Γ on Δ if the first order relation

$$\varphi^{\#}(\bar{x}, y_1, \ldots, y_k) \Leftrightarrow y_1 \in I^{r_1} \,\&\, \ldots \,\&\, y_k \in I^{r_k} \,\&\, \varphi(\bar{x}, H^{r_1}_{y_1}, \ldots, H^{r_k}_{y_k})$$

is in Γ — this is independent of the particular choice of a parametrization $\{I^n, H^n, \check{H}^n\}_{n\in\omega}$ by (9C.9) of EIAS. We call φ Δ on Δ if both φ and $\neg\varphi$ are Γ on Δ. Several closure properties of these classes of relations are listed in (9C.10) of EIAS and we will refer to them often in the sequel.

Recall also that if Λ is a class of relations,

$$o(\Lambda) = supremum\{\mathrm{rank}(\precsim)\colon \precsim \textit{ is a prewellordering in } \Lambda\}\,.$$

LEMMA. *If $\mathfrak{F}$ is a typical, nonmonotone class of operators on A, $\bar{Q} = Q_1, \ldots, Q_m$ are $\mathfrak{F}$-hyperdefinable and P is inductive on the structure $\langle A, Q_1, \ldots, Q_m\rangle$, then P is $\mathfrak{F}$-inductive.*

Proof. By Exercise 1.15 of EIAS we have

$$P(\bar{x}) \Leftrightarrow (\bar{a}, \bar{x}) \in I_\varphi$$

with a sequence of constants $\bar{a}$ and φ definable by a disjunction of simple existential and universal formulas of $\langle A, Q_1, \ldots, Q_m\rangle$, so that P is $\mathrm{Rel}(\mathfrak{F}; \bar{Q})$-inductive by Conditions (A), (D) and hence $\mathfrak{F}$-inductive by Theorem 4. ∎

THEOREM 10. *If $\mathfrak{F}$ is a typical, nonmonotone class of operators on A, then the collection of relations $\Gamma = \mathfrak{F}$-IND is a Spector class, $o(\Delta) = \|\mathfrak{F}\|$ and every second order relation in $\mathfrak{F}$ is Δ on Δ.*

Proof. Condition (E), Exercise 1.7 of EIAS and the lemma imply immediately that there is a coding scheme in $\Gamma = \mathfrak{F}$-IND and Theorem 3 yields all the necessary closure properties for Γ to be a Spector class. Theorem 9 then implies the Prewellordering property for Γ and the Parametrization property is immediate by Condition (F).

That $o(\Delta) = \|\mathfrak{F}\|$ is an immediate consequence of Theorem 8.

Finally, to prove that every relation in $\mathfrak{F}$ is Δ on Δ we imitate the argument in the proof of Theorem 5. Choosing $\varphi(\bar{x}, Y)$ of signature $(n, 1)$ for simplicity in notation, find operative relations ψ_1, ψ_2, ψ_3 such that

$$a \in I^1 \Leftrightarrow (\bar{c}_1, a) \in I_{\psi_1}\,,$$
$$(a, t) \in H^1 \Leftrightarrow (\bar{c}_2, a, t) \in I_{\psi_2}\,,$$
$$(a, t) \notin \check{H}^1 \Leftrightarrow (\bar{c}_3, a, t) \in I_{\psi_3}$$

and consider the system

$$\varphi_1(\bar{u}_1, a, S_1, \ldots, S_6) \Leftrightarrow \psi_1(\bar{u}_1, a, S_1)\,,$$
$$\varphi_2(\bar{u}_2, a, t, S_1, \ldots, S_6) \Leftrightarrow \psi_2(\bar{u}_2, a, t, S_2)\,,$$
$$\varphi_3(\bar{u}_3, a, t, S_1, \ldots, S_6) \Leftrightarrow \psi_3(\bar{u}_3, a, t, S_3)\,,$$
$$\varphi_4(\bar{x}, a, S_1, \ldots, S_6) \Leftrightarrow S_1(\bar{c}_1, a) \,\&\, (\forall t)[S_2(\bar{c}_2, a, t) \vee S_3(\bar{c}_3, a, t)] \,\&\, \varphi(\bar{x}, \{t\colon S_2(\bar{c}_2, a, t)\})\,,$$
$$\varphi_5(a, z, S_1, \ldots, S_6) \Leftrightarrow S_1(\bar{c}_1, a) \,\&\, z = d \,\&\, (\forall t)[S_2(\bar{c}_2, a, t) \vee S_3(\bar{c}_3, a, t)]\,,$$
$$\varphi_6(\bar{x}, a, S_1, \ldots, S_6) \Leftrightarrow S_5(a, d) \,\&\, \neg S_4(\bar{x}, a)\,.$$

Clearly $\varphi_1, \ldots, \varphi_6$ are all in $\mathfrak{F}$ and it is easy to verify (in the notation of Theorem 1) that

$$(\bar{x}, a) \in J_4 \Leftrightarrow a \in I^1 \,\&\, \varphi(\bar{x}, H^1_a)\,,$$
$$(\bar{x}, a) \in J_6 \Leftrightarrow a \in I^1 \,\&\, \neg\varphi(\bar{x}, H^1_a)\,. \quad ∎$$

This result yields many useful closure properties of $\mathfrak{F}$-IND for a typical $\mathfrak{F}$, in particular those that follow from the closure properties of the class of relations Δ on Δ and the calculus of inductive second order relations, see Theorem (9C.10) of EIAS.

We now verify that the main examples discussed in Section 1 (except for Π^0_1-induction) lead to typical, nonmonotone classes.

THEOREM 11. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$ be an almost acceptable structure, i.e. a structure which admits a hyperelementary coding scheme.*

(i) *For $m, k \geqslant 1$, the classes of Σ^m_k and Π^m_k operators on $\mathfrak{A}$ are typical, nonmonotone.*

(ii) *If $\mathfrak{C}$ is a hyperelementary coding scheme on $\mathfrak{A}$ and $k \geqslant 2$, then the classes of $\Sigma^0_k(\mathfrak{C})$ and $\Pi^0_k(\mathfrak{C})$ operators on $\mathfrak{A}$ are typical, nonmonotone.*

(iii) *If $\mathfrak{C}$, $\mathfrak{C}'$ are hyperelementary coding schemes on $\mathfrak{A}$ and $k \geqslant 2$, then*

$$\Pi^0_k(\mathfrak{C})\text{-IND} = \Pi^0_k(\mathfrak{C}')\text{-IND}\,, \tag{1}$$

$$\|\Pi^0_k(\mathfrak{C})\| = \|\Pi^0_k(\mathfrak{C}')\| \tag{2}$$

and similarly for $\Sigma^0_k(\mathfrak{C})$, $\Sigma^0_k(\mathfrak{C}')$.

Proof. To prove (i) and (ii) we must verify Conditions (A)-(F) for each of the classes of operators in question. Of these (A)-(E) follow trivially from the results we already have. To prove (F) for $\mathfrak{F} = \Sigma^m_k$ (for example), let $\mathfrak{A}' = (\mathfrak{A}, \leqslant, f)$, where $\leqslant$ and f are guaranteed by Condition (E) and recall that $\mathfrak{A}'$ admits a hyperelementary coding scheme, by Exercise 1.7 of EIAS. If Seq, lh, q are decoding relations and functions of such a coding scheme and if we further expand $\mathfrak{A}'$ to $\mathfrak{A}'' = (\mathfrak{A}', \mathrm{Seq}, lh, q)$, it is not hard to verify by well known methods that the class $\mathfrak{F}''$ of Σ_k^m operations on $\mathfrak{A}''$ satisfies Condition (F′) and hence Condition (F). But $\mathfrak{F}''\text{-IND} = \mathfrak{F}\text{-IND}$ by the Lemma and Theorem 4, so that $\mathfrak{F}$ also satisfies (F).

Proof of (iii). The definition we gave in Section 1 for the relations on a structure $\mathfrak{B}$ which are restricted relative to a coding scheme $\mathfrak{C}$ makes perfectly good sense if $\mathfrak{B}$ has functions as well as relations,

$$\mathfrak{B} = \langle B, P_1, \ldots, P_k, g_1, \ldots, g_m\rangle .$$

The prime formulas are more complicated in this case, as we can use $g_1, \ldots, g_m$ to construct complicated terms. We define $\Pi_k^0(\mathfrak{C})$ and $\Sigma_k^0(\mathfrak{C})$ for such structures as before. If G_h is the *graph* of a function h,

$$G_h(\bar{x}, y) \Leftrightarrow h(\bar{x}) = y ,$$

then the trivial equivalences

$$\varphi(h(\bar{x})) \Leftrightarrow (\exists y)[G_h(\bar{x}, y) \,\&\, \varphi(y)] \Leftrightarrow (\forall y)[G_h(\bar{x}, y) \Rightarrow \varphi(y)] ,$$
$$(\forall i \leqslant j)(\exists y)\varphi(i, y) \Leftrightarrow (\exists y)(\forall i \leqslant j)\varphi(i, (y)_i)$$

imply easily that *every $\Pi_k^0(\mathfrak{C})$ relation on $\mathfrak{B}$ is $\Pi_k^0(\mathfrak{C})$ on the associated structure*

$$\mathfrak{B}' = \langle B, P_1, \ldots, P_k, G_{g_1}, \ldots, G_{g_m}\rangle .$$

Suppose now that we are given two hyperelementary coding schemes $\mathfrak{C}$ and $\mathfrak{C}'$ on $\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$ with associated relations and functions N, $\leqslant$, Seq, lh, q and N', $\leqslant'$, Seq′, lh', q'. It is easy to verify that the unique isomorphism g of $\langle N, \leqslant\rangle$ with $\langle N', \leqslant'\rangle$ is hyperelementary on $\mathfrak{A}$. The trivial equivalences

$$(\forall i \leqslant' j)\varphi(i) \Leftrightarrow (\exists m)[g(m) = j \,\&\, (\forall n \leqslant m)\varphi(g(n))]$$
$$\Leftrightarrow (\forall m)[g(m) = j \Rightarrow (\forall n \leqslant m)\varphi(g(n))]$$

and their duals allow us to replace restricted quantification on $\leqslant'$ by restricted quantification on $\leqslant$, so that easily, $\Pi_k^0(\mathfrak{C}')$ relations on $\mathfrak{A}$ are $\Pi_k^0(\mathfrak{C})$ on $\mathfrak{A}' = \langle A, R_1, \ldots, R_l, \leqslant', \mathrm{Seq}', lh', q', g\rangle$ and hence they all are $\Pi_k^0(\mathfrak{C})$ on $\mathfrak{A}'' = \langle A, R_1, \ldots, R_l, \leqslant', \mathrm{Seq}', G_{lh'}, G_{q'}, G_g\rangle$. This means that $\Pi_k^0(\mathfrak{C}')$-inductive relations on $\mathfrak{A}$ are $\Pi_k^0(\mathfrak{C})$-inductive on $\mathfrak{A}''$ and hence they are $\Pi_k^0(\mathfrak{C})$-inductive on $\mathfrak{A}$, since $\mathfrak{A}''$ is a hyperelementary expansion of $\mathfrak{A}$.

The same argument works for Σ_k^0 and then (2) follows by (ii) and Theorem 10. ■

In view of this result we will refer to Σ_k^0- and Π_k^0-induction on an almost acceptable structure $\mathfrak{A}$, meaning $\Sigma_k^0(\mathfrak{C})$- and $\Pi_k^0(\mathfrak{C})$-induction for any hyperelementary $\mathfrak{C}$.

5. Structural characterizations. We aim here to characterize the class $\mathfrak{F}$-IND for a typical nonmonotone $\mathfrak{F}$ as the smallest Spector class Γ such that every relation in $\mathfrak{F}$ is Δ on Δ and which further satisfies a natural "reflection" or "compactness" property relative to $\mathfrak{F}$.

We call a Spector class Γ *compact relative to* $\mathfrak{F}$ or $\mathfrak{F}$-*compact* if for every relation R in Γ and every $\varphi(Y)$ in $\mathfrak{F}$ and every $R^0 \subseteq R$, R^0 in Δ,

$$\varphi(R) \Rightarrow \textit{there exists } R^* \in \Delta,\ R^0 \subseteq R^* \subseteq R,\ \textit{such that } \varphi(R^*) .$$

It is convenient to prove $\mathfrak{F}$-compactness equivalent to two other useful properties.

Theorem 12. *Let Γ be a Spector class on A and let $\mathfrak{F}$ be a reasonable, nonmonotone class of operations on A. Then the following propositions are equivalent:*

(i) *Γ is $\mathfrak{F}$-compact.*

(ii) *For every $R_1, \ldots, R_n$ in Γ and every $\varphi(Y_1, \ldots, Y_n)$ in $\mathfrak{F}$ and every $R_1^0, \ldots, R_n^0$ in Δ, $R_1^0 \subseteq R_1, \ldots, R_n^0 \subseteq R_n$,*

$$\varphi(R_1, \ldots, R_n) \Rightarrow \textit{there exist } R_1^*, \ldots, R_n^* \textit{ in } \Delta,\ R_1^0 \subseteq R_1^* \subseteq R_1, \ldots$$
$$\ldots, R_n^0 \subseteq R_n^* \subseteq R_n,\ \textit{such that } \varphi(R_1^*, \ldots, R_n^*) .$$

(iii) *For every relation R in Γ and every Γ-norm σ on R and every $\varphi(Y)$ in $\mathfrak{F}$,*

$$\varphi(R) \Rightarrow \textit{there exists } \xi < o(\Delta) \textit{ such that } \varphi(\{\bar{x} \in R\colon \sigma(\bar{x}) < \xi\}) .$$

Proof is round-robin style.

Proof of (i) ⇒ (ii). Suppose for simplicity of notation that $R_1 \subseteq A$ and $R_2 \subseteq A^2$. Choose distinct constants a, b in A and put

$$\psi(Y) \Leftrightarrow \varphi(\{y\colon (a, a, y) \in Y\}, \{(x, y)\colon (b, x, y) \in Y\}) ,$$
$$R = \{(a, a, y)\colon y \in R_1\} \cup \{(b, x, y)\colon (x, y) \in R_2\} ,$$
$$R^0 = \{(a, a, y)\colon y \in R_1^0\} \cup \{(b, x, y)\colon (x, y) \in R_2^0\} .$$

Now $\psi(Y)$ is in $\mathfrak{F}$ and $\psi(R)$ holds, so by (i) there is some $R^* \in \Delta$, $R^0 \subseteq R^* \subseteq R$ such that $\psi(R^*)$ holds; the conclusion of (ii) is satisfied with

$$R_1^* = \{y\colon (a, a, y) \in R^*\} ,\quad R_2^* = \{(x, y)\colon (b, x, y) \in R^*\} .$$

Proof of (ii) ⇒ (iii). Assume the hypotheses of (iii) and put

$$\psi(Y, Z, W) \Leftrightarrow \varphi(Y) \,\&\, (\forall\bar{x})(\forall\bar{y})[y \in Y \Rightarrow [(\bar{x}, \bar{y}) \in Z \vee (\bar{y}, \bar{x}) \in W]]$$
$$\&\, (\forall\bar{x})(\forall\bar{y})[[\bar{y} \in Y \,\&\, (\bar{x}, \bar{y}) \in Z] \Rightarrow \bar{x} \in Y] .$$

Clearly $\psi(Y, Z, W)$ is in $\mathfrak{F}$ and $\psi(R, \leqslant_\sigma^*, <_\sigma^*)$ holds, so by (ii) there are relations $R^* \subseteq R$, $Q_1^* \subseteq \leqslant_\sigma^*$, $Q_2^* \subseteq <_\sigma^*$ in Δ such that $\psi(R^*, Q_1^*, Q_2^*)$ holds. Since

$$\bar{y} \in R^* \Rightarrow (\bar{x}, \bar{y}) \in Q_1^* \vee (\bar{y}, \bar{x}) \in Q_2^*$$

and for $\bar{y} \in R^*$ we cannot have $(\bar{x}, \bar{y}) \in Q_1^* \,\&\, (\bar{y}, \bar{x}) \in Q_2^*$, as this would imply $\bar{x} \leqslant_\sigma^* \bar{y} \,\&\, \bar{y} <_\sigma^* \bar{x}$, we must have

$$\bar{y} \in R^* \Rightarrow (\forall \bar{x})[\bar{x} \leqslant_\sigma^* \bar{y} \Leftrightarrow Q_1^*(\bar{x}, \bar{y})] .$$

Thus

$$\bar{y} \in R^* \,\&\, \bar{x} \leqslant_\sigma^* \bar{y} \Rightarrow \bar{x} \in R^* .$$

This together with the Covering theorem 9C.6 of EIAS implies immediately that there is some $\xi < o(\Delta)$ such that

$$R^* = \{\bar{x} \in R\colon\ \sigma(\bar{x}) < \xi\} ,$$

which proves (iii).

Proof of (iii) $\Rightarrow$ (i). Assuming the hypothesis of (i), that R is in Γ, $R^0 \subseteq R$ and R^0 is in Δ and $\varphi(R)$ holds, choose a Γ-norm σ on R and using the Covering Theorem 9C.6 of EIAS choose some $\eta < o(\Delta)$ such that

$$R^0 \subseteq \{\bar{x} \in R\colon\ \sigma(\bar{x}) < \eta\} .$$

Put

$$\bar{x} \leqslant_\tau \bar{x}' \Leftrightarrow \bar{x}, \bar{x}' \in R \,\&\, [\sigma(\bar{x}) < \eta \vee \sigma(\bar{x}) \leqslant \sigma(\bar{x}')]$$

and verify easily that there is a Γ-norm τ on R such that

$$\bar{x} \leqslant_\tau \bar{x}' \Leftrightarrow \tau(\bar{x}) \leqslant \tau(\bar{x}') .$$

The point is that for this τ,

$$\bar{x} \in R^0 \Rightarrow \tau(\bar{x}) = 0 .$$

Assuming without loss of generality that $R \neq \emptyset$, choose $\bar{x}_0 \in R$ and notice that $\varphi(R) \,\&\, \bar{x}_0 \in R$ is true; hence by property (iii) there is some $\xi < o(\Delta)$ such that

$$\varphi(\{\bar{x} \in R\colon\ \tau(\bar{x}) < \xi\}) \,\&\, \tau(\bar{x}_0) < \xi$$

is true and the conclusion of (i) follows immediately by taking

$$R^* = \{\bar{x} \in R\colon\ \tau(\bar{x}) < \xi\} . \qquad \blacksquare$$

It is convenient to break the proof of the characterization we seek in two parts which have some independent interest.

THEOREM 13. *Let $\mathfrak{F}$ be a typical, nonmonotone class of operators on a set A. Then $\mathfrak{F}$-IND is $\mathfrak{F}$-compact.*

Proof. Assume that

$$R(\bar{x}) \Leftrightarrow (\bar{a}, \bar{x}) \in I_\psi$$

is an $\mathfrak{F}$-inductive relation, that $R^0 \subseteq R$, $R^0 \in \mathfrak{F}$-HYP and that $\varphi(Y)$ is some relation in $\mathfrak{F}$ such that $\varphi(R)$ holds. Consider the following system of simultaneous induction in $\mathrm{Rel}(\mathfrak{F}; R^0)$:

$$\varphi_1(\bar{u}, \bar{x}, S_1, S_2) \Leftrightarrow \psi(\bar{u}, \bar{x}, S_1) ,$$

$$\varphi_2(t, S_1, S_2) \Leftrightarrow t = b \,\&\, (\forall \bar{x})[R^0(\bar{x}) \Rightarrow S_1(\bar{a}, \bar{x})] \,\&\, \varphi(\{\bar{x}\colon\ S_1(\bar{a}, \bar{x})\}) .$$

In the notation of Theorem 1 we obviously have $J_1 = I_\psi$, so that $\{\bar{x}\colon\ J_1(\bar{a}, \bar{x})\} = R$ and by the definition, $b \in J_2$. Thus there is an ordinal $\lambda < \|\mathrm{Rel}(\mathfrak{F}; R^0)\| = \|\mathfrak{F}\| = o(\Delta)$ such that $b \in J_2^\lambda$. Put $R^* = \{\bar{x}\colon\ (\bar{a}, \bar{x}) \in J_1^{<\lambda}\}$ and notice that R^* is $\mathfrak{F}$-HYP by Theorems 1, 4 and 7. Clearly $R^* \subseteq R$ and $\varphi(R^*)$ holds, which proves (ii). $\blacksquare$

THEOREM 14. *Let Γ be a Spector class on A and $\varphi(\bar{x}, S)$ an operative relation which is Δ on Δ, let $\varkappa = o(\Delta)$. Then the set $I_\varphi^{<\varkappa}$ is in Γ, each $I_\varphi^{<\xi}$ for $\xi < \varkappa$ is in Δ and the norm*

$$\sigma(\bar{x}) = |\bar{x}|_\varphi \qquad (\bar{x} \in I_\varphi^{<\varkappa})$$

is a Γ-norm. If Γ is also $\mathfrak{F}$-compact for some reasonable $\mathfrak{F}$ which contains φ, then $\|\varphi\| \leqslant \varkappa$, so that $I_\varphi = I_\varphi^{<\varkappa}$ is in Γ.

Proof. Given Γ and φ satisfying the hypotheses, define the second order relation $\psi(Y, Z)$, with Y varying over binary relations on A and Z varying over $(n+1)$-ary relations, by

$$\begin{aligned}\psi(Y, Z) \Leftrightarrow\ & Y \text{ is a prewellordering} \\ & \&\, (\forall t)\{t \in \mathit{Field}(Y) \Rightarrow (\forall \bar{x})\big[Z(t, \bar{x}) \Leftrightarrow \big[(\exists s <_Y t) Z(s, \bar{x}) \vee \\ & \qquad \vee \varphi(\bar{x}, \{\bar{x}'\colon\ (\exists s <_Y t) Z(s, \bar{x}')\})\big]\big]\} ,\end{aligned}$$

where naturally

$$s <_Y t \Leftrightarrow Y(s, t) \,\&\, \neg Y(t, s) .$$

Clearly ψ is Γ on Δ, using the hypothesis and (6A.1), (9C.10) of EIAS, so fixing a parametrization $\{I^n, H^n, \bar{H}^n\}_{n \in \omega}$ of Δ as described in Section 4, the relation

$$\psi^{\#}(a, b) \Leftrightarrow a \in I^2 \,\&\, b \in I^{n+1} \,\&\, \psi(H_a^2, H_b^{n+1})$$

is in Γ.

If H_a^2 is a prewellordering and $t \in \mathit{Field}(H_a^2)$, let us temporarily denote by $|t|_a$ the ordinal (the rank) of t in H_a^2. It is clear from the definition that for prewellorderings H_a^2,

$$t \in \mathit{Field}(H_a^2) \,\&\, \xi = |t|_a \,\&\, \psi(H_a^2, Z) \Rightarrow Z_t = \{\bar{x}\colon\ Z(t, \bar{x})\} = I_\varphi^\xi .$$

An easy transfinite induction on $|t|_a$, using the version for Spector classes of the Collection Theorem (6D.3) of EIAS shows that for each $u \in Field(H_a^2)$ there is some $Z \in \Delta$ such that

$$(\forall t <_a u)(\forall \bar{x})\big[Z(t, \bar{x}) \Leftrightarrow [(\exists s <_a t) Z(s, \bar{x}) \vee \varphi(\bar{x}, \{\bar{x}' : (\exists s < t) Z(s, \bar{x}')\})]\big],$$

where of course $s \lesssim_a t \Leftrightarrow (s, t) \in H_a$. Then another application of Collection gives

$$(\forall a)\{[a \in I^2 \;\&\; H_a^2 \textit{ a prewellordering}] \Rightarrow (\exists b)\psi^{\#}(a, b)\}.$$

Thus if $\varkappa = o(\Delta)$,

$$\bar{x} \in I_\varphi^{<\varkappa} \Leftrightarrow (\exists a)(\exists b)[\psi^{\#}(a, b) \;\&\; (\exists t)[t \in Field(H_a^2) \;\&\; H_b^{n+1}(t, \bar{x})]]$$

and the set $I_\varphi^{<\varkappa}$ is in Γ.

A similar argument shows that the function

$$\sigma(\bar{x}) = |\bar{x}|_\varphi$$

is a Γ-norm on $I_\varphi^{<\varkappa}$, since e.g.

$$\bar{x} \leqslant_\sigma^* \bar{y} \Leftrightarrow (\exists a)(\exists b)\{\psi^{\#}(a, b) \;\&\; (\exists t)[t \in Field(H_a^2) \;\&\; H_b^{n+1}(t, \bar{x})$$
$$\& (\forall s)[s <_a t \Rightarrow \neg \breve{H}_b^{n+1}(s, \bar{y})]]\}.$$

This of course implies immediately that for each $\xi < o(\Delta)$, the set $I_\varphi^{<\xi}$ is in Δ.

Now assume the extra hypotheses for the second part of the Theorem. We have

$$\varphi(\bar{x}, I_\varphi^{<\varkappa}) \Rightarrow \text{for some } \xi < \varkappa,\ \varphi(\bar{x}, I_\varphi^{<\xi}) \text{ by } \mathfrak{F}\text{-Compactness}$$
$$\Rightarrow \text{for some } \xi < \varkappa,\ \bar{x} \in I_\varphi^{\xi},$$

so that $\|\varphi\| \leqslant \varkappa$ and the proof is complete. ■

These two results give immediately

THEOREM 15. *If $\mathfrak{F}$ is a typical, nonmonotone class of operators on A, then $\mathfrak{F}$-IND is the smallest $\mathfrak{F}$-compact Spector class on A such that every relation in $\mathfrak{F}$ is Δ on Δ.* ■

6. The main examples. We aim here to characterize Π_k^0-, Σ_k^0-, Π_k^m- and Σ_k^m-induction on an almost acceptable structure in terms of two very natural model theoretic notions of *compactness* and *reflection.*

A Spector class Γ is $\forall_k$*-compact* ($k \geqslant 1$) if for every $\forall_k$ formula $\varphi(Y)$ of the trivial structure $\langle A\rangle$ (i.e. with no relation constants other than $=$) and every $R \in \Gamma$, $R^0 \in \Delta$, $R^0 \subseteq R$,

$$\varphi(R) \Rightarrow \textit{there exists } R^* \in \Delta,\ R^0 \subseteq R^* \subseteq R, \textit{ such that } \varphi(R^*).$$

This is simply to say that Γ is compact relative to the class of all operators which are definable by $\forall_k$ formulas on $\langle A\rangle$. We define $\exists_k$-, Σ_k^m- and Π_k^m-compactness in the same way.

If B is a set in some Spector class Γ and $R \subseteq B^n$ is a relation on B, we say that R *is Δ on B* if both R and $B^n - R$ are in Γ. (Notice that this does not imply that R is in Δ, unless B is in Δ.) A Spector class Γ satisfies the Σ_k^m *reflection principle* or is Σ_k^m*-reflecting* ($m, k \geqslant 1$), if for every B in Γ and relations $R_1, \ldots, R_l$ which are Δ on B, for every $B^0 \subseteq B$, $B^0 \in \Delta$ and for every Σ_k^m sentence θ of the structure $\mathfrak{B} = \langle B, R_1, \ldots, R_l\rangle$,

$$\langle B, R_1, \ldots, R_l\rangle \models \theta \Rightarrow \textit{there exists } B^* \in \Delta,\ B^0 \subseteq B^* \subseteq B \textit{ such that}$$
$$\langle B^*, R_1 \restriction B^*, \ldots, R_l \restriction B^*\rangle \models \theta.$$

The notions of Π_k^m-, $\exists_k$- and $\forall_k$-reflection are defined similarly.

We first establish the connection between these two notions.

THEOREM 16 (i). *If Γ is a Spector class on A, and $m, k \geqslant 1$, then*

$$\Gamma \textit{ is } \Sigma_k^m\textit{-compact} \Leftrightarrow \Gamma \textit{ is } \Sigma_k^m\textit{-reflecting},$$

$$\Gamma \textit{ is } \Pi_k^m\textit{-compact} \Leftrightarrow \Gamma \textit{ is } \Pi_k^m\textit{-reflecting}.$$

(ii) *For each Spector class Γ and each $k \geqslant 1$, the following conditions are equivalent.*

(a) *Γ is $\forall_k$-compact.*

(b) *Γ is $\forall_{k+1}$-reflecting.*

(c) *For each $R_1, \ldots, R_l$ in Δ and each coding scheme $\mathfrak{C}$ in Δ, Γ is compact relative to the class of operators*

$$\mathfrak{F} = \textit{all } \Pi_k^0(\mathfrak{C}) \textit{ operators on } \mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle.$$

Proof. We prove only (ii), since the proof of (i) is similar and a bit simpler. In fact, to simplify notation let us just show the equivalence of (a), (b) and (c) with $k = 2$, since this argument is perfectly general.

Proof of (a) $\Rightarrow$ (b). Given $\mathfrak{B} = \langle B, R_1, \ldots, R_l\rangle$ with B in Γ and $R_1, \ldots, R_l$, $P_1 = B^{n_1} - R_1, \ldots, P_l = B^{n_l} - R_l$ in Γ and given a $\forall_3$ sentence

$$\theta \equiv (\forall \bar{x})(\exists \bar{y})(\forall \bar{z})\psi(\bar{x}, \bar{y}, \bar{z}, R_1, \ldots, R_l)$$

(with ψ quantifier free) such that $\mathfrak{B} \models \theta$ and given also some B^0 in Δ, $B^0 \subseteq B$, put

$$\varphi(X, Y, Z_1, \dots, Z_l, W_1, \dots, W_l)$$

$$\equiv (\forall \bar{x} \in X^n)(\forall \bar{y} \in X^m)[(\bar{x}, \bar{y}) \in Y \Leftrightarrow (\exists \bar{z} \in X^s) \neg \psi(\bar{x}, \bar{y}, \bar{z}, Z_1, \dots, Z_l)]$$

$$\& (\forall \bar{x} \in X^n)(\exists \bar{y} \in X^m)[(\bar{x}, \bar{y}) \notin Y]$$

$$\& (\forall \bar{x}_1 \in X^{n_1})[Z_1(\bar{x}_1) \Leftrightarrow \neg W_1(\bar{x}_1)]$$

$$\& \dots\dots\dots\dots\dots\dots$$

$$\& (\forall \bar{x}_l \in X^{n_l})[Z_l(\bar{x}_l) \Leftrightarrow \neg W_l(\bar{x}_l)] .$$

Now φ is $\forall_2$ on the trivial structure $\langle A \rangle$ and Γ is $\forall_2$-compact and $\varphi(B, C, R_1, \dots, R_l, P_1, \dots, P_l)$ holds, where

$$(\bar{x}, \bar{y}) \in C \Leftrightarrow \bar{x} \in B^n \;\&\; \bar{y} \in B^m \;\&\; (\exists \bar{z} \in B^s) \neg \psi(\bar{x}, \bar{y}, \bar{z}, R_1, \dots, R_l) ;$$

also $B, R_1, \dots, R_l, P_1, \dots, P_l$ are in Γ and so is C — this is the significant observation. Now choose $C^*, R_1^*, \dots, R_l^*, P_1^*, \dots, P_l^*$ in Δ so that $\varphi(B^*, C^*, \dots)$ holds and $B^0 \subseteq B^* \subseteq B$ and verify easily that

$$R_1^* = R_1 \upharpoonright B^*, \dots, R_l^* = R_l \upharpoonright B^* ,$$

$$\bar{x} \in (B^*)^n, \bar{y} \in (B^*)^m \Rightarrow \big[(\bar{x}, \bar{y}) \in C^* \Leftrightarrow \big(\exists \bar{z} \in (B^*)^s\big) \neg \psi(\bar{x}, \bar{y}, \bar{z}, R_1^*, \dots, R_l^*)\big]$$

and finally

$$\langle B^*, R_1^* \upharpoonright B, \dots, R_l^* \upharpoonright B^* \rangle \models \theta .$$

Proof of (b) $\Rightarrow$ (c). We need two lemmas.

LEMMA 1. *Let $\mathfrak{B} = \langle B, R_1, \dots, R_l, g_1, \dots, g_m \rangle$ be a structure, perhaps with functions, let $\varphi(\bar{x})$ be a quantifier free formula in the language of $\mathfrak{B}$. There is an $\exists_1$ formula $\varphi'(\bar{x})$ in the language of the associated structure with no functions*

$$\mathfrak{B}' = \langle B, R_1, \dots, R_l, G_{g_1}, \dots, G_{g_m} \rangle$$

such that for every substructure $\mathfrak{C}$ of $\mathfrak{B}$,

$$\mathfrak{C} \models \varphi(\bar{x}) \Leftrightarrow \mathfrak{C}' \models \varphi'(\bar{x}) .$$

Proof is trivial, using the substitutions

$$\varphi(h(\bar{t})) \Leftrightarrow (\exists y)[G_h(\bar{t}, y) \;\&\; \varphi(y)] \Leftrightarrow (\forall y)[G_h(\bar{t}, y) \Rightarrow \varphi(y)] .$$

In applying this observation we should keep in mind that the substructures of $\mathfrak{B}$ are (by definition) closed under $g_1, \dots, g_m$, so they correspond to the substructures of $\mathfrak{B}'$ which satisfy the $\forall_2$ sentence

$$(\forall \bar{x}_1)(\exists y) G_{g_1}(\bar{x}_1, y) \;\&\; \dots \;\&\; (\forall \bar{x}_m)(\exists y) G_{g_m}(\bar{x}_m, y) .$$

LEMMA 2. *Let Γ be a $\forall_3$-reflecting Spector class and assume that $B \in \Gamma$, $R_1, \dots, R_l$ are Δ on B, $g_1, \dots, g_m$ are functions in Δ, $\mathfrak{C}$ is a coding scheme on B with associated relations and functions $\leqslant$, Seq, lh, q in Δ and θ is a $\Pi_3^0(\mathfrak{C})$ sentence such that*

$$\mathfrak{B} = \langle B, R_1, \dots, R_l, g_1, \dots, g_m, \leqslant, \mathrm{Seq}, lh, q \rangle \models \theta .$$

Then for every $B^0 \subseteq B$, B^0 in Δ there is some B^* in Δ, $B^0 \subseteq B^* \subseteq B$, such that B^* is closed under $g_1, \dots, g_m, lh, q$ and the coding $\langle\ \rangle^{\mathfrak{C}}$, and

$$\mathfrak{B}^* = \langle B^*, R_1 \upharpoonright B^*, \dots, R_l \upharpoonright B^*, g_1 \upharpoonright B^*, \dots$$
$$\dots, g_m \upharpoonright B^* \leqslant, \mathrm{Seq}, lh \upharpoonright B^*, q \upharpoonright B^* \rangle \models \theta .$$

Proof. Assume for simplicity that

$$\theta \equiv (\forall \bar{x})(\exists \bar{y})(\exists i)(\forall \bar{z})(\forall j \leqslant i)(\exists m \leqslant j) \psi(\bar{x}, \bar{y}, i, \bar{z}, j, m)$$

with ψ quantifier free, the argument for this case being perfectly general. Define on B

$$P_1(\bar{x}, \bar{y}; i, \bar{z}, j, m) \Leftrightarrow \psi(\bar{x}, \bar{y}, i, \bar{z}, j, m) ,$$

$$P_2(\bar{x}, \bar{y}, i, \bar{z}, j) \Leftrightarrow (\exists m \leqslant j) P_1(\bar{x}, \bar{y}, i, \bar{z}, j, m) ,$$

$$P_3(\bar{x}, \bar{y}, i, \bar{z}) \Leftrightarrow (\forall j \leqslant i) P_2(\bar{x}, \bar{y}, i, \bar{z})$$

and notice that P_1, P_2, P_3 are Δ on B. Let

$$\theta' \equiv (\forall \bar{x})(\exists \bar{y})(\exists i)(\forall \bar{z}) P_3(\bar{x}, \bar{y}, i, \bar{z})$$

$$\& (\forall \bar{x})(\forall \bar{y})(\forall i)(\forall \bar{z})(\forall j)(\forall m)[P_1(\bar{x}, \bar{y}, i, \bar{z}, j, m) \Leftrightarrow \psi'(\bar{x}, \bar{y}, i, \bar{z}, j, m)]$$

$$\& (\forall \bar{x}(\forall \bar{y})(\forall i)(\forall \bar{z})(\forall j)[P_2(\bar{x}, \bar{y}, i, \bar{z}, j) \Leftrightarrow (\exists m \leqslant j) P_1(\bar{x}, \bar{y}, i, \bar{z}, j, m)]$$

$$\& (\forall \bar{x})(\forall \bar{y})(\forall i)(\forall \bar{z})[P_3(\bar{x}, \bar{y}, i, \bar{z}) \Leftrightarrow (\forall j \leqslant i) P_2(\bar{x}, \bar{y}, i, \bar{z}, j)]$$

$$\& (\forall \bar{x}_1)(\exists y) G_{g_1}(\bar{x}_1, y)$$

$$\& \dots\dots\dots\dots\dots\dots$$

$$\& (\forall x)(\forall i)(\exists y) G_q(x, i, y)$$

$$\& \langle \emptyset \rangle = \langle \emptyset \rangle \;\&\; (\forall u)(\forall x)\big[\mathrm{Seq}(u) \Rightarrow (\exists v)[\mathrm{Seq}(v) \;\&\; v = u\hat{\ }\langle x \rangle]\big] ,$$

where ψ' is the $\exists_1$ formula associated with ψ by Lemma 1. Since θ' is $\forall_3$, there is some $B^* \in \Delta$ such that $B^0 \cup N \subseteq B^* \subseteq B$ $\big(N = \mathit{Field}(\leqslant)\big)$ and such that the restriction of $\mathfrak{B}'$ to B^* satisfies θ'. Since B^* is closed under the functions $g_1, \dots, g_m, lh, q$, it induces a substructure $\mathfrak{B}^*$ of $\mathfrak{B}$ as in the statement of the lemma and then the definition of θ' and Lemma 1 make it obvious that $\mathfrak{B}^* \models \theta$.

To prove now (b) ⇒ (c), assume that Γ is $\forall_3$-reflecting and $R_1, \ldots, R_l \in \Delta$, $\mathfrak{C}$ is a coding scheme with associated relations and functions $\leqslant$, Seq, lh, q in Δ, $\varphi(Y)$ is $\Pi_2^0(\mathfrak{C})$ on $\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$ and $\varphi(R)$ holds, for some R in Γ. We must show that for each $R^0 \subseteq R$, R^0 in Δ, there is some $R^* \in \Delta$, $R^0 \subseteq R^* \subseteq R$, such that $\varphi(R^*)$. Say

$$\varphi(Y) \Leftrightarrow (\forall \bar{u})(\exists \bar{v})\psi(\bar{u}, \bar{v}, Y),$$

with ψ quantifier free. We aim to apply Lemma 2 with a suitable $\mathfrak{B}$. Take

$$B_0 = \{\langle 0, t\rangle \colon t \in A\},$$

$$B_1 = \{\langle 1, \bar{x}\rangle \colon R(\bar{x})\},$$

$$B = B_0 \cup B_1 .$$

This B will be the domain of the structure $\mathfrak{B}$. Notice that $B \in \Gamma$. The projection map

$$\pi(t) = (t)_2$$

gives a one-to-one correspondence of B_0 with A and we can use it to represent $\mathfrak{A}$ as a reduct of a substructure of $\mathfrak{B}$. For each relation P and function g put

$$P^\pi(x_1, \ldots, x_s) \Leftrightarrow P\big(\pi(x_1), \ldots, \pi(x_s)\big),$$

$$g^\pi(x_1, \ldots, x_s) = \pi^{-1}\big(g(\pi(x_1), \ldots, \pi(x_s))\big)$$

and, to begin with, put in $\mathfrak{B}$ the relations and functions $R_1^\pi, \ldots, R_l^\pi, \leqslant^\pi$, Seq^π, lh^π, q^π.

Choose a Γ-norm σ on R and define Q_σ on $(B_0)^n \times B_1$ by

$$Q_\sigma(\langle 0, t_1\rangle, \ldots, \langle 0, t_n\rangle, \langle 1, \bar{x}\rangle) \Leftrightarrow (t_1, \ldots, t_n) \in R \;\&\; \sigma(t_1, \ldots, t_n) \leqslant \sigma(\bar{x}).$$

Easily Q_σ is Δ on B.

Finally let $\mathfrak{C}'$ be a coding scheme on B with associated relations $\leqslant^\pi$, Seq′ which are Δ on B and functions lh', q' which are in Δ — it is not hard to construct such a $\mathfrak{C}'$ using $\mathfrak{C}$ and π — and take

$$\mathfrak{B} = \langle B, B_0, B_1, R_1^\pi, \ldots, R_l^\pi, \leqslant^\pi, \mathrm{Seq}^\pi, lh^\pi, q^\pi, Q_\sigma, \mathrm{Seq}', lh', q'\rangle .$$

We can define R^π by an $\exists_1$ formula of $\mathfrak{B}$,

$$\begin{aligned} R^\pi(y_1, \ldots, y_n) &\Leftrightarrow R\big(\pi(y_1), \ldots, \pi(y_n)\big) \\ &\Leftrightarrow (\exists \bar{x})\big[R(\bar{x}) \;\&\; \sigma\big(\pi(y_1), \ldots, \pi(y_n)\big) \leqslant \sigma(\bar{x})\big] \\ &\Leftrightarrow (\exists x) Q_\sigma(y_1, \ldots, y_n, x). \end{aligned}$$

Using this definition we can express in $\mathfrak{B}$ the fact that $\varphi(R)$ holds by the sentence

$$\theta \equiv (\forall \bar{u} \in (B_0)^s)(\exists \bar{v} \in (B_0)^t)\psi^\pi\big(\bar{u}, \bar{v}, \{\bar{y} \colon (\exists x) Q_\sigma(\bar{y}, x)\}\big),$$

where ψ^π comes from ψ by replacing each relation constant P by P^π and each function symbol g by g^π. We can find a $\Pi_3^0(\mathfrak{C}')$ sentence θ' equivalent to θ by advancing the quantifier $(\exists x)$ past the restricted quantifiers in ψ^π via the trivial equivalence

$$(\forall i \leqslant^\pi j)(\exists x)\chi(i, x) \Leftrightarrow (\exists x)(\forall i \leqslant^\pi j)\chi\big(i, (x)_i'\big);$$

in fact θ is equivalent to θ' on every substructure of $\mathfrak{B}$ closed under $\langle\ \rangle^{\mathfrak{C}'}$.

Now apply Lemma 2 to $\mathfrak{B}$ and θ' to get some $B^* \in \Delta$ such that the induced subtructure $\mathfrak{B}^* \models \theta$ and such that

$$B_0 \cup \{\langle 1, \bar{x}\rangle \colon \bar{x} \in R^0\} \subseteq B^*,$$

and take

$$R^* = \{\bar{y} \colon (\exists \bar{x})[\langle 1, \bar{x}\rangle \in B^* \;\&\; \sigma(\bar{y}) \leqslant \sigma(\bar{x})]\}.$$

Clearly $R^* \in \Delta$, $R^0 \subseteq R^* \subseteq R$ and the fact that $\mathfrak{B}^*$ satisfies θ simply means that $(\forall \bar{u})(\exists \bar{v})\psi(\bar{u}, \bar{v}, R^*)$ holds, which is what we needed to show.

The implication (c) ⇒ (a) is trivial. ∎

Some of the annoying technical details of this proof would not be necessary if we had allowed structures with relations and functions in the definition of $\forall_k$-reflection. Perhaps it is worth doing this extra work to get as elegant a characterization of Π_k^0-IND as possible, especially since the definition of this class is a bit complicated.

THEOREM 17. (i) *For each $k \geqslant 1$, a Spector class Γ is $\exists_{k+1}$-compact if and only if Γ is $\forall_k$-compact.*

(ii) *Every Spector class is $\exists_2$-compact.* (Essentially Grilliot [1971].)

Proof. (i) is completely trivial; if $\varphi(Y)$ is $\exists_{k+1}$ on $\langle A\rangle$, i.e.

$$\varphi(Y) \Leftrightarrow (\exists \bar{y})\psi(\bar{y}, Y)$$

with $\psi(\bar{y}, Y)$ a $\forall_k$ relation and if $\varphi(R)$ holds for some $R \in \Gamma$ and $R^0 \subseteq R$, R^0 in Δ, then

$$\begin{aligned} \varphi(R) &\Rightarrow \text{for some } \bar{y}, \psi(\bar{y}, R) \\ &\Rightarrow \text{for some } \bar{y} \text{ and some } R^* \in \Delta,\ R^0 \subseteq R^* \subseteq R \text{ and } \psi(\bar{y}, R^*) \\ &\qquad (\text{by } \forall_k\text{-compactness}) \\ &\Rightarrow \text{for some } R^* \in \Delta,\ R^0 \subseteq R^* \subseteq R \text{ and } \varphi(R^*). \end{aligned}$$

Proof of (ii). By Theorem 16, it is enough to prove that every Spector class is $\forall_2$-reflecting, so assume that $B \in \Gamma$, $R_1, \dots, R_l$ are Δ on Γ, $B^0 \subseteq B$, $B^0 \in \Delta$ and

$$\theta \equiv (\forall x)(\exists y)\psi(x, y)$$

is a $\forall_2$ sentence such that

$$\mathfrak{B} = \langle B, R_1, \dots, R_l\rangle \models \theta .$$

(We are assuming for simplicity of notation that θ has one $\forall$ and one $\exists$ quantifier, the more general case being only a bit messier.)

We may assume that B is not in $\neg\Gamma$, since otherwise we can take $B^* = B$. Let $\sigma\colon B \to o(\Delta)$ be a Γ-norm on B, put

$$B^{<\xi} = \{x \in B\colon \sigma(x) < \xi\}$$

and for each $\xi < o(\Delta)$ define

$$f(\xi) = \text{least } \eta < \xi \text{ such that } (\forall x \in B^{<\xi})(\exists y \in B^{<\eta})\psi(x, y) .$$

To see that $f(\xi) < o(\Delta)$, put

$$\begin{aligned} Q(u, v) \Leftrightarrow u \in B \mathbin{\&} v \in B \mathbin{\&} \sigma(u) < \sigma(v) \\ \mathbin{\&} (\forall x <^*_\sigma u)(\exists y <^*_\sigma v)\psi(x, y) \\ \mathbin{\&} (\forall v')[\sigma(u) < \sigma(v') < \sigma(v) \Rightarrow (\exists x <^*_\sigma u)(\forall y <^*_\sigma v')\neg\psi(x, y)] , \end{aligned}$$

and verify easily that Q is in Γ and

$$Q(u, v) \Rightarrow f(\sigma(u)) = \sigma(v) .$$

Thus to show $\xi < o(\Delta) \Rightarrow f(\xi) < o(\Delta)$, it is enough to verify that for each $u \in B$, $(\exists v)Q(u, v)$, which is immediate since the contrary hypothesis yields

$$v \in B \Leftrightarrow (\exists x)\big[\sigma(x) < \sigma(u) \mathbin{\&} (\forall y)[(y \in B \mathbin{\&} \psi(x, y)) \Rightarrow \neg(y <^*_\sigma v)]\big]$$

which implies that B is in $\neg\Gamma$.

The Covering theorem for Spector classes, 9C.2 of EIAS, implies that there is some η such that $B^0 \subseteq B^{<\eta}$. Put

$$\xi_0 = \eta , \quad \xi_{n+1} = f(\xi_n) , \quad \xi = \text{limit}_n \xi_n .$$

Again $\xi < o(\Delta)$, since otherwise

$$v \in B \Leftrightarrow (\exists n \geqslant 2)(\forall u)\{(\forall i < n-1)Q((u)_{i+1}, (u)_{i+2}) \Rightarrow \neg((u)_n <^*_\sigma v)\}$$

which implies B is in $\neg\Gamma$. Now take

$$B^* = B^{<\xi} = \bigcup_n B^{<\xi_n}$$

and verify immediately that $\langle B^*, R_1 \restriction B^*, \dots, R_l \restriction B^*\rangle \models \theta$. ■

Grilliot proves in his [1971] that a specific Spector class satisfies (essentially) property (c) of Theorem 16 with $k = 1$. His argument is perfectly general but more complicated than the one above, since he is verifying directly the more complex property of compactness rather than reflection.

A Spector class Γ on A is *Σ^m_k-admissible* ($m, k \geqslant 1$) if every Σ^m_k second order relation on the trivial structure A is Δ on Δ. Clearly, the corresponding notion of *Π^m_k-admissibility* is equivalent to Σ^m_k-admissibility.

THEOREM 18. *Let $\mathfrak{A} = \langle A, R_1, \dots, R_l\rangle$ be an almost acceptable structure and take all inductions below on $\mathfrak{A}$.*

(i) Σ^0_2-IND $=$ *the smallest Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$*
$=$ *the class of all inductive relations on $\mathfrak{A}$.*

(ii) *For each $k \geqslant 2$,*
Π^0_k-IND $= \Sigma^0_{k+1}$-IND
$=$ *the smallest $\forall_k$-compact Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$*
$=$ *the smallest $\forall_{k+1}$-reflecting Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$.*

(iii) *For each $m, k \geqslant 1$,*
Σ^m_k-IND $=$ *the smallest Σ^m_k-admissible, Σ^m_k-compact Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$*
$=$ *the smallest Σ^m_k-admissible, Σ^m_k-reflecting Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$,*
Π^m_k-IND $=$ *the smallest Π^m_k-admissible, Π^m_k-compact Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$*
$=$ *the smallest Π^m_k-admissible, Π^m_k-reflecting Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$.*

Proof. To prove (i), notice that on the one hand Σ^0_2-IND is a Spector class on $\mathfrak{A}$ by Theorems 11 and 10 and on the other hand every Spector class Γ on A such that $R_1, \dots, R_l \in \Delta$ is compact relative to the class of $\Sigma^0_2(\mathfrak{C})$ operators on $\mathfrak{A}$ ($\mathfrak{C}$ a hyperelementary coding scheme on $\mathfrak{A}$) by Theorems 17 and 16 and hence contains Σ^0_2-IND $= \Sigma^0_2(\mathfrak{C})$-IND by Theorem 15. The proof of (ii) is equally simple.

Proof of (iii). Σ^m_k-IND is Σ^m_k-admissible and Σ^m_k-compact by Theorem 15. For the converse assume that Γ is Σ^m_k-admissible and Σ^m_k-compact and $R_1, \dots, R_l \in \Delta$, let $\varphi(\bar{x}, \bar{Y})$ be a second order relation which is Σ^m_k on $\mathfrak{A} = \langle A, R_1, \dots, R_l\rangle$. Then

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow \psi(R_1, \dots, R_l, \bar{x}, \bar{Y})$$

with ψ some Σ_k^m relation on the trivial structure $\langle A\rangle$ and relative to any parametrization of Δ,

$$\varphi^{\#}(\bar{x}, \bar{y}) \Leftrightarrow \psi^{\#}(r_1, \ldots, r_l, \bar{x}, \bar{y}),$$

with $r_1, \ldots, r_l$ codes of $R_1, \ldots, R_l$, so that φ is Δ on Δ. A very similar argument shows that Γ is compact relative to the class of all Σ_k^m operators on $\mathfrak{A}$, so that Σ_k^m-IND $\subseteq \Gamma$ by Theorem 15. ■

To prove that these Spector classes are all distinct, it is convenient to introduce the natural notion of inaccessibility. A Spector class Γ is *Σ_k^m-inaccessible* $(m, k \geqslant 1)$ if for every $R \in \Delta$ there is a Σ_k^m-compact Spector class Γ^* such that

$$R \in \Gamma^* \subseteq \Delta .$$

Similarly for Π_k^m-, $\forall_k$- and $\exists_k$-inaccessibility.

THEOREM 19. (i) *For each $k \geqslant 2$, if a Spector class Γ is $\forall_k$-compact, then it is $\exists_k$-inaccessible.*

(ii) *If $0 < m < m'$, $k, k' \geqslant 1$ and Γ is Σ_k^m-admissible and $\Sigma_{k'}^{m'}$- or $\Pi_{k'}^{m'}$-compact, then Γ is Σ_k^m- and Π_k^m-inaccessible.*

(iii) *If $m \geqslant 1$ and $1 \leqslant k < k'$ and Γ is Σ_k^m-admissible and $\Sigma_{k'}^m$- or $\Pi_{k'}^m$-compact, then Γ is Σ_k^m- and Π_k^m-inaccessible.*

Proof. Suppose Γ is $\forall_k$-compact, $k \geqslant 2$, let $R \in \Delta$, let $\mathfrak{C}$ be a coding scheme on A in Δ and put

$$\Gamma^* = \text{all } \Sigma_k^0(\mathfrak{C})\text{-inductive relations on } \langle A, R\rangle .$$

Clearly $R \in \Gamma^*$ and Γ^* is $\exists_k$-compact by Theorems 17 and 18. It remains to prove that $\Gamma^* \subseteq \Delta$ and for this it is enough (by Theorem 14) to show that if $\varphi(\bar{x}, S)$ is $\Sigma_k^0(\mathfrak{C})$ *on* $\langle A, R\rangle$, then $\|\varphi\| < \varkappa = o(\Delta)$. Given such a φ, the set $I_\varphi^{<\varkappa}$ is in Γ and the norm

$$\sigma(\bar{x}) = |\bar{x}|_\varphi$$

is a Γ-norm by Theorem 14, and since Γ is $\forall_{k-1}$-compact, it is compact by (c) of Theorem 16 relative to the class of $\Pi_{k-1}^0(\mathfrak{C})$ operators on $\langle A, R\rangle$, hence, easily, it is compact relative to the class of $\Sigma_k^0(\mathfrak{C})$ operators on $\langle A, R\rangle$. Hence we have $\|\varphi\| \leqslant \varkappa$ by Theorem 14 and

$$(\forall \bar{x})[\varphi(\bar{x}, I_\varphi^{<\varkappa}) \Rightarrow \bar{x} \in I_\varphi^{<\varkappa}]$$

holds. But this is a $\Pi_k^0(\mathfrak{C})$ condition on $\langle A, R\rangle$ and Γ is compact relative to the class of $\Pi_k^0(\mathfrak{C})$ operations on $\langle A, R\rangle$, again by Theorem 16, so that for some $\lambda < \varkappa$,

$$(\forall \bar{x})[\varphi(\bar{x}, I_\varphi^{<\lambda}) \Rightarrow \bar{x} \in I_\varphi^{<\lambda}],$$

i.e. $\|\varphi\| \leqslant \lambda < \varkappa$.

This proves (i) and the proofs of (ii) and (iii) are similar and a bit simpler. ■

Theorems 18 and 19 give the following sequence of proper inclusions for the Spector classes obtained by elementary nonmonotone induction on an almost acceptable structure $\mathfrak{A}$:

$$\begin{aligned}\text{IND} = \Sigma_2^0\text{-IND} &\not\subseteq \Pi_2^0\text{-HYP} \not\subseteq \Pi_2^0\text{-IND}\\ &= \Sigma_3^0\text{-IND} \not\subseteq \Pi_3^0\text{-HYP} \not\subseteq \Pi_3^0\text{-IND} = \ldots\end{aligned}$$

In particular we have the increasing sequence of ordinals

$$\varkappa^{\mathfrak{A}} = \|\Sigma_2^0\text{-IND}\| < \|\Pi_2^0\text{-IND}\| = \|\Sigma_3^0\text{-IND}\| < \|\Pi_3^0\text{-IND}\| = \|\Sigma_4^0\text{-IND}\| < \ldots$$

These theorems also imply that on an almost acceptable structure $\mathfrak{A}$ and for $1 \leqslant m < m'$ or $1 \leqslant m = m'$ and $1 \leqslant k < k'$,

$$\Sigma_k^m\text{-IND} \cup \Pi_k^m\text{-IND} \not\subseteq \Sigma_{k'}^{m'}\text{-HYP} \cap \Pi_{k'}^{m'}\text{-HYP}$$

and the corresponding inequalities for the closure ordinals. A very simple reflection argument (like that in Theorem 19) together with Theorem 18 shows that the classes Σ_k^m-IND and Π_k^m-IND are distinct, and in fact their closure ordinals are distinct and we have

$$\Sigma_k^m\text{-IND} \subseteq \Pi_k^m\text{-HYP} \quad \textit{or} \quad \Pi_k^m\text{-IND} \subseteq \Sigma_k^m\text{-HYP} .$$

But it appears to be independent of the axioms of Zermelo–Fraenkel set theory which of these inclusions hold. Aanderaa [1973] showed that on the structure N of arithmetic,

$$\Pi_1^1\text{-IND} \not\subseteq \Sigma_1^1\text{-HYP},$$

$$\Sigma_2^1\text{-IND} \not\subseteq \Pi_2^1\text{-HYP},$$

$$\text{if } V = L, \quad \textit{then} \quad \Sigma_3^1\text{-IND} \not\subseteq \Pi_3^1\text{-HYP},$$

$$\text{if PD}, \quad \textit{then} \quad \Pi_3^1\text{-IND} \not\subseteq \Sigma_3^1\text{-HYP},$$

where $V = L$ is Gödel's axiom of constructibility and PD is the rather esoteric hypothesis of Projective Determinacy — see Aanderaa's paper and the references given there for a discussion of these notions. ■

7. Computation of companions. Suppose A is a transitive, infinite set (e.g. an ordinal) and $\mathfrak{F}$ is a typical, nonmonotone class of operators on A which contains the membership relation $\in \upharpoonright A$. In this case we can apply the theory of companions developed in Chapter 9 of EIAS and characterize $\mathfrak{F}$-IND and $\|\mathfrak{F}\|$ in terms of suitable admissible sets having A as a member.

Changing slightly the terminology in Section 9E of EIAS, we call a *companion of a Spector class* Γ on a transitive set A any pair $\langle\mathcal{M}, \Delta\rangle$ with the following properties:

(i) $\mathcal{M}$ is a transitive set and $A \in \mathcal{M}$.

(ii) Δ is the collection of $\Sigma_1(R)$ relations on $\mathcal{M}$, for some relation R on $\mathcal{M}$.

(iii) $\mathcal{M}$ is admissible, resolvable and projectible on A relative to R.

(iv) $o(\mathcal{M}) = o(\Delta)$.

(v) If $P \subseteq A^n$, then P is in $\Gamma \Leftrightarrow P$ is in Δ.

The Companion theorem (9E.1 and 9E.3 of EIAS) asserts that *every Spector class Γ on a transitive set has exactly one companion $\langle\mathcal{M}(\Delta), \Gamma^*\rangle$*, where in fact

(vi) $\mathcal{M}(\Delta) = \bigcap\{\mathcal{M}\colon \mathcal{M}$ is admissible and for each $X \subseteq A$, if $X \in \Delta$ then $x \in \mathcal{M}\}$.

We call $\mathcal{M}(\Delta)$ the *companion set* of Γ and Γ^* the *companion class* of Γ.

Just knowing that the companion exists gives us almost no information about a particular Spector class. The Companion theorem is useful because it often helps us to find concrete descriptions of the companions of specific Spector classes. This was illustrated in the computations of the *next admissible set* and the *next strongly Q-admissible set*, Theorems 9F.2 and 9F.3 of EIAS. Here we will compute the companions of $\mathfrak{F}$-IND for a typical nonmonotone $\mathfrak{F}$ and Π^0_k-IND ($k \geqslant 2$), Σ^m_k-IND, Π^m_k-IND ($m, k \geqslant 1$) on an almost acceptable structure $\mathfrak{A}$.

If $\mathcal{M}$ is admissible, $A \in \mathcal{M}$ and φ is a second order relation on A, then $\varphi^{\mathcal{M}}$ is the *restriction of φ to $\mathcal{M}$*, i.e.

$$\varphi^{\mathcal{M}}(x_1, \ldots, x_n, Y_1, \ldots, Y_k) \Leftrightarrow Y_1, \ldots, Y_k \in \mathcal{M} \;\&\; \varphi(x_1, \ldots, x_k, Y_1, \ldots, Y_k).$$

For each class of operations $\mathfrak{F}$ on A, the $\Delta_0(\mathfrak{F})$ relations on $\mathcal{M}$ are simply the relations which are $\Delta_0(\varphi_1^{\mathcal{M}}, \ldots, \varphi_m^{\mathcal{M}})$ for some $\varphi_1, \ldots, \varphi_m$ in $\mathfrak{F}$, and similarly for the $\Sigma_1(\mathfrak{F})$, $\Pi_1(\mathfrak{F})$ and $\Delta_1(\mathfrak{F})$ relations.

We call $\mathcal{M}$ *admissible relative to* $\mathfrak{F}$ (or $\mathfrak{F}$-*admissible*) if it satisfies $\Delta_0(\mathfrak{F})$-*Separation* and $\Delta_0(\mathfrak{F})$-*Collection*, i.e. the schemes Δ_0-Sep and Δ_0-Coll of 9D of EIAS with φ in $\Delta_0(\mathfrak{F})$.

We call $\mathcal{M}$ *compact relative to* $\mathfrak{F}$ or $\mathfrak{F}$-*compact*, if for every relation R on A which is $\Sigma_1(\mathfrak{F})$ on $\mathcal{M}$ and every $\varphi(Y)$ in $\mathfrak{F}$ and every $R^0 \subseteq R$, $R^0 \in \mathcal{M}$,

$$\varphi(R) \Rightarrow \textit{there exists some } R^* \in \mathcal{M},\ R^0 \subseteq R^* \subseteq R, \textit{ such that } \varphi(R^*).$$

First an analog of Theorem 14.

THEOREM 20. *Let $\mathfrak{F}$ be a reasonable, nonmonotone class of operations on the infinite, transitive set A, such that $\varepsilon \restriction A$ is in $\mathfrak{F}$ and let $\mathcal{M}$ be an $\mathfrak{F}$-admissible set with ordinal $\varkappa = o(\mathcal{M})$ such that $A \in \mathcal{M}$. If $\varphi(\bar{x}, S)$ is an operative relation in $\mathfrak{F}$, then each $I_\varphi^{<\xi}$ for $\xi < \varkappa$ is a member of $\mathcal{M}$ and $I_\varphi^{<\varkappa}$ is $\Sigma_1(\mathfrak{F})$. If $\mathcal{M}$ is also $\mathfrak{F}$-compact, then $\|\varphi\| \leqslant \varkappa$, so that $I_\varphi = I_\varphi^{<\varkappa}$ is $\Sigma_1(\mathfrak{F})$.*

Proof of the first assertion is easy by standard arguments on admissible sets and we will omit it — cf. Theorem 9D.5 and Lemma 9F.1 of EIAS.

To prove the second assertion, notice first that every $\mathfrak{F}$-admissible, $\mathfrak{F}$-compact set has the following property: *if $R \subseteq A^n$ is $\Sigma_1(\mathfrak{F})$ and $\sigma\colon R \to o(\mathcal{M})$ is a $\Sigma_1(\mathfrak{F})$-norm* (i.e. both $\leqslant^*_\sigma$ and $<^*_\sigma$ are $\Sigma_1(\mathfrak{F})$) *and $\varphi(R)$ holds for some $\varphi(Y)$ in $\mathfrak{F}$, then there is some $\xi < \varkappa$ so that $\varphi(\{\bar{x} \in R\colon \sigma(\bar{x}) < \xi\})$ holds.* This is quite easy by repeating the proofs of (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) of Theorem 12 in the context of an admissible set. Using this we can easily finish the proof as in Theorem 14. ■

Using this simple result and the key Theorem 15, we can now compute the companion of $\mathfrak{F}$-IND.

THEOREM 21. *Let $\mathfrak{F}$ be a typical, nonmonotone class of operators on the transitive, infinite set A, such that $\varepsilon \restriction A$ is in $\mathfrak{F}$. Then the companion set of $\mathfrak{F}$-IND is*

$$\mathcal{M}_{\mathfrak{F}} = \bigcap\{\mathcal{M}\colon \mathcal{M} \textit{ is } \mathfrak{F}\textit{-admissible}, \mathfrak{F}\textit{-compact and } A \in \mathcal{M}\}$$

and the companion class of $\mathfrak{F}$-IND is

$$\mathfrak{F}\text{-IND}^* = \textit{all } \Sigma_1(\mathfrak{F}) \textit{ relations on } \mathcal{M}_{\mathfrak{F}}.$$

Moreover, $\mathcal{M}_{\mathfrak{F}}$ is $\mathfrak{F}$-admissible and $\mathfrak{F}$-compact.

Proof. Let $\langle\mathcal{M}^*, \Sigma_1(R)\rangle$ be the companion of $\mathfrak{F}$-IND. Theorem 20 and the definition of $\mathcal{M}^*$ imply immediately that

$$\mathcal{M}^* \subseteq \mathcal{M}_{\mathfrak{F}}.$$

If $\varphi(Y)$ is in $\mathfrak{F}$, with Y varying over subsets of A^n and $\{I^n, H^n, \check{H}^n\}_{n=0}^\infty$ is a parametrization of $\Delta = \mathfrak{F}$-HYP given by Theorem 9C.8 of EIAS, then for $Y \in \mathcal{M}^*$,

$$\begin{aligned}
\varphi(Y) &\Leftrightarrow (\exists a \in A)\{a \in I^n \;\&\; \varphi^{\#}(a) \;\&\; Y = H^n_a\}\\
&\Leftrightarrow (\exists a \in A)\{a \in I^n \;\&\; \varphi^{\#}(a) \;\&\; (\forall x \in Y)[H^n(a, x)]\\
&\qquad \& \;(\forall x \in A)[\check{H}^n(a, x) \Rightarrow x \in Y]\};
\end{aligned}$$

since I^n, H^n, $\neg\check{H}^n$, $\varphi^{\#}$ are in Γ (the last by Theorem 10) and all relations in Γ are $\Sigma_1(R)$ on $\mathcal{M}^*$ by the Companion theorem, this means that $\varphi^{\mathcal{M}^*}$ is $\Sigma_1(R)$ on $\mathcal{M}^*$. The same argument works, of course, when φ has more than one relation argument and also arguments in A, and it also works for the negation $\neg\varphi$ of any relation in $\mathfrak{F}$; thus all relations in $\mathfrak{F}$ are $\Delta_1(R)$ on $\mathcal{M}^*$ and then a simple induction proves that *every $\Delta_0(\mathfrak{F})$ relation on $\mathcal{M}^*$ is $\Delta_1(R)$*, hence every $\Sigma_1(\mathfrak{F})$ relation on $\mathcal{M}^*$ is $\Sigma_1(R)$. This implies immediately that $\mathcal{M}^*$ is $\mathfrak{F}$-admissible and also $\mathfrak{F}$-compact, since Γ is $\mathfrak{F}$-compact, so that

$$\mathcal{M}_{\mathfrak{F}} = \mathcal{M}^*$$

and by Theorem 20 every $\mathfrak{F}$-inductive relation on A is $\Sigma_1(\mathfrak{F})$ on $\mathcal{M}_{\mathfrak{F}}$.

Since $\mathcal{M}_{\mathfrak{F}}$ is admissible relative to $\mathfrak{F}$ and also admissible, resolvable and projectible on A relative to R and since for $P \subseteq A^n$,

$$P \text{ is } \Sigma_1(\mathfrak{F}) \Leftrightarrow P \text{ is } \mathfrak{F}\text{-inductive} \Leftrightarrow P \text{ is } \Sigma_1(R)\,,$$

Lemma 9E.2 of EIAS guarantees that for every relation P on $\mathcal{M}^*$,

$$P \text{ is } \Sigma_1(\mathfrak{F}) \Leftrightarrow P \text{ is } \Sigma_1(R)$$

which completes the proof. ■

In particular, this computation characterizes $\|\mathfrak{F}\|$ as the ordinal of the *next $\mathfrak{F}$-admissible, $\mathfrak{F}$-compact set*, i.e. the smallest $\mathfrak{F}$-admissible, $\mathfrak{F}$-compact set which has A as a member.

For the main examples of induction in the higher order language over a structure, we can reformulate the characterization of Theorem 21 in terms of more familiar reflection properties. We do this first for Π_k^0-induction, $k \geqslant 2$.

A sentence θ in the language of a structure $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$ is Σ_2 is θ is of the form $(\exists x)(\forall y)\psi$, where ψ has only bounded quantifiers $(\exists u \in v)$, $(\forall u \in v)$, i.e. where ψ defines a Δ_0 relation. The Σ_k and Π_k sentences are defined similarly for all $k \geqslant 1$.

An admissible set $\mathcal{M}$ is *Σ_k-reflecting* if for every Σ_k sentence θ of the language of $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$,

(*) $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \theta \Rightarrow$ *there exists a transitive $w \in \mathcal{M}$ such that* $\langle w, \varepsilon \restriction w\rangle \models \theta$.

It is implicit in the notation we use that if constants $w_1, \ldots, w_k$ occur in θ, then $w_1, \ldots, w_k$ are all members of w, otherwise θ would not be defined in $\langle w, \varepsilon \restriction w\rangle$. The notion of Π_k-reflection is defined similarly.

We will need a lemma which relates this notion of reflection to $\mathfrak{F}$-compactness.

THEOREM 22. *Let A be a transitive, infinite set, let $R_1, \ldots, R_l$ be relations on A such that the structure*

$$\mathfrak{A} = \langle A, \varepsilon \restriction A, R_1, \ldots, R_l\rangle$$

is almost acceptable, let $\mathfrak{C}$ be a hyperelementary coding scheme on $\mathfrak{A}$ and for $k \geqslant 2$ put

$$\mathfrak{F} = \textit{all } \Pi_k^0(\mathfrak{C}) \textit{ second order relations on } \mathfrak{A}\,.$$

(i) *If $\mathcal{M}$ is a Π_{k+1}-reflecting, admissible set such that $A, R_1, \ldots, R_l \in \mathcal{M}$, then $\mathcal{M}$ is $\mathfrak{F}$-compact.*

(ii) *The companion set of Π_k^0-IND is Π_{k+1}-reflecting.*

Proof. We take $k = 2$ to simplify notation.

Proof of (i). Suppose $(\forall x \in A)(\exists y \in A)\psi(x, y, R)$ holds, where ψ is restricted on $\mathfrak{A}$ (relative to $\mathfrak{C}$) and R is Σ_1 on $\mathcal{M}$ and let $R^0 \in \mathcal{M}$, $R^0 \subseteq R$. Then

$$\bar{u} \in R \Leftrightarrow (\exists z)\chi(z, \bar{u})$$

with some Δ_0 formula χ and hence $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$ satisfies the sentence

$$\theta \equiv (\forall x \in A)(\exists y \in A)\psi\big(x, y, \{\bar{u}\colon\ (\exists z)\chi(z, \bar{u})\}\big) \mathbin{\&} (\forall \bar{u} \in R^0)(\exists z)\chi(z, \bar{u})\}\,.$$

We can prove θ equivalent to a Π_3 sentence by advancing the quantifier $(\exists z)$ past the restricted "number" quantifiers in ψ, using closure of $\mathcal{M}$ under the formation of finite tuples. Thus there is a transitive $w \in \mathcal{M}$ such that $w \supseteq A \cup R^0$, $\langle w, \varepsilon \restriction w\rangle \models \theta$. Now take

$$R^* = \{\bar{u}\colon\ (\exists z \in w)\chi(z, \bar{u})\}$$

and use the absoluteness of Δ_0 formulas for transitive sets to show easily that

$$R^0 \subseteq R^*, \qquad (\forall x \in A)(\exists y \in A)\psi(x, y, R^*)\,.$$

Proof of (ii). We show first that $\mathcal{M}_{\mathfrak{F}}$ has the following reflection property: if θ is Π_3, $\langle \mathcal{M}_{\mathfrak{F}}, \varepsilon \restriction \mathcal{M}_{\mathfrak{F}}\rangle \models \theta$ and $w^0 \in \mathcal{M}_{\mathfrak{F}}$, then there is some $w^* \in \mathcal{M}_{\mathfrak{F}}$, $w^0 \subseteq w$ such that $\langle w^*, \varepsilon \restriction w^*\rangle \models \theta$ (w^* need not be transitive).

Let us assume for simplicity of notation that

$$\theta \equiv (\forall x)(\exists y)(\forall z)(\forall u \in x)(\exists v \in y)\psi(x, y, z, u, v)\,,$$

where ψ is quantifier free, the general case being no harder in principle. Choose a Δ_1 projection of $\mathcal{M}_{\mathfrak{F}}$ on A

$$\pi\colon\quad D \twoheadrightarrow \mathcal{M}_{\mathfrak{F}}$$

and define on A

$$R(x, y, z) \Leftrightarrow x, y, z \in D \mathbin{\&} \big(\forall u \in \pi(x)\big)\big(\exists v \in \pi(y)\big)\psi\big(\pi(x), \pi(y), \pi(z), u, v\big)\,,$$

$$Q(x, y, z, u) \Leftrightarrow x, y, z, u \in D \mathbin{\&} \big(\exists v \in \pi(y)\big)\psi\big(\pi(x), \pi(y), \pi(z), \pi(u), v\big)\,,$$

$$S(x, y, z, u, v) \Leftrightarrow x, y, z, u, v \in D \mathbin{\&} \psi\big(\pi(x), \pi(y), \pi(z), \pi(u), \pi(v)\big)\,.$$

Taking $\Gamma = \Pi_2^0$-IND, D is in Γ and the relations R, Q, S are all Δ on D, since both R, Q, S and $D^3 - R$, $D^4 - Q$, $D^5 - S$ are Σ_1 on $\mathcal{M}_{\mathfrak{F}}$. The relation E defined by

$$s\,E\,t \Leftrightarrow s, t \in D \mathbin{\&} \pi(s) \in \pi(t)$$

is also Δ on D, and the assumption that $\langle \mathcal{M}_{\mathfrak{F}}, \varepsilon \upharpoonright \mathcal{M}_{\mathfrak{F}} \rangle$ satisfies θ means that the structure $\langle D, E, R, Q, S\rangle$ satisfies the sentence

$$\begin{aligned}(\forall x)(\exists y)(\forall z)R(x, y, z) \;\&\; (\forall x)(\forall y)(\forall z)[R(x, y, z)\\ \Leftrightarrow (\forall u)[u\, E\, x \Rightarrow Q(x, y, z, u)]]\\ \&\; (\forall x)(\forall y)(\forall z)(\forall u)[Q(x, y, z, u)\\ \Leftrightarrow (\exists v)[v\, E\, y \;\&\; S(x, y, z, u, v)]]\\ \&\; (\forall x)(\forall y)(\forall z)(\forall u)[S(x, y, z, u, v)\\ \Leftrightarrow \psi^*(x, y, z, u, v)]\,,\end{aligned}$$

where ψ^* is obtained from ψ by replacing "ϵ" by "E". This is a $\forall_3$ sentence and Γ is $\forall_3$-reflecting by Theorem 18, so taking

$$D^0 = \{t \in D\colon\ \pi(t) \in w^0\}\,,$$

there is some set D^* in $\Delta = \Pi^0_2$-HYP, $D^0 \subseteq D^* \subseteq D$, such that the structure $\langle D^*, E\upharpoonright D^*, R\upharpoonright D^*, Q\upharpoonright D^*, S\upharpoonright D^*\rangle$ satisfies the same sentence. Now take

$$w^* = \{\pi(t)\colon\ t \in D^*\}$$

and verify easily that $w^0 \subseteq w^*$ and $\langle w^*, \varepsilon\upharpoonright w^*\rangle \models \theta$.

To prove that $\mathcal{M}_{\mathfrak{F}}$ is Π_3-reflecting, given θ such that $\langle \mathcal{M}_{\mathfrak{F}}, \varepsilon\upharpoonright \mathcal{M}_{\mathfrak{F}}\rangle \models \theta$, take w^0 to be transitive and having as members all the constants which occur in θ and find $w^* \supseteq w^0$ such that $\langle w^*, \varepsilon\upharpoonright w^*\rangle \models \theta$. Let $f\colon w^* \to w$ collapse w^* to a transitive set. Then f is an isomorphism of $\langle w^*, \varepsilon\upharpoonright w^*\rangle$ with $\langle w, \varepsilon\upharpoonright w\rangle$ which leaves all the constants in θ fixed, so we have $\langle w, \varepsilon\upharpoonright w\rangle \models \theta$ as required. ∎

This result gives immediately the characterization of Π^0_k-IND which we seek.

THEOREM 23. *Let A be a transitive, infinite set, let $R_1, \ldots, R_l$ be relations on A such that the structure*

$$\mathfrak{A} = \langle A, \varepsilon\upharpoonright A, R_1, \ldots, R_l\rangle$$

is almost acceptable. (This holds when A is closed under pairing or when A is an infinite ordinal or when $A = V_\xi$ = the set of all sets of rank $<\xi$, with $\xi \geqslant \omega$.)

For each $k \geqslant 2$, the companion set of Π^0_k-IND on $\mathfrak{A}$ is

$$\mathfrak{A}^+(\Pi^0_k) = \bigcap\{\mathcal{M}\colon\ \mathcal{M} \text{ is admissible, } \Pi_{k+1}\text{-reflecting and } A, R_1, \ldots, R_l \in \mathcal{M}\}$$

and the companion class of Π^0_k-IND is

$$\Pi^0_k\text{-IND}^* = \text{all } \Sigma_1 \text{ relations on } \mathfrak{A}^+(\Pi^0_k)\,.$$

Moreover, $\mathfrak{A}^+(\Pi^0_k)$ is Π_{k+1}-reflecting.

Proof. From Theorem 21 we know that the companion of Π^0_k-IND is $\langle \mathcal{M}_{\mathfrak{F}}, \Sigma_1\rangle$, where

$$\mathcal{M}_{\mathfrak{F}} = \bigcap\{\mathcal{M}\colon\ \mathcal{M} \text{ is admissible and } \mathfrak{F}\text{-compact and } A \in \mathcal{M}\}\,,$$

with $\mathfrak{F}$ = all $\Pi^0_k(\mathfrak{C})$ operations on $\mathfrak{A}$, and $\mathfrak{C}$ any hyperelementary coding scheme on $\mathfrak{A}$. Part (i) of Theorem 22 implies that $\mathcal{M}_{\mathfrak{F}} \subseteq \mathfrak{A}^+(\Pi^0_k)$ and Part (ii) of Theorem 22 yields $\mathfrak{A}^+(\Pi^0_k) \subseteq \mathcal{M}_{\mathfrak{F}}$, so we have $\mathcal{M}_{\mathfrak{F}} = \mathfrak{A}^+(\Pi^0_k)$. ∎

The corresponding characterization of the companions of Σ^m_k-IND and Π^m_k-IND are a bit messier. If $\mathcal{M}$ is admissible, $A \in \mathcal{M}$ and $\mathfrak{F}$ is a class of operators on A, the $\Sigma^m_k(\mathfrak{F})$ and $\Pi^m_k(\mathfrak{F})$ formulas of the structure $\langle \mathcal{M}, \varepsilon\upharpoonright \mathcal{M}\rangle$ are defined in the obvious way: we allow prime formulas of the form "$\varphi(\bar{x}, Y)$" with φ in $\mathfrak{F}$ in the usual definition of Σ^m_k and Π^m_k formulas. We say that $\mathcal{M}$ is *$\Sigma^m_k(\mathfrak{F})$-reflecting* if for every $\Sigma^m_k(\mathfrak{F})$ sentence θ,

$$\langle \mathcal{M}, \varepsilon\upharpoonright \mathcal{M}\rangle \models \theta \Rightarrow \textit{there exists a transitive } w \in \mathcal{M} \textit{ such that } \langle w, \varepsilon\upharpoonright w\rangle \models \theta\,,$$

and similarly for Π^m_k-reflection.

THEOREM 24. *Let A be a transitive, infinite set, let $R_1, \ldots, R_l$ be relations on A such that the structure*

$$\mathfrak{A} = \langle A, \varepsilon\upharpoonright A, R_1, \ldots, R_l\rangle$$

is almost acceptable.

For each $k, m \geqslant 1$, let

$$\mathfrak{F} = \text{all } \Sigma^m_k \text{ second order relations on } \mathfrak{A}$$

and put

$$\mathfrak{A}^+(\Sigma^m_k) = \bigcap\{\mathcal{M}\colon\ \mathcal{M} \text{ is } \mathfrak{F}\text{-admissible, } \Sigma^m_k(\mathfrak{F})\text{-reflecting and } A, R_1, \ldots, R_l \in \mathcal{M}\}\,.$$

Then $\mathfrak{A}^+(\Sigma^m_k)$ is $\mathfrak{F}$-admissible and $\Sigma^m_k(\mathfrak{F})$-reflecting and the companion of Σ^m_k-IND on $\mathfrak{A}$ is $\langle \mathfrak{A}^+(\Sigma^m_k), \Sigma_1(\mathfrak{F})\rangle$.

Similarly, put

$$\mathfrak{A}^+(\Pi^m_k) = \bigcap\{\mathcal{M}\colon\ \mathcal{M} \text{ is } \mathfrak{F}\text{-admissible, } \Pi^m_k(\mathfrak{F})\text{-reflecting and } A, R_1, \ldots, R_l \in \mathcal{M}\}\,.$$

Then $\mathfrak{A}^+(\Pi^m_k)$ is $\mathfrak{F}$-admissible and $\Pi^m_k(\mathfrak{F})$-reflecting and the companion of Π^m_k-IND on $\mathfrak{A}$ is $\langle \mathfrak{A}^+(\Pi^m_k), \Sigma_1(\mathfrak{F})\rangle$.

Proof is exactly like that of Theorem 23, after showing first the analog of Theorem 22. ∎

This characterization of Σ_k^m- and Π_k^m-induction is really not much simpler than that given directly by Theorem 21 in terms of $\mathfrak{F}$-compactness. We can simplify it, however, by eliminating the relativization to $\mathfrak{F}$, if $m = k = 1$ and A is countable.

By Σ_k^m- and Π_k^m-reflection we naturally mean $\Sigma_k^m(\mathfrak{F})$- and $\Pi_k^m(\mathfrak{F})$-reflection with $\mathfrak{F}$ the empty set of relations.

THEOREM 25. *Let A be a countable, transitive set, and let $R_1, \ldots, R_l$ be relations on A such that the structure*

$$\mathfrak{A} = \langle A, \varepsilon \restriction A, R_1, \ldots, R_l\rangle$$

is almost acceptable. Then the companion set of Σ_1^1-IND on $\mathfrak{A}$ is the smallest Σ_1^1-reflecting, admissible set having $A, R_1, \ldots, R_l$ as elements and the companion class of Σ_1^1-IND consists of all Σ_1 relations on the companion set.

Similarly, the companion set of Π_1^1-IND on $\mathfrak{A}$ is the smallest Π_1^1-reflecting, admissible set having $A, R_1, \ldots, R_l$ as elements and the companion class of Π_1^1-IND consists of all Σ_1 relations on the companion set.

Proof. In view of Theorem 24, it will be enough to verify that whenever $\mathcal{M}$ is admissible, Σ_1^1- or Π_1^1-reflecting and $A, R_1, \ldots, R_l \in \mathcal{M}$, then every Π_1^1 relation φ on $\mathfrak{A}$ is Δ_1 on $\mathcal{M}$. The key for this computation is the main result of Barwise-Gandy-Moschovakis [1971]. It is well-known and easy to verify that there is a fixed Π_3 sentence θ with no constants, such that for every transitive set $\mathcal{M}$,

$$\mathcal{M} \text{ is admissible} \Leftrightarrow \langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \theta .$$

Thus, if $\mathcal{M}$ is admissible and Π_3-reflecting, then $\mathcal{M}$ is *inaccessible*, i.e. for each $w^0 \in \mathcal{M}$ there is an admissible $w \in \mathcal{M}$, $w^0 \subseteq w$.

By Theorem 3.1 of Barwise-Gandy-Moschovakis [1971] (relativized) or Theorem 8A.1 of EIAS, if $\varphi(\bar{x}, \bar{Y})$ is Π_1^1 on $\mathfrak{A}$, then $\varphi(\bar{x}, \bar{Y})$ is inductive on $\mathfrak{A}$. Hence by the Abstract Spector-Gandy Theorem 7D.2 of EIAS (relativized), there is a fixed elementary relation $\psi(Z, \bar{x}, \bar{Y})$ on an acceptable expansion $\mathfrak{A}' = (\mathfrak{A}, P_1, \ldots, P_k)$, with $P_1, \ldots, P_k$ hyperelementary on $\mathfrak{A}$, such that

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow (\exists Z)[Z \textit{ is hyperelementary on } (\mathfrak{A}, \bar{Y}) \;\&\; \psi(\omega, \bar{x}, \bar{Y})] .$$

Now ψ translates into a Δ_0 relation on $\mathcal{M}$ and by the main result of Barwise-Gandy-Moschovakis Theorem 9F.2 of EIAS, the hyperelementary relations on $\mathfrak{A}$ are exactly the members of the smallest admissible set $\mathfrak{A}^+$ having $A, R_1, \ldots, R_l$ as elements. Put

$$\begin{aligned} &\mathrm{Adm}(w) \Leftrightarrow w \textit{ is admissible},\\ &Q(w, \bar{Y}) \Leftrightarrow \mathrm{Adm}(w) \;\&\; A, R_1, \ldots, R_l, \bar{Y} \in w\\ &\qquad\qquad \& \;(\forall u \in w)[(A, R_1, \ldots, R_l, \bar{Y} \in u) \Rightarrow \neg \mathrm{Adm}(u)] \end{aligned}$$

and notice that $Q(w, \bar{Y})$ is Δ_0 on $\mathcal{M}$ and

$$\begin{aligned} \varphi(\bar{x}, \bar{Y}) &\Leftrightarrow (\exists w)[Q(w, \bar{Y}) \;\&\; (\exists Z \in w)\psi(Z, \bar{x}, \bar{Y})]\\ &\Leftrightarrow (\forall w)[Q(w, \bar{Y}) \Rightarrow (\exists Z \in w)\psi(Z, \bar{x}, \bar{Y})] . \quad \blacksquare \end{aligned}$$

The argument in this proof is essentially the same as that used in the proof of Lemma 10.6 of Aczel-Richter 1973.

One of the prime motivations for computing the companions of Spector classes is the desire to study and understand the structure of various kinds of admissible sets. A good example is the main result of Barwise-Gandy-Moschovakis [1971], Theorem 9F.2 of EIAS, which can be considered as a *construction* of the *next admissible set*. Similarly, Theorem 9F.3 of EIAS constructs the *next strongly* Q-*admissible set* (Q a monotone, unary quantifier on the almost acceptable structure $\mathfrak{A} = \langle A, \varepsilon \restriction A\rangle$ and Exercise 9.16 of EIAS constructs the *next strongly β set*. From this point of view, Theorem 23 constructs the *next Π_{k+1}-reflecting set* and Theorem 25 constructs the *next Π_1^1-reflecting set* and the *next Σ_1^1-reflecting set*, but only when we start with a *countable* almost acceptable structure $\mathfrak{A} = \langle A, \varepsilon \restriction A, R_1, \ldots, R_l\rangle$. The computation of the companion of Π_1^1-IND for uncountable $\mathfrak{A}$ given by Theorem 24 does not yield obviously natural and interesting admissible sets. There is, however, a generalization of Theorem 25 which is suggested by a result of Aczel and Richter and which is worth discussing briefly, as one more example of the applicability of the methods we have been studying.

If $\mathcal{M}, \mathcal{N}$ are admissible and $\mathcal{M} \subseteq \mathcal{N}$, we say that $\mathcal{M}$ is $\mathcal{N}$-*stable* if for every Σ_1 sentence θ of $\langle \mathcal{N}, \varepsilon \restriction \mathcal{N}\rangle$ whose constants are all in $\mathcal{M}$,

$$\textit{if } \langle \mathcal{N}, \varepsilon \restriction \mathcal{N}\rangle \models \theta, \quad \textit{then } \langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \theta .$$

We abbreviate this condition by

$$\mathcal{M} \precsim_{\Sigma_1} \mathcal{N} .$$

Theorem 6.4 of Aczel-Richter [1973] establishes that a countable, admissible set $\mathcal{M}$ is Π_1^1-reflecting if and only if $\mathcal{M} \precsim_{\Sigma_1} \mathcal{M}^+$, where

$$\begin{aligned} \mathcal{M}^+ &= \textit{the next admissible set}\\ &= \textit{the smallest admissible set } \mathcal{N} \textit{ such that } \mathcal{M} \in \mathcal{N} . \end{aligned}$$

Thus Theorem 25 constructs the *next $\mathcal{M}$ which is $\mathcal{M}^+$-stable*, starting with a countable, almost acceptable $\mathfrak{A} = \langle A, \varepsilon \restriction A, R_1, \ldots, R_l\rangle$. We now outline a construction of this set starting with an arbitrary almost acceptable $\mathfrak{A}$.

The *same quantifier* $\mathrm{G} = \mathrm{G}^{\mathfrak{C}}$ on a set A and relative to a coding scheme $\mathfrak{C}$ on A was defined in 5C of EIAS by

$$(\mathrm{G}z)R(z) \Leftrightarrow \{(\forall s_1)(\exists t_1)(\forall s_2)(\exists t_2)\ldots\} \bigvee_m R(\langle s_1, t_1, \ldots, s_m, t_m\rangle)\,.$$

If $\mathcal{M}$ is an admissible set, then we always take $\mathrm{G} = \mathrm{G}^{\mathcal{M}}$ relative to the ordinary set theoretic tuple functions, unless there is notice of the contrary. A G_1 *formula* on $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$ is a formula

$$\varphi \equiv (\mathrm{G}z)\psi(z)$$

where ψ is elementary on $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$. By Theorem 6C.7 of EIAS, the relations definable by G_1 formulas on $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$ are precisely the inductive relations of this structure.

An admissible set $\mathcal{M}$ is G_1-*reflecting* if for every G_1 sentence θ of $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$,

$\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \theta \Rightarrow$ *there is some admissible* $w \,\varepsilon\, \mathcal{M}$ *such that* $\langle w, \varepsilon \restriction w\rangle \models \theta$.

Sets which are G_1-reflecting are obviously Π_m-reflecting for every m.

THEOREM 26. *An admissible set* $\mathcal{M}$ *is* G_1-*reflecting if and only if* $\mathcal{M} \prec_{\Sigma_1} \mathcal{M}^+$. (Essentially Aczel-Richter [1973].)

Proof. The argument given in the proof of Theorem 6.4 of Aczel-Richter [1973] works here too, since it only uses properties of Π_1^1 sentences on countable admissible sets which hold for G_1 sentences on all admissible sets. Notice first that if $\mathcal{M}$ is G_1-reflecting or $\mathcal{M}^+$-stable, then $\mathcal{M}$ is inaccessible.

With each G_1 formula $\varphi(\bar{x})$ *which has no constants* we can effectively associate a Σ_1 formula $\varphi^+(\bar{x})$, also with no constants, such that for every admissible set $\mathcal{M}$ and $\bar{x} = x_1, \ldots, x_n \in \mathcal{M}$,

$$\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \varphi(\bar{x}) \Leftrightarrow \langle \mathcal{M}^+, \varepsilon \restriction \mathcal{M}^+\rangle \models \varphi^+(\bar{x})\,.$$

The construction of φ^+ is implicit in the proofs of Theorem 5C.1 and Lemma 9F.1 of EIAS. Now if $\mathcal{M}$ is $\mathcal{M}^+$-stable and $\varphi(\bar{x})$ is a G_1 formula of $\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle$, for $\bar{x}$ in $\mathcal{M}$ we have

$$\begin{aligned}
\langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \varphi(\bar{x}) &\Rightarrow \langle \mathcal{M}^+, \varepsilon \restriction \mathcal{M}^+\rangle \models \varphi^+(\bar{x})\\
&\Rightarrow \langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \varphi^+(\bar{x}) \text{ since } \mathcal{M} \prec_{\Sigma_1} \mathcal{M}\\
&\Rightarrow \langle w^0, \varepsilon \restriction w^0\rangle \models \varphi^+(\bar{x}) \text{ for some } w^0 \,\varepsilon\, \mathcal{M}, \text{ since } \varphi^+(\bar{x}) \text{ is } \Sigma_1\\
&\Rightarrow \langle w^+, \varepsilon \restriction w^+\rangle \models \varphi^+(\bar{x}) \text{ for some admissible } w \supseteq w^0, \text{ since}\\
&\qquad\qquad \varphi^+(\bar{x}) \text{ persists upwards}\\
&\Rightarrow \langle w, \varepsilon \restriction w\rangle \models \varphi(\bar{x})\,.
\end{aligned}$$

Conversely, with each Σ_1 formula $\psi(\bar{x})$ we can effectively associate a G_1 formula $\psi^-(\bar{x})$, such that for every admissible $\mathcal{M}$ and $\bar{x} = x_1, \ldots, x_n \in \mathcal{M}$,

$$\langle \mathcal{M}^+, \varepsilon \restriction \mathcal{M}^+\rangle \models \psi(\bar{x}) \Leftrightarrow \langle \mathcal{M}, \varepsilon \restriction \mathcal{M}\rangle \models \psi^-(\bar{x})\,.$$

This too is implicit in the uniformity of the proofs of Theorems 9E.1 (especially Lemmas 8, 10) and 5C.2 of EIAS and it implies exactly as above that if $\mathcal{M}$ is G_1-reflecting, then it is $\mathcal{M}^+$-stable. ■

THEOREM 27. *Let* A *be a transitive set and let* $R_1, \ldots, R_l$ *be relations on* A *such that the structure*

$$\mathfrak{A} = \langle A, \varepsilon \restriction A, R_1, \ldots, R_l\rangle$$

is almost acceptable. Let

$$\mathrm{G}_1 = \textit{all inductive second order relations on } \mathfrak{A}\,.$$

Then G_1 *is a typical, nonmonotone class, the companion set of* G_1-IND *is the smallest admissible set* $\mathcal{M}$ *which is* $\mathcal{M}^+$-*stable and has* $A, R_1, \ldots, R_l$ *as elements and the companion class of* $\mathfrak{F}$-IND *consists of all* Σ_1 *relations on the companion set.*

Proof (outline). That G_1 is a typical, nonmonotone class of operators on A is obvious from the results of Chapter 6 of EIAS.

Choose a hyperelementary coding scheme on $\mathfrak{A}$ and define the game quantifier G' relative to this coding scheme as in 5C of EIAS,

$$(\mathrm{G}'z)R(z, \bar{x}) \Leftrightarrow \{(\forall s_1)(\exists t_1)(\forall s_2)(\exists t_2)\ldots\} \bigvee_{n \in \omega} R(\langle s_1, t_1, \ldots, s_m, t_m\rangle', \bar{x})\,.$$

A relation $\varphi(\bar{x}, \bar{Y})$ is in $\mathfrak{F}$ if and only if

$$\varphi(\bar{x}, \bar{Y}) \Leftrightarrow (\mathrm{G}'z)\psi(z, \bar{x}, \bar{Y})\,,$$

with $\psi(z, \bar{x}, \bar{Y})$ elementary on $\mathfrak{A}$.

If $\mathcal{M}$ is an inaccessible admissible set such that $A, R_1, \ldots, R_l \in \mathcal{M}$, we can prove as in Theorem 25 that every inductive $\varphi(\bar{x}, \bar{Y})$ is Δ_1 on $\mathcal{M}$. Letting G be the standard game quantifier on $\mathcal{M}$ we also have

$$\begin{aligned}
(\mathrm{G}'z)\psi(z, \bar{x}, \bar{Y}) \Leftrightarrow \{(\forall s_1)(\exists t_1)\ldots\} \bigvee_{m \in \omega} [(\forall i < m)[t_i \in A \,\&\, s_m \notin A] \vee{}\\
\vee \psi(\langle s_1, \ldots, t_m\rangle', \bar{x}, \bar{Y})] \Leftrightarrow (\mathrm{G}z)\psi^*(z, \bar{x}, \bar{Y})
\end{aligned}$$

with some $\psi^*(z, \bar{x}, \bar{Y})$ which is Δ_1 on $\mathcal{M}$, and from this it follows easily that if $\mathcal{M}$ is G_1-reflecting, then $\mathcal{M}$ is $\mathfrak{F}$-compact. Thus the companion set $\mathcal{M}_{\mathrm{G}_1}$ is contained in every G_1-reflecting $\mathcal{M}$ with $A, R_1, \ldots, R_l \in \mathcal{M}$, and the proof will be complete if we can show that $\mathcal{M}_{\mathrm{G}_1}$ is G_1-reflecting. This follows by the method of the proof of (ii) in Theorem 22, using the $\mathfrak{F}$-compactness of $\mathcal{M}_{\mathfrak{F}}$ and we will omit the details. ■

We end this section with a corollary of Theorems 23, 25 and 27 which yields an easy comparison of these results with some of the theorems in Aczel-Richter [1973].

An ordinal $\varkappa$ is *Σ_k-reflecting* if the set $L_\varkappa$ of sets constructible before $\varkappa$ is Σ_k-reflecting, and similarly for Π_k-, Σ_k^m- and Π_k^m-reflection. For each infinite λ, let

$$\sigma_k[\lambda] = \text{least } \Sigma_k\text{-reflecting } \varkappa > \lambda\,,$$

and define $\pi_k[\lambda]$, $\sigma_k^m[\lambda]$, $\pi_k^m[\lambda]$ in the same way.

If $\lambda < \varkappa$, then λ is *$\varkappa$-stable* if $L_\lambda \prec_{\Sigma_1} L_\varkappa$. We will refer to ordinals $\varkappa$ which are *$\varkappa^+$-stable*, where $\varkappa^+$ = the least admissible ordinal $>\varkappa$. It is well-known (and easy to prove by an absoluteness argument) that $(L_\varkappa)^+ = L_{\varkappa^+}$, so that $\varkappa$ is $\varkappa^+$-stable if and only if $L_\varkappa$ is G_1-reflecting.

THEOREM 28. *Let λ be an infinite ordinal.*

(i) *For each $k \geqslant 2$, the companion set of Π_k^0-IND on $\langle\lambda, \varepsilon \restriction \lambda\rangle$ is $L_\varkappa$, with $\varkappa = \pi_{k+1}[\lambda] = \sigma_{k+2}[\lambda]$.*

(ii) *If λ is countable, then the companion set of Σ_1^1-IND on $\langle\lambda, \varepsilon \restriction \lambda\rangle$ is $L_\varkappa$ with $\varkappa = \sigma_1^1[\lambda]$ and the companion set of Π_1^1-IND on $\langle\lambda, \varepsilon \restriction \lambda\rangle$ is $L_\varkappa$ with $\varkappa = \pi_1^1[\lambda]$.*

(iii) *The companion set of G_1-IND on $\langle\lambda, \varepsilon \restriction \lambda\rangle$ is $L_\varkappa$ with*

$$\varkappa = \textit{least ordinal greater than } \lambda \textit{ which is } \varkappa^+\textit{-stable}\,.$$

(For $\lambda = \omega$ these results are due to Aczel-Richter.)

Proof is immediate from Theorems 23, 25 and 27 together with the following absoluteness results: if $\mathcal{M}$ is admissible and $\varkappa = o(\mathcal{M})$, then

$$\mathcal{M} \text{ is } \Pi_k\text{-reflecting} \Rightarrow L \text{ is } \Pi_k\text{-reflecting}\,,$$

and similarly with Σ_1^1, Π_1^1 and G_1 in place of Π_k. All these follow easily from the fact that the function $\xi \to L_\xi$ is Δ_1 on $\mathcal{M}$. ∎

Theorems A and B of Aczel-Richter [1973] are (i) and (ii) above with $\lambda = \omega$, while (iii) with $\lambda = \omega$ is immediate from Theorem 6.4 of this Aczel-Richter paper. Theorem C of their paper is an ordinal version of our Theorem 24 (with $A = \omega$). The most general result in Aczel-Richter [1973] (among their results which compare with ours) is Theorem D. This is their analog of Theorem 21 (with $A = \omega$) in this paper. It gives a somewhat different characterization of the companion that our result.

8. The case of Π_1^0-induction. Suppose $\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$ is an acceptable structure and we take

$$\mathfrak{F} = \textit{all } \Pi_1^0(\mathrm{C}) \textit{ second order relations on } \mathfrak{A}\,,$$

where C is some elementary coding scheme on $\mathfrak{A}$. Using Theorem 4 we can prove as in Theorem 11 that the class $\mathfrak{F}$-IND and the ordinal $\|\mathfrak{F}\|$

do not depend on the particular choice of C, so we can call $\mathfrak{F}$-IND simply Π_1^0-IND on $\mathfrak{A}$.

It is well known that on the structure N of arithmetic Π_1^0-IND coincides with Σ_2^0-IND = IND = *all Π_1^1 relations on N*. In general, however, Π_1^0-IND need not be a Spector class — we will mention counterexamples further on.

Let us collect in a definition the key properties of a Spector class which Π_1^0-IND always satisfies. A class of relations Γ on a set A is a *semi-Spector class* if the following conditions hold (in the terminology of Section 9A of EIAS).

(i) Γ contains $=$, $\neq$, and a coding scheme C on A.

(ii) Γ is closed under &, $\vee$, $\forall$ and trivial combinatorial substitutions.

(iii) Γ is parametrized and normed.

(iv) Whenever $P(y, \bar{x})$, $Q(y, \bar{x})$ are disjoint relations in Γ, i.e.

$$(\forall y)(\forall \bar{x})[\neg P(y, \bar{x}) \vee \neg Q(y, \bar{x})]\,,$$

then the relation $R(\bar{x})$ satisfying

$$R(\bar{x}) \Leftrightarrow (\forall y)[P(y, \bar{x}) \vee Q(y, \bar{x})] \;\&\; (\exists y)P(y, \bar{x})$$

is in Γ.

In other words, Γ is a semi-Spector class if it has all the properties of a Spector class, except that instead of closure under $\exists$, Γ satisfies the weaker *closure under deterministic* $\exists$ formulated in (iv).

It is very easy to verify using the results of Sections 1-3 that *on each acceptable $\mathfrak{A}$, Π_1^0-IND is a semi-Spector class*. The converse, however, is non-trivial and we will only state it here.

THEOREM 29. *If $\mathfrak{A} = \langle A, R_1, \ldots, R_l\rangle$ is an acceptable structure, then Π_1^0-IND is the smallest semi-Spector class Γ on A such that $R_1, \ldots, R_l$ are in Δ.* (Essentially Grilliot [1971].)

What Grilliot proved (essentially) in Theorems 4 and 8 of his [1971] is that on an acceptable $\mathfrak{A}$,

$$\Pi_1^0\text{-IND} = \textit{the class of all relations on } A \textit{ which are semi-prime computable in } E, R_1, \ldots, R_l\,,$$

where E is the total type-2 object which represents quantification over A. Prime computability is one of the notions of abstract recursion theory introduced in Moschovakis [1969]. On the other hand, the characterization of the relations on the reals which are semirecursive in ${}^3E$ given in Moschovakis [1973] lifts easily to an arbitrary acceptable structure $\mathfrak{A}$ in the following form: *the class of relations semi-prime computable in*

$E, R_1, \ldots, R_l$ is the smallest semi-Spector class Γ on A such that $R_1, \ldots, R_l \in \Delta$. Thus these two results taken together give a proof of Theorem 29.

This proof makes heavy and unnecessary use of the theory of prime computability. A more direct proof of Theorem 29 can be constructed by imitating the Grilliot argument in the context of an arbitrary semi-Spector class rather than the specific class of relations semi-prime computable in $E, R_1, \ldots, R_l$. We will not give this argument here since it is long and quite involved and since (in the only version that I know now) it still appeals to same results about (first order) prime computability. In this paper, as in EIAS, we have attempted as much as possible to stay away from recursion theoretic techniques in favor of more "elegant", coding-free methods. It would be very nice to have a proof of Theorem 29 in this spirit.

One example where Π_1^0-IND is not a Spector class is the structure $\boldsymbol{R}$ of analysis, see Section 1D of EIAS. Grilliot's Theorem implies that on $\boldsymbol{R}$, Π_1^0-IND is the class of all relations semi-hyperanalytic in some real α, and this is not closed under $\mathfrak{E}$ by Corollary 10.2 of Moschovakis [1967]. Examples of countable acceptable structures on which Π_1^0-IND is not a Spector class can be constructed easily from this, using the methods of Section 8D of EIAS.

Semi-Spector classes arise naturally in the theory of recursion in higher types over ω and it would be useful to develop a theory of companions for them.

References

S. Aanderaa

[1973] *Inductive definitions and their closure ordinals*, Proc. of the 1972 Oslo Colloquium on Generalized Recursion Theory, Amsterdam 1973.

P. Aczel

[1973] *Stage comparison theorems and game playing with inductive definitions*, ibid. Oslo Proceedings.

P. Aczel and W. Richter

[1971] *Inductive definitions and analogues of large cardinals*, Conference in Mathematical Logic, London 1970, (1971), pp. 1–10.

[1973] *Inductive definitions and reflecting properties of admissible ordinals*, ibid. Oslo Proceedings.

J. Barwise, R. O. Gandy and Y. N. Moschovakis

[1971] *The next admissible set*, J. Symbolic Logic 36 (1971), pp. 108–120.

T. J. Grilliot

[1971] *Inductive definitions and computability*, Trans. Amer. Math. Soc. 158 (1971), pp. 309–317.

Y. N. Moschovakis

[1967] *Hyperanalytic predicates*, Trans. Amer. Math. Soc. 129 (1967), pp. 249–282.

[1969] *Abstract first order computability I and II*, Trans. Amer. Math. Soc. 138 (1969), pp. 427–504.

[1973] *Structural characterizations of classes of relations*, ibid. Oslo Proceedings.

[EIAS] *Elementary Induction on Abstract Structures*, Amsterdam 1973.

W. Richter

[1971] *Recursively Mahlo ordinals and inductive definitions*, Logic Colloquium 1969, Amsterdam 1971, pp. 273–288.

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