On cluster sets of arbitrary functions

by

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Abstract. In the article two general theorems concerning cluster sets of arbitrary functions are proved. The first theorem (Theorem 1) generalizes a theorem of U. Hunter on the sets of asymmetry of arbitrary functions and the second (Theorem 8) is an analogy of Theorem 1. It refers to the set of all points at which two cluster sets defined in different ways do not intersect. By these general theorems some theorems on cluster sets and approximate cluster sets of arbitrary functions defined on Euclidean spaces are proved. For example, a characterization of the sets of all the points of asymmetry of arbitrary functions defined on Euclidean spaces is given.

1. Introduction and notation. The symbol E_n denotes the n-dimensional Euclidean space with the usual scalar product $\langle \ldots, \ldots \rangle$ and the norm $|\ldots|$. The origin in E_n is denoted by o. We put $E_1^* = E_1 \cup \{+\infty, -\infty\}$. The vector determined by the points $x \in E_n$, $y \in E_n$ is denoted by xy. The angle determined by the vectors u, v is denoted by $\widehat{u, v}$. For $o \neq t \in E_n$ and $0 < \alpha \leqslant \pi$ the open cone $U(t, \alpha)$ is the set of all points $o \neq x \in E_n$ for which $\widehat{ox, ot} < \alpha$. The whole space E_n is also termed a cone. The open sphere of the centre $x \in E$ and the radius r is denoted by $K(x, r)$. The symbol μ denotes the outer Lebesgue measure in E_n. If $M \subset E_n$ and $x \in E_n$, then we denote by M_x the image of M under the translation taking the origin into x. If $Y \subset E_n$, $M \subset E_n$, $x \in E_n$ and $\mu\big(M \cap K(x, h)\big) > 0$ for arbitrary $h > 0$, then we put

$$\overline{D}(Y, x, M) = \limsup_{h \to 0+} \mu\big(K(x, h) \cap Y \cap M\big)/\mu\big(K(x, h) \cap M\big),$$

$$\underline{D}(Y, x, M) = \liminf_{h \to 0+} \mu\big(K(x, h) \cap Y \cap M\big)/\mu\big(K(x, h) \cap M\big).$$

We write $\overline{D}(Y, x) = \overline{D}(Y, x, E_n)$ and in the case of $n = 1$ we put $D^+(Y, x) = \overline{D}\big(Y, x, (x, \infty)\big)$ and $D^-(Y, x) = \overline{D}\big(Y, x, (-\infty, x)\big)$. If U is the cone in E_n, then we put $D^U Y(x) = \overline{D}(Y, x, U_x)$. Similarly we define the symbols $\underline{D}(Y, x)$, $D_+(Y, x)$, $D_-(Y, x)$, $D_U(Y, x)$.

Let $M \subset E_n$, $D \subset E_n$, $x \in E_n$. Let T be a topological space and let $f\colon D \to T$ be a mapping. Then we define the partial cluster set $C(f, x, M)$ as the set of all points $y \in T$ such that $x \in \big(f^{-1}(V) \cap M\big)'$ for any neighbourhood V of the point y. We write $C(f, x, E_n) = C(f, x)$, $C(f, x, U_x) = C_U(f, x)$ and in the case of $n = 1$ we put

$$C_+(f, x) = C(f, x, (x, \infty)), \quad C_-(f, x) = C(f, x, (-\infty, x)).$$

Further, we define the partial approximate cluster set $W(f, x, M)$ as the set of all points $y \in T$ such that $\bar{D}(M \cap f^{-1}(V), x) > 0$ for any neighbourhood V of the point y. We write $W(f, x) = W(f, x, E_n)$, $W_U(f, x) = W(f, x, U_x)$ and in the case of $n = 1$ we put

$$W_+(f, x) = W(f, x, (x, \infty)), \quad W_-(f, x) = W(f, x, (-\infty, x)).$$

The set of points $x \in E_n$ for which there exists a cone U such that $C_U(f, x) \neq C(f, x)$ we call the *set of points of asymmetry of the mapping* f and denote by $A(f)$. We have

$$A(f) = \{x / C(f, x) \neq \bigcap_U C_U(f, x)\}$$

and in the case of $n = 1$

$$A(f) = \{x / C_+(f, x) \neq C_-(f, x)\}.$$

Similarly the set $A_{ap}(f)$ of points of approximate asymmetry of the mapping f is the set

$$A_{ap}(f) = \{x / W(f, x) \neq \bigcap W_U(f, x)\}$$

and in the case of $n = 1$

$$A_{ap}(f) = \{x / W^+(f, x) \neq W^-(f, x)\}.$$

The sets of points of asymmetry and approximate asymmetry of arbitrary functions have been investigated by several authors.

W. H. Young [13] proved that for an arbitrary function $f\colon E_1 \to E_1$ the set $A(f)$ is countable. In the same article he also proved an analogy of the preceding theorem in E_2. M. Kulbacka [10] proved that for an arbitrary function $f\colon E_1 \to E_1$ the set $A_{ap}(f)$ is of the first category and of measure zero. L. Belowska [2] proved that there exists a function f for which the set $A_{ap}(f)$ is uncountable. The theorem of Kulbacka has been generalized by T. Świątkowski [12] and U. Hunter [8]. U. Hunter [8] generalized the theorem of M. Kulbacka by proving that for an arbitrary function $f\colon E_2 \to E_1$ the set $A_{ap}(f)$ is also of the first category and of measure zero. In [7] U. Hunter proved a general theorem concerning the set of points of asymmetry of an arbitrary function. For a further general theorem of this kind see [12].

In the second part of the present article a general theorem (Theorem 1) concerning the sets of points of asymmetry of arbitrary functions is proved. This theorem generalizes the basic theorem of ([7]). We frequently use it to deduce theorems concerning mappings from theorems concerning sets.

In the third part we characterize the sets of points of asymmetry of arbitrary mappings $f\colon E_n \to T$ where T is an infinite locally compact separable metric space.

In the fourth part we prove several theorems concerning approximate cluster sets of arbitrary functions. We prove two theorems (Theorem 3, Theorem 6), which improve Hunter's theorem on the sets of points of approximate asymmetry. In this part we use some well-known theorems concerning the boundary behaviour of arbitrary functions in E_n.

It is possible to say that in parts 2-4 the set of all points at which two cluster sets defined in different ways are not equal is investigated.

In parts 5-6 the set of all points at which two cluster sets do not intersect is investigated.

In part five a general theorem concerning sets of this type (Theorem 8) is proved.

In part six several theorems are proved on the basis of that general theorem. We prove that the set of all points x for which $W^+(f, x) \cap W^-(f, x) = \emptyset$ is countable for an arbitrary function $f\colon E_1 \to E_1^*$. This theorem generalizes a theorem of Kempisty [9]. Further, we prove an analogy of this theorem for functions defined in E_n, $n > 1$. The last theorem of this part (Theorem 11) generalizes Bagemihl's theorem concerning crookedly ambiguous points ([1], p. 213).

In part seven several theorems which describe the boundary behaviour of arbitrary functions in E_2 in terms of the angle approximate cluster sets are proved. Theorem 12 generalizes both the theorem of Bruckner and Goffman ([4], p. 517), and the theorem of Goffman and Sledd ([6], Theorem 4).

2. A general theorem concerning sets of points of asymmetry. This part is based on Hunter's paper [7]. Theorem 1 strengthens and generalizes the basic theorem of [7]. The main difference between Theorem 1 and Hunter's theorem is that by Hunter's theorem it can only be proved that a certain set is small and Theorem 1 enables us to prove in addition that this set is a Borel set.

An arbitrary mapping $u\colon \exp S \to \exp S$, for which $M \subset u(M)$ and $\bigcup_{k=1}^{n} u(M_k) = u(\bigcup_{k=1}^{n} M_k)$, where the sets $M, M_1, \ldots, M_n$ are arbitrary subsets of S, is called a *closure operation* on S. We put $M'_u = \{x\colon x \in u(M - \{x\})\}$. Clearly the relation $\bigcup_{k=1}^{n} (M_k)'_u = (\bigcup_{k=1}^{n} M_k)'_u$ holds.

Let the symbol T denote some fixed topological space in this part. If u is a closure operation on a set S, $f\colon S\to T$ is an arbitrary mapping and $x\in S$, we define the *cluster set* $C(f,x,u)$ as the set of all points $y\in T$ such that $x\in\big(f^{-1}(V)\big)_u$ for any neighbourhood V of the point y. If a set $M^x\subset E_n$ is given for every point $x\in E_n$, there exists a unique closure operation $u=u[\{M^x\}]$ such that $C(f,x,M^x)=C(f,x,u)$ for any $x\in E_n$ and any mapping $f\colon D\to T$, where D is a subset of E_n. The closure operation $u=u[\{M^x\}]$ is defined by the relation $u(L)=L\cup\{x\colon x\in\overline{M^x\cap L}\}$. Further, there exists a unique closure operation $v=v[\{M^x\}]$ such that $W(f,x,M^x)=C(f,x,v)$ for any $x\in E_n$ and any mapping $f\colon D\to T$ where D is a subset of E_n. The closure operation $v=v[\{M^x\}]$ is defined by the relation $v(L)=L\cup\{x\colon \overline{D}(M^x\cup L,x)>0\}$.

If U is a cone in E_n, we put $e[U]=u[\{U_x\}]$ and $d[U]=v[\{U_x\}]$. The closure operations $e[U]$, $d[U]$ are topologies but we do not need this fact. The topology $e=e[E_n]$ is the Euclidean topology and $d=d[E_n]$ is the density topology. In the case of $n=1$ we define, in a natural way, the topologies e^+, e^-, d^+, d^-. The relations $C_U(f,x)=C(f,x,e[U])$, $W_U(f,x)=C(f,x,d[U])$, $C_+(f,x)=C(f,x,e^+)$ etc. hold for any $x\in E_n$ and any mapping $f\colon D\to T$, where D is a subset of E_n.

Let closure operations u,v on S be given. Then, if $M\subset S$, we define the set of points of asymmetry of the set M with respect to closure operations u,v as the set $A(M,u,v)=(M'_u-M'_v)\cup(M'_v-M'_u)$. If $f\colon S\to T$ is an arbitrary mapping, we define the set of points of asymmetry of the mapping f with respect to u,v as the set

$$A(f,u,v)=\{x\colon C(f,x,u)\neq C(f,x,v)\}.$$

THEOREM 1. *Suppose we are given closure operations u,v on a set S and a locally compact topological space T having a countable basis of open sets. Let $f\colon S\to T$ be an arbitrary mapping. Then there exist sequences $\{M_n\}_{n=1}^\infty$, $\{L_n\}_{n=1}^\infty$ of subsets of S such that*

$$A(f,u,v)=\bigcup_{n=1}^\infty\big(A(M_n,u,v)\cap A(L_n,u,v)\big). \tag{1}$$

Proof. Let $\mathfrak{B}$ be a countable basis of open sets of T. Let $\{(U_n,V_n)\}_{n=1}^\infty$ be a sequence of all pairs $U_n\in\mathfrak{B}$, $V_n\in\mathfrak{B}$ such that $\overline{U}_n\subset V_n$ and $\overline{U}_n$ is compact. We shall prove that relation (1) holds for $M_n=f^{-1}(U_n)$, $L_n=f^{-1}(V_n)$.

Let $x\in A(f,u,v)$. Then we may suppose without loss of generality that there exists a $y\in T$ such that $y\in C(f,x,u)-C(f,x,v)$. Then there exists a $V\in\mathfrak{B}$ such that $y\in V$ and $x\notin\big(f^{-1}(V)\big)'_v$. Since every locally compact topological space is regular, there exists a $U\in\mathfrak{B}$ such that $y\in U$, $\overline{U}$ is compact and $\overline{U}\subset V$. Let k be an integer such that $U=U_k$ and $V=V_k$. Then we have $x\notin(L_k)'_v$ and therefore $x\notin(M_k)'_v$. Since $x\in(L_k)'_u$ and $x\in(M_k)'_u$, we have $x\in A(M_k,u,v)\cap A(L_k,u,v)$. Consequently

$$A(f,u,v)\subset\bigcup_{n=1}^\infty\big(A(M_n,u,v)\cap A(L_n,u,v)\big).$$

Let $x\in A(M_k,u,v)\cap A(L_k,u,v)$ for some integer k. Then we may suppose without loss of generality that $x\in(M_k)'_u$. Then, since $M_k\subset L_k$, we have $x\in(L_k)'_u$ and therefore $x\notin(L_k)'_v$. We shall prove that $\overline{U}_k\cap C(f,x,u)\neq\emptyset$. Suppose that $\overline{U}_k\cap C(f,x,u)=\emptyset$. Then for every point $z\in\overline{U}_k$ there exists a neighbourhood V_z such that $x\notin\big(f^{-1}(V_z)\big)'_u$. Since $\overline{U}_k$ is compact, there exists a finite set $K\subset\overline{U}_k$ such that $\overline{U}_k\subset\bigcup_{z\in K}V_z$. Hence $f^{-1}(\overline{U}_k)\subset\bigcup_{z\in K}f^{-1}(V_z)$ and therefore $x\notin(M_k)'_u\subset\big(f^{-1}(\overline{U}_k)\big)'_u$ and that is a contradiction. Hence there exists a $y\in\overline{U}_k\cap C(f,x,u)$. Since $y\in V_k$ and $x\notin\big(f^{-1}(V_k)\big)'_v$, we have $y\notin C(f,x,v)$ and consequently $x\in A(f,u,v)$ Therefore we have

$$A(f,u,v)\supset\bigcup_{n=1}^\infty\big(A(M_n,u,v)\cap A(L_n,u,v)\big),$$

and this completes the proof.

3. A characterization of the sets of points of asymmetry of arbitrary real functions defined on E_n. The main purpose of this part is to characterize the sets of points of asymmetry of arbitrary functions defined on E_n. We prove a more general theorem from which it follows that the same characterization holds for the sets of points of asymmetry of more general mappings, e.g. of real functions which we consider as mappings $E_n\to E_1^*$ (i.e. if we permit the limit values $+\infty$, $-\infty$). Suppose that $n>1$ is a fixed integer.

If $G\subset E_n$ and there exists a system of Cartesian coordinates and a Lipschitz function $f\colon E_{n-1}\to E_1$ such that G is the set of all points whose coordinates fulfil the equation $x'_n=f(x'_1,\ldots,x'_{n-1})$, then the set G is called a *Lipschitz surface*. If $M\subset E_n$ and there exists a sequence $\{G_n\}_{n=1}^\infty$ of Lipschitz surfaces in E_n such that $M\subset\bigcup_{n=1}^\infty G_k$, then the set M is called a *sparse set*. It is obvious that every sparse set is a set of the first category and of measure zero. Every subset of a sparse set is a sparse set and the union of a sequence of sparse sets is a sparse set.

The essential part of the proof of the following proposition is contained in [11], p. 265.

PROPOSITION 1. *Suppose we are given $M\subset E_n$ and a cone $U=U(t,\alpha)$ $(|t|=1)$ in E_n. Denote by A the set of all $x\in M$ such that $x\notin\overline{M\cap U_x}$. Then*

(i) *A is a sparse set.*

(ii) *If $a > \frac{1}{2}\pi$, then the set A is countable.*

Proof. For every positive integer k, we denote by P_k the set of all points $x \in M$ such that $M \cap U_x \cap K(x, 1/k) = \emptyset$. We express each P_k as the union of a sequence $\{P_{k,m}\}_{m=1}^{\infty}$ of sets with diameters less than $1/k$. Clearly $A = \bigcup_{k,m=1}^{\infty} P_{k,m}$. Choose a new system of Cartesian coordinates such that $(0, \ldots, 0, 0)$ and $(0, \ldots, 0, 1)$ are the new coordinates of the origin and the point t, respectively. For an arbitrary pair of points $x \in P_{k,m}$, $y \in P_{k,m}$ with the new coordinates $(x_1', \ldots, x_n')$ and $(y_1', \ldots, y_n')$ we have $x \notin U_y$ and $y \notin U_x$. From this it follows that $\langle y-x, t\rangle \leqslant \cos a|y-x|$ and $\langle x-y, t\rangle \leqslant \cos a|x-y|$ and therefore

$$|y-x|\cos a \geqslant |\langle y-x, t\rangle| . \tag{2}$$

If $a > \frac{1}{2}\pi$, (2) is absurd, hence no set $P_{k,m}$ contains two different points. Therefore the set A is countable and this proves (ii).

If $a \leqslant \frac{1}{2}\pi$, it follows from (2) that

$$|y_n'-x_n'| \leqslant \cos a\big(|(y_1', \ldots, y_{n-1}')-(x_1', \ldots, x_{n-1}')|+|y_n'-x_n'|\big),$$

$$|y_n'-x_n'| \leqslant (\cos a/1-\cos a)(y_1', \ldots, y_{n-1}')-(x_1', \ldots, x_{n-1}')| .$$

Hence there exists a Lipschitz function f defined on a subset of E_{n-1} such that for the new coordinates $z_1', \ldots, z_n'$ of any point $z \in P_{k,m}$ we have $z_n' = f(z_1', \ldots, z_{n-1}')$. Since for every Lipschitz function defined on a subset of a metric space there exists an extension on the whole space ([3]), $P_{k,m}$ is a subset of a Lipschitz surface. Hence the set A is a sparse set.

Proposition 2. *Suppose we are given $M \subset E_n$ and a cone $U = U(t, a)$ in E_n. Then*

(i) *$A(M, e, e[U])$ is a sparse set.*

(ii) *If $a > \frac{1}{2}\pi$, then the set $A(M, e, e[U])$ is countable.*

Proof. By definition, $A(M, e, e[U]) = \{x\colon\ x \in M',\ \ x \notin \overline{M \cap U_x}\}$. Hence $z \in \overline{M}$ and $z \notin \overline{M \cap U_z}$ for an arbitrary point $z \in A(M, e, e[U])$. Therefore, according to Proposition 1, $A(M, e, e[U])$ is a sparse set.

Proposition 3. *Suppose we are given $M \subset E_n$ and a cone $U = U(t, a)$ in E_n. Then $A(M, e, e[U])$ is a F_σ set.*

Proof. Since $A(M, e, e[U]) = M' \cap \{x\colon x \notin \overline{M \cap U_x}\}$, it is sufficient to prove that the set $P = \{x\colon x \notin \overline{M \cap U_x}\}$ is a F_σ set. If for every positive integer k, we denote by P_k the set of all points x such that $U_x \cap K(x, 1/k) \cap M = \emptyset$, then clearly $P = \bigcup_{k=1}^{\infty} P_k$. If $\{x_m\}_{m=1}^{\infty}$ is a sequence such that $\lim x_m = x$, we have clearly

$$U_x \cap K(x, 1/k) \subset \bigcup_{m=1}^{\infty} \big(U_{x_m} \cap K(x_m, 1/k)\big) . \tag{3}$$

If $x_m \in P_k$ for every positive integer m, it follows from (3) that $U_x \cap K(x, 1/k) \cap M = \emptyset$ and therefore $x \in P_k$. Hence all the sets P_k are closed and consequently P is a F_σ set.

Lemma 1. *If $F \subset E_n$ is a closed set, then there exists a set $M \subset E_n$ such that $F = M'$.*

Proof. It is sufficient to add to each isolated point x of M in an obvious way a sequence of points converging to x.

Theorem 2. *Let P be a locally compact topological space having a countable basis of open sets. Then*

(i) *If $f\colon E_n \to P$ is an arbitrary mapping, then the set $A(f)$ of all points of asymmetry of the mapping f is a sparse F_σ set.*

(ii) *If A is a sparse F_σ set and $N \subset P$ is an infinite set, then there exists a mapping $f\colon E_n \to P$ such that $f(E_n) \subset N$ and $A = A(f)$.*

Proof. (i) Let $\mathfrak{U}$ be the set of all cones $U(t, a)$ in E_n for which the coordinates of the point t and the number a are rational. Then evidently $A(f) = \bigcup_{U \in \mathfrak{U}} \{x\colon C(f, x) \neq C_U(f, x)\}$. We have $A(f, e, e[U]) = \{x\colon C(f, x) \neq C_U(f, x)\}$. By Theorem 1 there exist sequences $\{M_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$ of subsets of E_n such that $A(f, e, e[U]) = \bigcup_{k=1}^{\infty} A(M_k, e, e[U]) \cap A(L_k, e, e[U])$. From Propositions 2 and 3 it follows that each set $A(f, e, e[U])$ is a sparse F_σ set and consequently $A(f)$ is a sparse F_σ set.

(ii) Since A is a sparse set, there exists a sequence $\{H_v\}_{v=1}^{\infty}$ of Lipschitz surfaces such that $A \subset \bigcup_{v=1}^{\infty} H_v$. Further, $A = \bigcup_{k=1}^{\infty} D_k$ where all D_k are closed sets. Then $A = \bigcup_{v,k=1}^{\infty} (H_v \cap D_k)$ and therefore we may write $A = \bigcup_{m=1}^{\infty} A_m$, where any set A_m is a closed subset of a Lipschitz surface G_m. Any set G_m is closed in E_n and there exists a homeomorphism $h_m\colon E_{n-1} \to G_m$. Any set $h_m^{-1}(A_m)$ is closed in E_{n-1} and by Lemma 1 there exists a $D_m \subset E_{n-1}$ such that $h_m^{-1}(A_m) = D_m'$. Writing $C_m = h_m(D_m)$, we have $C_m \subset G_m$ and $A_m = C_m'$. Since N is an infinite subset of P, there clearly exist two univalent sequences $\{r_k\}_{k=1}^{\infty}$ and $\{q_k\}_{k=1}^{\infty}$ in N for which, if we put $R = \bigcup_{k=1}^{\infty} \{r_k\}$ and $Q = \bigcup_{k=1}^{\infty} \{q_k\}$, we infer that the set $R' = Q'$ contains at most one point and $R' \cap R = \emptyset$, $Q' \cap Q = \emptyset$, $R \cap Q = \emptyset$.

Evidently there exists a sequence $\{F_k\}_{k=1}^{\infty}$ of disjoint dense subsets of E_n such that $E_n - \bigcup_{t=1}^{\infty} C_t = \bigcup_{k=1}^{\infty} F_k$.

We put

$$f(x) = \begin{cases} r_k, & \text{if} \quad x \in \bigcup_{t=1}^{\infty} C_t, \quad \text{where } k = \min\{t, x \in C_t\}, \\ q_k, & \text{if} \quad x \in F_k. \end{cases}$$

If $y \in A$, writing $j = \min\{k, y \in A_k\}$, we infer that $y \in C'_j$ and $y \notin C'_k$ for $1 \leqslant k < j$. Then $r_j \in C(f, y)$. Since G_j is a Lipschitz surface, there exists a cone U such that $U_y \cap G_j = \emptyset$ and therefore $U_y \cap C_j = \emptyset$. Then $r_j \notin C_U(f, y)$ and consequently $y \in A(f)$. If $y \notin A$, then $y \notin C'_k$ for all integers k and therefore $\bar{R} \cap C(f, y) = \emptyset$. Hence $C_U(f, y) = C(f, y) = \bar{Q}$ for an arbitrary cone U in E_n and therefore $y \notin A(f)$.

Note. If we replace in Theorem 2 "E_n" by "E_1" and "sparse F_σ" by "countable", then we obtain a theorem whose proof is quite analogous to the proof of Theorem 2.

4. Theorems concerning approximate cluster sets of arbitrary functions. The main purpose of this part is to prove Theorems 3 and 7 which improve Hunter's theorem concerning the sets of approximate asymmetry of arbitrary functions in E_n. In this part we suppose that $n \geqslant 1$ is a fixed integer. We shall state some definitions.

The half-space $E_n \times (0, \infty)$ is denoted by H. Any point $(x, 0) \in E_{n+1}$ is identified with the point $x \in E_n$. If U is a cone in E_{n+1} with the vertex at the origin such that $\bar{U} \subset H \cup \{0\}$, then U is called a cone in H. If $t \in H$, $x \in E_n$ and f is a function in H, we put $C_t(f, x) = C(f, x, P)$, where P is the half-line issuing from the point x in the direction of the vector ot.

A point $x \in E_n$ is termed a P-point of a set $M \subset E_n$, if there exists a $\delta > 0$ such that for any $\varepsilon > 0$ there exist spheres $K(x, h)$, $K(y, r)$ such that $K(y, r) \subset K(x, h) - M$, $h \leqslant \varepsilon$ and $\delta < r/h$. A set $M \subset E_n$ is termed a P-set if an arbitrary point $x \in M$ is a P-point of the set M. A set $N \subset E_n$ is termed a P_σ-set if it is the union of a sequence of P-sets. Every P_σ-set is evidently a set of the first category and of measure zero. On the contrary, there exist sets of the first category and of measure zero which are not P_σ-sets. This assertion is stated without proof in [5]. We shall use the following theorems concerning the boundary behaviour of functions in E_n:

THEOREM A [5]. *Let f be an arbitrary function defined on H. Let A be the set of all points $x \in E_n$ for which there exist cones U, V in H such that $C_U(f, x) \neq C_V(f, x)$. Then A is a P_σ-set.*

THEOREM B [4]. *Let f be a continuous function defined on H. Let $t \in H$ and A be the set of all points $x \in E_n$ for which $C(f, x, H) \neq C_t(f, x)$. Then A is a set of the first category.*

Denote by φ the mapping defined on H by the relation $\varphi(x, r) = K(x, r)$. Let a set $M \subset E_n$ be given. Then we denote by f_M the function defined on H by the relation $f_M(z) = \mu(M \cap \varphi(z))/\mu(\varphi(z))$.

LEMMA 2. *Let $U(t, \alpha)$, $U(t, \beta)$ be cones in E_n, $0 < \alpha < \beta \leqslant \pi$. Let $x \in E_n$, $y \in U_x(t, \alpha)$, $|x - y| = r$. Then, writing $d = \frac{1}{2} r \sin(\beta - \alpha)$, we have $K(y, d) \subset U(t, \beta) \cap K(x, 2r)$.*

Proof. Let $z \in K(y, d)$. Then evidently $\sin(\widehat{xz, xy}) \leqslant d/r$ and according to the definition of d we obtain $\widehat{xz, xy} < \beta - \alpha$. Hence $\widehat{xz, ot} \leqslant \widehat{xz, xy} + \widehat{xy, ot} < \beta$, and therefore $z \in U(t, \beta)$. Consequently $K(y, d) \subset U(t, \beta)$ and, since evidently $K(y, d) \subset K(x, 2r)$, $K(y, d) \subset U(t, \beta) \cap K(x, 2r)$.

LEMMA 3. *Let $U(t, a)$ be a cone in E_n, $0 < a < b$. Then there exists a cone V in H with the following property: If $z \in E_n$ and $(x, x_{n+1}) \in V_z$ then $x \in U_z(t, \frac{1}{2}a)$ and $|x - z| a < x_{n+1} < |x - z| b$.*

Proof. It is evidently sufficient to put $V = U((t, t_{n+1}), \beta)$, where $a < t_{n+1}/|t| < b$ and $\beta > 0$ is a sufficiently small number.

LEMMA 4. *Let $U = U(t, \alpha)$ be a cone in E_n. Then there exists a cone V in H with the following property: If $M \subset E_n$, $x \in M$ and $D^U M(x) = 0$ then $C_V(f_M, x) = \{0\}$.*

Proof. We choose a cone V in H according to Lemma 3 for $a = \frac{1}{8} \sin \frac{1}{2}\alpha$ and $b = \frac{1}{2} \sin \frac{1}{2}\alpha$. Suppose that $D^U M(x) = 0$ and $C_V(f, x) \neq \{0\}$. Then there exist a number $a > 0$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of points of V_x such that $\lim x_k = x$ and $f_M(x_k) = \mu(\varphi(x_k) \cap M)/\mu(\varphi(x_k)) > a$ for all integers $k \geqslant 1$. If $x_k = (s_k, r_k)$, then $\varphi(x_k) = K(s_k, r_k)$ and, by the choice of the cone V, we have $s_k \in U_x(t, \frac{1}{2}\alpha)$ and

$$(4) \qquad \tfrac{1}{8}|s_k - x| \sin(\tfrac{1}{2}\alpha) < r_k < \tfrac{1}{2}|s_k - x| \sin(\tfrac{1}{2}\alpha).$$

Put $R_k = U \cap K(x, 2|s_k - x|)$. Then, according to Lemma 2, $K_k = K(s_k, r_k) \subset R_k$. On account of (4) we infer that there exists an $\varepsilon > 0$ such that $\mu K_k / \mu R_k > \varepsilon$ and therefore

$$\mu(M \cap R_k)/\mu R_k \geqslant \mu(M \cap K_k)/\mu R_k = \big(\mu(M \cap K_k)/\mu K_k\big) \cdot (\mu K_k / \mu R_k) > a\varepsilon$$

for all k. Hence $D^U M(x) > 0$ and this is a contradiction.

The following lemma is obvious.

LEMMA 5. *Let $M \subset E_n$, $x \in E_n$. Then, putting $b = (0, \ldots, 0, 1) \in H$, we have $\bar{D} M(x) = \max C_b(f_M, x)$.*

LEMMA 6. *If $M \subset E_n$, $x \in E_n$, and if $\mu(M \cap K(x, h)) > 0$ for any $h > 0$, then $1 \in C(f_M, x, H)$.*

Proof. Since $\mu(M \cap K(x, 1/k)) > 0$ for any integer $k \geqslant 1$, there exists a point $y_k \in K(x, 1/k)$ such that $\bar{D}M(y_k) = 1$. Hence there exists an $0 < a_k < 1/k$ such that $\mu(M \cap K(y_k, a_k))/\mu(K(y_k, a_k)) > 1 - 1/k$. Then $\lim(y_k, a_k) = x$ and $\lim f_M(y_k, a_k) = 1$ and therefore $1 \in C_H(f, x)$.

Proposition 4. *Let $M \subset E_n$. Denote by A the set of all $x \in E_n$ for which $\bar{D}M(x) > 0$ and $D^U M(x) = 0$. Then A is a P_σ-set.*

Proof. Let $x \in A$. By Lemma 5 $C_b(f, x) \neq \{0\}$ and therefore there exists a cone W in H such that $C_W(f, x) \neq \{0\}$. On the other hand, by Lemma 4 there exists a cone V in H such that $C_V(f, x) = \{0\}$. Therefore $C_W(f, x) \neq C_V(f, x)$ and consequently, on account of Theorem A, the set A is a P_σ-set.

Proposition 5. *Let $M \subset E_n$. Denote by A the set of all $x \in E_n$ such that $\bar{D}M(x) \neq 1$ and $\mu(M \cap K(x, h)) > 0$ for any $h > 0$. Then A is a set of the first category.*

Proof. Let $x \in A$. From Lemma 6 it follows that $1 \in C(f_M, x, H)$ and from Lemma 5 it follows that $1 \notin C_b(f_M, x)$. Therefore $C_b(f_M, x) \neq C(f_M, x, H)$ and, since f_M is evidently continuous in H, from Theorem B it follows that A is a set of the first category.

Proposition 6. *Let $M \subset E_n$. Denote by A the set of all $x \in E_n$ for which $1 \neq \bar{D}M(x) > 0$. Then A is a set of the first category and of measure zero.*

Proof. From Proposition 5 it follows that A is a set of the first category. Choose a measurable set $G \supset M$ such that $\mu(K \cap G) = \mu(K \cap M)$ for any sphere K. Then $\bar{D}(G, x) \neq 1$ and $\underline{D}(E_n - G, x) \neq 1$ for any point $x \in A$. Hence, by the density theorem, A is a set of measure zero.

Lemma 7. *Let $M \subset E_n$ and let $U(t, a)$ be a cone in E_n. Then the function $f(x) = D^U M(x)$ is of Baire class 2.*

Proof. Put $g(x, r) = \mu(U_x(t, a) \cap K(x, r) \cap M)/\mu(U_x(t, a) \cap K(x, r))$ for $x \in E_n$ and $r > 0$. The function g is evidently continuous in $E_n \times (0, \infty)$ and $f(x) = \limsup_{r \to 0+} g(x, r)$. Put

$$a_{j,m}(x) = \max_{1/j \leqslant r \leqslant 1/m} g(x, r) \quad \text{and} \quad b_k(x) = \sup_{0 < r \leqslant 1/k} g(x, r).$$

Then

$$f(x) = \lim b_k(x) \quad \text{and} \quad b_k(x) = \lim_{j \to \infty} a_{j,k}(x).$$

Since all the functions $a_{j,m}$ are continuous, all the functions b_k are of Baire class 1. Therefore the function $f(x)$ is of Baire class 2.

Theorem 3. *Let P be a separable locally compact metric space. Let $f\colon E_n \to P$ be an arbitrary mapping. Then the set $A_{ap}(f)$ of all points of approximate asymmetry of f is a P_σ-set of the class $F_{\sigma\delta\sigma}$.*

Proof. Let $\mathfrak{U}$ be the set of all cones $U(t, a)$ in E_n for which the coordinates of the point t and the number a are rational. Since $A(f, d, d[U]) = \{x\colon W(f, x) \neq W_U(f, x)\}$, clearly

$$A_{ap}(f) = \bigcup_{U \in \mathfrak{U}} A(f, d, d[U]). \tag{5}$$

By Theorem 1 there exist sequences $\{M_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$ of subsets of E_n such that

$$A(f, d, d[U]) = \bigcup_{k=1}^{\infty} \big(A(M_k, d, d[U]) \cap A(L_k, d, d[U])\big). \tag{6}$$

By definition, for any $M \subset E_n$ we have

$$A(M, d, d[U]) = \{x\colon \bar{D}M(x) > 0\} \cap \{x\colon D^U M(x) = 0\}. \tag{7}$$

From Proposition 4 and (7) it follows that each set $A(M, d, d[U])$ is a P_σ-set. From Lemma 7 and (7) it follows that $A(M, d, d[U])$ is the intersection of a $G_{\delta\sigma}$ set and a $F_{\sigma\delta}$ set. Hence $A(M, d, d[U])$ is a $F_{\sigma\delta\sigma}$ set. Consequently the theorem follows from (5) and (6).

From Propositions 5 and 6 which refer to sets we deduce, by a usual method, two theorems which refer to mappings.

In the rest of this part, let P be a fixed topological space having a countable basis of open sets and let $f\colon E_n \to P$ be an arbitrary mapping. If $x \in E_n$, we denote by $M(f, x)$ the set of all points $y \in P$ such that $\mu(f^{-1}(V) \cap K(x, h)) > 0$ for any $h > 0$ and any neighbourhood V of the point y. Evidently $M(f, x)$ is the set of all points $y \in P$ for which there exists a set $B \subset E_n$ such that $\lim_{t \to x,\, t \in B} f(t) = y$ and $\mu(B \cap K(x, h)) > 0$ for any $h > 0$. Further, we denote by $H(f, x)$ the set of all points $y \in P$ such that $\bar{D}(f^{-1}(V))(x) = 1$ for any neighbourhood V of the point y. The set $H(f, x)$ coincides with the set of all points $y \in P$ for which there exists a set $B \subset E_n$ such that $\lim_{t \to x,\, t \in B} f(t) = y$ and $\bar{D}B(x) = 1$.

Lemma 8. *Let $Z \subset E_n$ be a measurable set, $\underline{D}Z(x) > 0$. Then*

$$H(f, x) \subset W(f, x, Z) \subset W(f, x) \subset M(f, x).$$

Proof. It is clearly sufficient to prove $H(f, x) \subset W(f, x, Z)$. For an arbitrary set $M \subset E_n$ we clearly have

$$\bar{D}M(x) \leqslant \bar{D}(M - Z)(x) + \bar{D}(M \cap Z)(x) \leqslant 1 - \underline{D}Z(x) + \bar{D}(M \cap Z)(x).$$

Therefore the relation $\bar{D}M(x) = 1$ implies $\bar{D}(M \cap Z)(x) > 0$. From this and from the definitions our assertion immediately follows.

Theorem 4. *The set $B = \{x\colon H(f, x) \neq M(f, x)\}$ is a set of the first category.*

Proof. Let a sequence $\{G_k\}_{k=1}^{\infty}$ form a basis of open sets of P. Denote by B_k the set of all $x \in E_n$ such that $\overline{D}(f^{-1}(G_k))(x) < 1$ and $\mu(f^{-1}(G_k) \cap K(x, h)) > 0$ for any $h > 0$. Clearly $B \subset \bigcup_{k=1}^{\infty} B_k$. From Proposition 5 it follows that each B_k is a set of the first category and therefore B is also a set of the first category.

THEOREM 5. *The set $C = \{x\colon H(f, x) \neq W(f, x)\}$ is a set of the first category and of measure zero.*

Proof. Let a sequence $\{G_k\}_{k=1}^{\infty}$ form a basis of open sets of P. Denote by C_k the set of all $x \in E_n$ such that $0 < \overline{D}(f^{-1}(G_k))(x) < 1$. Clearly $C \subset \bigcup_{k=1}^{\infty} C_k$. On account of Proposition 6 we immediately obtain our assertion.

On the basis of Lemma 8 and Theorem 5 we immediately obtain the following theorem.

THEOREM 6. *The set of all points $x \in E_n$ for which*

$$W(f, x) \neq \bigcap \{W(f, x, Z)\colon Z \text{ is a measurable set, } \underline{D}Z(x) > 0\}$$

is a set of the first category and of measure zero.

PROPOSITION 7. *Let $U = U(t, \beta)$ be a cone in E_n, $n > 1$, $M \subset E_n$. Put $A = \{x\colon D^U M(x) = 0, \underline{D}M(x) > 0\}$. Then*

(i) *A is a sparse set.*

(ii) *If $\beta > \frac{1}{2}\pi$, then A is countable.*

Proof. Denote by A_m the set of all points $x \in E_n$ such that

$$D^U M(x) = 0 \tag{8}$$

and

$$\mu(K(y, r) \cap M)/\mu(K(y, r)) > 1/m \quad \text{for} \quad r < 1/m . \tag{9}$$

Clearly $A = \bigcup_{m=1}^{\infty} A_m$. Put $V = U(t, \alpha)$, where α is a number such that

$$0 < \alpha < \beta \quad \text{if } \beta \leqslant \tfrac{1}{2}\pi \quad \text{and} \quad \tfrac{1}{2}\pi < \alpha < \beta \quad \text{if } \beta > \tfrac{1}{2}\pi .$$

We shall prove that the relation $x \in A_m$ implies $x \notin \overline{A_m \cap V_x}$. Suppose that this assertion does not hold. Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_k = x$ and $x_k \in V_x \cap A_m$ for each k. Put

$$d_k = \tfrac{1}{2}|x_k - x| \sin(\beta - \alpha) .$$

On account of Lemma 2 we obtain

$$K_k = K(x_k, d_k) \subset (U_x \cap K(x, 2|x - x_k|)) = R_k .$$

The number $p = \mu K_k/\mu R_k$ clearly does not depend on k. On account of (9) we infer that, for sufficiently large numbers k, $\mu(K_k \cap M)/\mu(K_k) > 1/m$ and consequently

$$\mu(R_k \cap M)/\mu R_k \geqslant \mu(K_k \cap M)/\mu R_k = (\mu(K_k \cap M)/\mu K_k)(\mu K_k/\mu W_k) > p/m .$$

Hence $D^U M(x) > 0$ and this contradicts (8).

Consequently from Proposition 1 it follows that each set A_m is a sparse set and in the case of $\beta > \frac{1}{2}\pi$ each set A_m is countable. Therefore the relation $A = \bigcup_{m=1}^{\infty} A_m$ implies (i) and (ii).

THEOREM 7. *For all points $x \in E_n$, except those of a sparse set A, (a countable set A, respectively), the following assertion holds:*

If there exists an $a = \lim\text{ap}_{U, t\to x} f(t)$ for a cone U in E_n (a cone $U = U(t, \alpha)$ in E_n such that $\alpha > \frac{1}{2}\pi$, respectively), then $a \in H(f, x)$.

Proof. Let a sequence $\{G_k\}_{k=1}^{\infty}$ form a basis of open sets of P. Let $\mathfrak{U}$ be the set of all cones $U(t, \alpha)$ in E_n such that the coordinates of the point t and number α are rational. Denote by A_U the set of all $x \in E_n$ such that there exists an $a = \lim\text{ap}_{U, t\to x} f(t)$ and $a \notin H(f, x)$. Denote by $A_{U,k}$ the set of all $x \in E_n$ for which $D^U(E_n - f^{-1}(G_k))(x) = 0$ and $\overline{D}(f^{-1}(G_k)) \neq 1$. Then, for each cone U in E_n, $A_U \subset \bigcup_{k=1}^{\infty} A_{U,k}$. If $x \in A_{U,k}$, then

$$\underline{D}(E_n - f^{-1}(G))(x) > 0 .$$

Therefore, by Proposition 7, each set $A_{U,k}$ is a sparse set (in the case of $U = U(t, \alpha)$ where $\alpha > \frac{1}{2}\pi$ it is countable). Hence it clearly suffices to put $A = \bigcup \{A_U\colon U \in \mathfrak{U}\}$, $(A = \bigcup \{A_U\colon U \in \mathfrak{U}, U = U(t, \alpha), \alpha > \frac{1}{2}\pi\}$, respectively).

5. A general theorem concerning the intersection of cluster sets. Let u, v be closure operations on a set S. If $M \subset S$, then we put

$$D(M, u, v) = ((S - M'_u) \cap (S - (S - M)'_v)) \cup ((S - M'_v) \cap (S - (S - M)'_u)) .$$

If T is a topological space and $f\colon S \to T$ is an arbitrary mapping, then we put $D(f, u, v) = \{x\colon C(f, x, u) \cap C(f, x, v) = \emptyset\}$.

THEOREM 8. *Suppose we are given closure operations u, v on a set S and a compact topological space T having a countable basis of open sets. Let $f\colon S \to T$ be an arbitrary mapping. Then there exist sequences $\{M_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$ of subsets of S such that*

$$D(f, u, v) = \bigcup_{k=1}^{\infty} (D(M_k, u, v) \cap D(L_k, u, v)) .$$

Proof. Let $\mathcal{B}$ be a countable basis of open sets of T. Let $x \in D(f, u, v)$. Then for each point $y \in T$ there exists a $U_y \in \mathcal{B}$ such that $y \in U_y$ and either $x \notin \big(f^{-1}(U_y)\big)'_u$ or $x \notin \big(f^{-1}(U_y)\big)'_v$. Since T is compact, there exists a finite set $K \subset T$ such that $T = \bigcup_{y \in K} U_y$. Put $G = \bigcup_{y \in L} U_y$ and $H = \bigcup_{y \in K-L} U_y$, where L is the set of all $y \in K$ such that $x \notin \big(f^{-1}(U_y)\big)'_u$. Then clearly the following relations hold:

$$f^{-1}(G) = \bigcup_{y \in L} f^{-1}(U_y), \quad f^{-1}(H) = \bigcup_{y \in K-L} f^{-1}(U_y), \quad f^{-1}(G) \cup f^{-1}(H) = S,$$
$$x \notin \big(f^{-1}(G)\big)'_u, \quad x \notin \big(f^{-1}(H)\big)'_v.$$

Therefore

$$x \in D\big(f^{-1}(G), u, v\big) \cap D\big(f^{-1}(H), u, v\big).$$

Let $\{(G_k, H_k)\}_{k=1}^{\infty}$ be a sequence of all pairs (G, H) such that G and H are finite unions of elements of $\mathcal{B}$ and $G \cup H = T$. Put $M_k = f^{-1}(G_k)$, $L_k = f^{-1}(H_k)$. Then

$$D(f, u, v) \subset \bigcup_{k=1}^{\infty} \big(D(M_k, u, v) \cap D(L_k, u, v)\big).$$

On the other hand, suppose that $x \in D(M_k, u, v) \cap D(L_k, u, v)$ for a positive integer k. Let $y \in T$. Then either $y \in G_k$ or $y \in H_k$. In both cases we infer that either $y \notin C(f, x, u)$ or $y \notin C(f, x, v)$. Hence $y \notin C(f, x, u) \cap C(f, x, v)$. Therefore we have $x \in D(f, u, v)$. Consequently

$$D(f, u, v) \supset \bigcup_{k=1}^{\infty} \big(D(M_k, u, v) \cap D(L_k, u, v)\big)$$

and this completes the proof.

6. Applications of Theorem 8.

Lemma 9. *The set $D(M, d^+, d^-)$ is countable for each set $M \subset E_1$.*

Proof. If $x \in D(M, d^+, d^-)$, then either $D_+M(x) = 1$ and $D^-M(x) = 0$, or $D^+M(x) = 0$ and $D_-M(x) = 1$. Set

$$f(y) = \begin{cases} \mu\big(M \cap (0, y)\big) & \text{if} \quad y \geqslant 0, \\ -\mu\big(M \cap (y, 0)\big) & \text{if} \quad y < 0. \end{cases}$$

Then either $1 = \underline{f}^+(x) > \bar{f}^-(x) = 0$ or $0 = \bar{f}^+(x) < \underline{f}^-(x) = 1$, where $\underline{f}^-(x)$, $\underline{f}^+(x)$, $\bar{f}^-(x)$, $\bar{f}^+(x)$ are Dini derivatives of f at the point x. On account of the well-known theorem on Dini derivatives ([11], p. 261) we infer that $D(M, d^+, d^-)$ is countable.

Theorem 9. *Let T be a compact topological space having a countable basis of open sets. Let $f\colon E_1 \to T$ be an arbitrary mapping. Then the set $D = \{x\colon W^+(f, x) \cap W^-(f, x) = \emptyset\}$ is countable.*

Proof. By Theorem 8 there exist sequences $\{M_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$ of subsets of E_1 such that

$$D = D(f, d^+, d^-) = \bigcup_{k=1}^{\infty} \big(D(M_k, d^+, d^-) \cap D(L_k, d^+, d^-)\big).$$

On account of this relation and Lemma 9 we obtain the assertion of the theorem.

Lemma 10. *Let V and $U = U(t, \beta)$ be cones in E_n, $n > 1$, and let $M \subset E_n$. Then*

(i) *$D(M, d[U], d[V])$ is a sparse set of type $F_{\sigma\delta}$.*

(ii) *If $\beta > \frac{1}{2}\pi$, then $D(M, d[U], d[V])$ is countable.*

Proof. By the definitions

$$D(M, d[U], d[V]) = \{x\colon D^U M(x) = 0,\ D^V(E_n - M)(x) = 0\} \cup$$
$$\cup \{x\colon D^V M(x) = 0,\ D^U(E_n - M)(x) = 0\}.$$

On account of Lemma 7 we infer that $D(M, d[U], d[V])$ is of the type $F_{\sigma\delta}$. The rest of the assertion of the lemma follows from Proposition 7.

Theorem 10. *Let T be a compact topological space having a countable basis of open sets. Let $n > 1$, and let $f\colon E_n \to T$ be an arbitrary mapping. Let D be the set of all $x \in E_n$ for which there exist cones U, V in E_n (cones $U = U(t, \alpha)$ and $V = U(t, \beta)$, where $\max(\alpha, \beta) > \frac{1}{2}\pi$, respectively) such that $W_U(f, x) \cap W_V(f, x) = \emptyset$. Then D is a sparse set of type $F_{\sigma\delta\sigma}$ (a countable set, respectively).*

Proof. Let R be the set of all pairs $U(t, \alpha)$, $U(s, \beta)$ of cones in E_n such that the coordinates of the points t, s and the numbers α, β are rational. Let $P \subset R$ be the set of all pairs $U(t, \alpha)$, $U(s, \beta)$ such that $\max(\alpha, \beta) > \frac{1}{2}\pi$. If U, V are cones in E_n, then we put $D_{U,V} = \{x\colon W_U(f, x) \cap W_V(f, x) = \emptyset\}$. Clearly

$$D = \bigcup \{D_{U,V}\colon (U, V) \in R\} \tag{11}$$

($D = \bigcup \{D_{U,V}\colon (U, V) \in P\}$, respectively). On account of Lemma 10 and Theorem 8 we infer that each set $D_{U,V}$ is a sparse set of the type $F_{\sigma\delta\sigma}$ and is countable for $(U, V) \in P$. Since the sets R and P are countable, the assertion of the theorem follows from (11).

By an arc at a point $x \in E_2$ we shall mean a simple continuous curve

$$\varphi\colon z = z(t) \quad (0 \leqslant t < 1)$$

such that $z(t) \neq x$ for $0 \leqslant t < 1$ and $\lim_{t \to 1-} z(t) = x$. We put $k(\varphi) = z(0)$.

We say that the arcs φ, ψ at a point x are associated provided there exists an angle $U = U(t, \alpha)$ in E_2 such that $\alpha < \frac{1}{2}\pi$, $\varphi \subset U_x$ and $\psi \subset U_x$.

Lemma 11. *Let $M \subset E_2$. Denote by D the set of all points $x \in E_2$ for which there exist associated arcs φ, ψ at x such that $\varphi \subset M$ and $\psi \subset E_2 - M$. Then the set D is a sparse set.*

Proof. Denote by $\mathcal{F}$ the set of all $F = (K, U, A, B)$ with the following properties:

(i) $K = K(s, r)$ is a disc in E_2, $U = U(t, \alpha)$ is an angle in E_2 and $\alpha < \frac{1}{2}\pi$.

(ii) $A \cap B = \emptyset$ and there exist angles $V = U(v, \beta)$, $W = U(w, \gamma)$ such that $A = \overline{\mathrm{Fr} K \cap V_s}$, $B = \overline{\mathrm{Fr} K \cap W_s}$.

(iii) The coordinates of the points s, t, v, w and the numbers r, α, β, γ are rational.

If $F = (K, U, A, B) \in \mathcal{F}$, we denote by D_F the set of all points $x \in K$ for which there exist arcs φ_x, ψ_x at x such that the following relations hold: $\varphi_x \subset \overline{U_x \cap K}$, $\psi_x \subset \overline{U_x \cap K}$, $\varphi_x \subset M$, $\psi_x \subset E_2 - M$, $\mathrm{Fr} K \cap \varphi_x = \{k(\varphi_x)\}$, $k(\varphi_x) \in A \subset U_x$, $\mathrm{Fr} K \cap \psi_x = \{k(\psi_x)\}$, $k(\psi_x) \in B \subset U_x$. It is easy to see that $D = \bigcup_{F \in \mathcal{F}} D_F$. Let $F \in \mathcal{F}$ be given. Let us use the same notation as in the definition of the set $\mathcal{F}$. Then without loss of generality we may suppose, considering the points t, v, w as complex numbers, that $v = kwe^{i\omega}$ for some $k > 0$ and $0 < \omega < \pi$. Let $Z = U(z, \delta)$ be the complementary angle to U such that $z = te^{\pi i/2}$ and $\delta = \frac{1}{2}\pi - \alpha$. Let $x \in D_F$. Then we shall prove that $x \notin (D_F \cap Z_x)'$. Assume, on the contrary, that $x \in (D_F \cap Z_x)'$. Then clearly there exists a $y \in D_F \cap Z_x$ such that $(A \cup B) \subset U_y$. Now it is easy to see that the arcs φ_x and ψ_y intersect, and this is a contradiction. Therefore by Proposition 1 the set D_F is a sparse set. Since the set $\mathcal{F}$ is countable, the set D is a sparse set as well.

Theorem 11. *Let T be a compact topological space having a countable basis of open sets, and let $f\colon E_2 \to T$ be an arbitrary mapping. Denote by D the set of all points $x \in E_2$ for which there exist associated arcs φ_x and ψ_x at x such that $C(f, x, \varphi_x) \cap C(f, x, \psi_x) = \emptyset$. Then the set D is a sparse set.*

Proof. Set

$$M^x = \varphi_x \quad \text{and} \quad N^x = \psi_x \quad \text{if} \quad x \in D$$

and

$$M^x = N^x = E_2 \quad \text{if} \quad x \in E_2 - D.$$

Set $u = u[\{M^x\}]$ and $v = u[\{N^x\}]$. The definition of these symbols is in the first part. From Lemma 11 it follows that $D(M, u, v)$ is a sparse set for an arbitrary $M \subset E_2$. Since $D = \{x\colon C(f, x, u) \cap C(f, x, v) = \emptyset\}$, we obtain by Theorem 8 that D is a sparse set.

Note. If we replace the word "associated arcs φ_x and ψ_x at x" by "arcs φ_x and ψ_x at x having non-collinear semitangents at x" in Theorem 11, then we obtain Bagemihl's theorem on crookedly ambiguous points [1].

7. The boundary behaviour of arbitrary functions in the plane and the approximate cluster sets. The main purpose of this part is to prove Theorem 12. Theorem 12 improves both a theorem from [4] which asserts that the set A from Theorem 12 is a set of the first category and a theorem from [6] which asserts that this set is a set of measure zero. Theorem 13 is a further application of Theorem 8.

In this part we shall use the following notation. If $\alpha \in E_1$, then we denote by $P(\alpha)$ the half-line in E_2 issuing from the origin and containing the point $e^{i\alpha}$. If numbers α, β, $0 \leqslant \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi$, $\alpha \neq \beta$, are given, then we denote by $U(\alpha, \beta)$ the angle determining by half-lines $P(\alpha)$ and $P(\beta)$. If the numbers α and β are rational, we shall say that the angle $U(\alpha, \beta)$ is rational.

Lemma 12. *Let $U = U(\alpha, \beta)$ and $V = U(\gamma, \delta)$ be angles such that $0 < \alpha < \beta < \gamma < \delta < \pi$. Then there exist positive integers k, l, m, n and a number $0 < p < 1$ such that the following assertion holds: If $x \in E_1$, $y \in E_1$ and $y - x = r > 0$, then*

(i) *$U_x \cap V_y \subset K(y, lr)$ and $\mu\big(K(y, lr) \cap V_y\big)/\mu(U_x \cap V_y) < m$.*

(ii) *$U_x \cap V_y \cap K(x, r/k) = \emptyset$ and $\mu T/\mu\big(K(x, r/k) \cap U_x\big) < n$, where T is the triangle determining by half-lines $P_x(\alpha)$, $P_x(\beta)$, $P_y(\gamma)$.*

(iii) *If we put $z = x + pr$, then $P_x(\alpha) \cap P_y(\delta) \cap P_z(\gamma) \neq \emptyset$.*

(iv) *Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $p \leqslant a_i \leqslant \frac{1}{2}(p+1)$ for each positive integer i. Define the sequence $\{y_i\}_{i=0}^{\infty}$ by relations $y_0 = x + r$ and $y_i = x + a_i(y_{i-1} - x)$. Then $\lim\limits_{i \to \infty} y_i = x$ and $T \subset \bigcup\limits_{i=0}^{\infty} V_{y_i}$. Further, there exists a positive integer N such that any point $z \in T$ is contained in at most N different angles U_{y_i}.*

Proof. The existence of the numbers k, l, m, n is obvious. Condition (iii) determines the number p. We shall prove that the number p satisfies condition (iv). Clearly $y_i = x + r\prod\limits_{k=1}^{i} a_k$ and therefore $x + rp^i \leqslant y_i \leqslant x + r\big(\frac{1}{2}(p+1)\big)^i$. Consequently $\lim\limits_{i \to \infty} y_i = x$. From (iii) it follows that $T \subset \bigcup\limits_{i=0}^{\infty} V_{y_i}$. Clearly there exists a number M such that $V_y \cap U_x \cap V_t = \emptyset$, where $t = x + r\big(\frac{1}{2}(p+1)\big)^M$. It is obviously sufficient to choose $N = M$.

Lemma 13. *Let $W \subset H$ be an angle in E_2 and let $M \subset H$ be an arbitrary set. Put $A = \{x \in E_1\colon D^W M(x) = 0,\ \overline{D}(M, x, H) > 0\}$. Then A is a P_σ-set of type $F_{\sigma\delta\sigma}$.*

Proof. Let $x \in A$. Then clearly there exists a rational angle $V = U(\gamma, \delta)$ such that $0 < \gamma < \delta < \pi$ and $D^V M(x) = 0$. If we put

$Z_\varepsilon = U(\varepsilon, \gamma-\varepsilon) \cup U(\delta+\varepsilon, \pi-\varepsilon)$, then for any sufficiently small $\varepsilon > 0$ we have

$$\bar{D}(Z_\varepsilon \cap M, x) \geqslant \bar{D}(M, x) - \bar{D}(M - Z_\varepsilon, x) \geqslant \bar{D}M(x) - 2\varepsilon/\pi > 0,$$

and therefore $D^U M(x) > 0$ for an angle U in H. We may clearly suppose that the cone U is rational. Put $A_{U,V} = \{x\colon D^U M(x) > 0,\ D^V M(x) = 0\}$ for any pair of angles U, V in H. Let $\mathfrak{U}$ be the set of all pairs of rational angles in H such that $\bar{U} \cap \bar{V} = \{0\}$. Then $A = \bigcup \{A_{U,V}\colon (U, V) \in \mathfrak{U}\}$. We shall prove that $A_{U,V}$ is a P_σ-set for any pair of angles $(U, V) \in \mathfrak{U}$. Assume, on the contrary, that there exists a pair $(U, V) \in \mathfrak{U}$ such that $A_{U,V}$ is not a P_σ-set. Then we can suppose without loss of generality that $U = U(\alpha, \beta)$ and $V = U(\gamma, \delta)$, where $0 < \alpha < \beta < \gamma < \delta < \pi$. Let the sense of the letters k, l, m, n, p, T be the same as in Lemma 12 and put $d = 1/mnN$. For any positive integers a, b denote by $B_{a,b}$ the set of all points $x \in E_1$ such that $D^U M(x) > 1/a$ and

$$\mu(V_x \cap M \cap K(x, h))/\mu(V_x \cap K(x, h)) < d/a \quad \text{for any } h < 1/b.$$

Then clearly $A_{U,V} \subset \bigcup_{a,b=1}^{\infty} B_{a,b}$. Since $A_{U,V}$ is not a P_σ-set, there exist positive integers a, b such that $B_{a,b}$ is not a P-set. Hence there exists a point $x \in B_{a,b}$ which is not a P-point of the set $B_{a,b}$. Then there exists a number $s_0 > 0$ such that

$$B_{a,b} \cap (x+ps, x+(p+1)s/2) \neq \emptyset \quad \text{for any } s < s_0. \tag{12}$$

Choose a number $r > 0$ such that $s_0 > r$, $lr < 1/b$ and

$$\mu(U_x \cap K(x, r/k) \cap M)/\mu(U_x \cap K(x, r/k)) > 1/a. \tag{13}$$

On account of (12) we infer that there exists a sequence of numbers $\{a_i\}_{i=1}^{\infty}$ such that $p \leqslant a_i \leqslant (p+1)/2$ for any integer $i \geqslant 1$ and $y_i \in B_{a,b}$ for any integer $i \geqslant 0$, where $\{y_i\}_{i=0}^{\infty}$ is the sequence defined in Lemma 12, (iv). Put $C_i = V_{y_i} \cap U_x$ for any integer $i \geqslant 0$. On account of Lemma 12, (iv) we have $T = \bigcup_{i=0}^{\infty} C_i$ and

$$\mu\Big(\bigcup_{i=1}^{\infty} C_i\Big) \geqslant \Big(\sum_{i=1}^{\infty} \mu C_i\Big)/N. \tag{14}$$

By Lemma 12, (ii) we obtain $K(x, r/k) \subset \bigcup_{i=1}^{\infty} C_i$ and

$$\begin{aligned} \mu(U_x \cap K(x, r/k) \cap M)/\mu(K(x, r/k) \cap U_x) &\leqslant n \sum_{i=1}^{\infty} \mu(C_i \cap M)/T \\ &\leqslant Nn \sum_{i=1}^{\infty} \mu(C_i \cap M)\Big/\sum_{i=1}^{\infty} \mu C_i. \end{aligned} \tag{15}$$

Since $lr < 1/b$ and $y_i \in B_{a,b}$ for any positive integer $i \geqslant 0$, we obtain by Lemma 12, (i) $\mu(C_i \cap M)/\mu C_i < md/a$. Hence by (15)

$$\mu(U_x \cap K(x, r/k) \cap M)/\mu(K(x, r/k) \cap U_x) \leqslant Nnmd/a = 1/a$$

and this contradicts (13). Therefore $A_{U,V}$ is a P_σ-set for any $(U, V) \in \mathfrak{U}$, and therefore A is a P_σ-set. On account of Lemma 7 we immediately infer that A is a $F_{\sigma\delta\sigma}$ set.

THEOREM 12. *Let T be a locally compact topological space having a countable basis of open sets. Let $f\colon H \to T$ be an arbitrary mapping. Denote by A the set of all points $x \in E_1$ for which there exists an angle U in H such that $W(f, x, H) \neq W_U(f, x)$. Then A is a P_σ-set of type $F_{\sigma\delta\sigma}$.*

Proof. Let $g\colon \bar{H} \to T$ be an arbitrary extension of the mapping f. Let $\mathfrak{U}$ be the set of all rational angles in H. Since

$$\{x \in E_1\colon W(f, x, H) \neq W_U(f, x)\} = E_1 \cap A(g, d[H], d[U])$$

for any angle U in H, it is evident that

$$A = E_1 \cap \bigcup_{U \in \mathfrak{U}} A(g, d[H], d[U]). \tag{16}$$

If U is an angle in H, then by Theorem 1 there exist sequences $\{M_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$ of subsets of $\bar{H}$ such that

$$A(g, d[H], d[U]) = \bigcup_{k=1}^{\infty} \big(A(M_k, d[H], d[U]) \cap A(L_k, d[H], d[U])\big);$$

hence, on account of Lemma 13, we infer that $E_1 \cap A(g, d[H], d[U])$ is a P_σ-set of type $F_{\sigma\delta\sigma}$. Therefore the assertion of the theorem follows from (16).

The following lemma clearly holds.

LEMMA 14. *Let $U = U(\alpha, \beta)$, $V = U(\gamma, \delta)$ be angles in H such that $0 < \alpha < \beta < \gamma < \delta < \pi$. Then there exist positive integers s, t, a such that the following assertion holds: If $x \in E_1$, $y \in E_1$ and $y - x = r > 0$, then*

$$(U_x \cap V_y) \subset (K(x, rs) \cap K(y, rt)),$$
$$\mu(U_x \cap V_y)/\mu(U_x \cap K(x, rs)) < 1/a,$$
$$\mu(U_x \cap V_y)/\mu(U_y \cap K(y, rt)) < 1/a.$$

LEMMA 15. *Let $U = U(\alpha, \beta)$ and $V = U(\gamma, \delta)$ be angles in H such that $0 < \alpha < \beta < \gamma < \delta < \pi$, and let $M \subset \bar{H}$. Then the set $E_1 \cap D(M, d[U], d[V])$ is countable.*

Proof. Put

$$K = \{x \in E_1\colon D^U M(x) = 0,\ D^V(\bar{H} - M)(x) = 0\}$$

and

$$L = \{x \in E_1\colon D^V M(x) = 0,\ D^U(\bar{H} - M)(x) = 0\}.$$

Since $E_1 \cap D(M, d[U], d[V]) = K \cup L$, it is clearly sufficient to prove that the set K is countable. Let the sense of the letters s, t, a be the same as in Lemma 14, and let n be an integer. Then we denote by K_n the set of all points $x \in E_1$ such that

$$\mu(U_x \cap K(x,h) \cap M)/\mu(U_x \cap K(x,h)) < 1/2a$$

and

$$\mu(V_y \cap K(y,h) \cap M)/\mu(V_y \cap K(y,h)) < 1/2a$$

for any $h < 1/n$. Clearly $K \subset \bigcup_{n=1}^{\infty} K_n$. If $x \in K_n$, $y \in K_n$ and $0 < y-x < 1/(n\max(s,t))$, we infer by Lemma 14 and by the definition of the sets K_n that

$$\mu(U_x \cap V_y \cap M)/\mu(U_x \cap V_y) < \tfrac{1}{2}$$

and

$$\mu(U_x \cap V_y \cap (\bar{H}-M))/\mu(U_x \cap V_y) < \tfrac{1}{2}$$

and this is a contradiction. Hence all the sets K_n are isolated and therefore the set K is countable.

THEOREM 13. *Let T be a locally compact topological space having a countable basis of open sets. Let $f\colon H \to T$ be an arbitrary mapping. Denote by D the set of all $x \in E_1$ for which there exist angles U, V in H such that $W_U(f,x) \cap W_V(f,x) = \emptyset$. Then D is countable.*

Proof. Let $g\colon \bar{H} \to E_1$ be an arbitrary extension of the mapping f. Let $\mathfrak{U}$ be the set of all pairs $U = U(\alpha, \beta)$, $V = U(\gamma, \delta)$ of rational angles in H such that $0 < \alpha < \beta < \gamma < \delta < \pi$. Since

$$\{x \in E_1\colon W_U(f,x) \cap W_V(f,x) = \emptyset\} = D(g, d[U], d[V]) \cap E_1$$

for any pair $(U, V) \in \mathfrak{U}$, evidently

$$D = E_1 \cap \bigcup_{(U,V)\in\mathfrak{U}} D(g, d[U], d[V]) . \tag{17}$$

If $(U, V) \in \mathfrak{U}$, then by Theorem 8 there exist sequences $\{M_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$ of subsets of $\bar{H}$ such that

$$D(g, d[U], d[V]) = \bigcup_{k=1}^{\infty} \big(D(M_k, d[U], d[V]) \cap D(L_k, d[U], d[V])\big).$$

Hence, on account of Lemma 15, we infer that $E_1 \cap D(g, d[U], d[V])$ is countable. Consequently, from (17) it follows that the set D is countable.

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