On the equivalence of certain coincidence theorems and fixed point theorems

by

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Abstract. The properties that are preserved when certain multifunctions on trees are composed are used to show that some coincidence theorems for multifunctions on compact spaces into trees are equivalent to some classical fixed point theorems on trees. These methods are also used to derive some new coincidence theorems.

1. Introduction. In [3] Helga Schirmer proved a multipart coincidence theorem for certain multifunctions into trees and derived some fixed point theorems from this theorem. In section 2 of this paper we prove the fixed point theorem by using the order structure of a tree and then by applying this theorem and two classical fixed point theorems on trees we obtain the coincidence theorem referred to above. This accomplishes two purposes. First it provides a new proof of the coincidence theorem, and secondly it shows that the coincidence theorem and the fixed point theorems are equivalent.

In [4] Schirmer extended results of Schweigert [5] and Ward [9] on homeomorphisms and monotone maps which leave an endpoint fixed to a coincidence theorem for u.s.c. biconnected multifunctions. In section 3 of this paper we prove some results for biconnected (but not necessarily u.s.c.) multifunctions which leave an endpoint fixed and use these to prove a fixed point theorem for biconnected multifunctions of the Schweigert-Ward type which was a corollary of Schirmer's theorem. We then use this theorem to prove Schirmer's coincidence theorem. A corollary to the results on biconnected multifunctions is a slightly improved version of a theorem of Smithson [7]. Finally, in the last section we use these methods to derive two new coincidence theorems.

A *multifunction* $F\colon X \to Y$ is a point to set correspondence (i.e. $F(x)$ is a nonempty subset of Y for each $x \in X$). If $A \subset X$, then $F(A) = \bigcup \{F(x)\colon x \in A\}$ and if $B \subset Y$, then $F^{-1}(B) = \{x\colon F(x) \cap B \neq \emptyset\}$ where $\emptyset$ is the empty set. The multifunction F is said to be *upper semi-continuous* (u.s.c.) in case $F^{-1}(A)$ is closed for each closed $A \subset Y$. Also F is *lower semi-continuous* (l.s.c.) in case $F^{-1}(V)$ is open for each open $V \subset Y$ and F is *continuous* in case it is both u.s.c. and l.s.c. Further, F is *point closed* (*connected*, etc.) in case $F(x)$ is closed (connected, etc.). A multifunction F is *connected* in case $F(A)$ is connected for each connected $A \subset X$, and F is *monotone* in case $F^{-1}(y)$ is connected for each $y \in Y$. Finally, F is *biconnected* in case F and F^{-1} are connected.

A well known result that we shall need later is that each u.s.c., or l.s.c., point connected multifunction is connected.

A *tree* is a continuum X in which each distinct pair of points in X can be separated by the removal of a third point. A partial order, $\leqslant$, is defined on X as follows: Let $e \in X$ be fixed, then $x \leqslant y$, $x, y \in X$, iff (i) $x = e$, (ii) $x = y$ or (iii) x separates e and y. Now set $L(x) = \{y \in X\colon y \leqslant x\}$ and $M(x) = \{y \in X\colon x \leqslant y\}$. Then Ward [10] characterized trees in terms of this partial order as follows:

LEMMA 1.1. *A continuum X is a tree iff it admits a partial order $\leqslant$ satisfying*:

(i) *$L(x)$ and $M(x)$ are closed for every $x \in X$,*
(ii) *if $x < y$, then there is a $z \in X$ such that $x < z < y$,*
(iii) *$L(x) \cap L(y)$ is nonempty and totally ordered for each $x, y \in X$,*
(iv) *$M(x) \setminus \{x\}$ is open for each $x \in X$.*

A *chain* is a totally ordered subset of a partially ordered set. If X is a tree, the element e used in the definition of the partial order is the least element in X, and in the sequel we refer to the above partial order as the partial order on X with least element e, and will assume each tree is endowed with this partial order. A point $x \in X$ (X a tree) is an *endpoint* in case $X \setminus \{x\}$ is connected. Some further properties of this partial order are:

LEMMA 1.2. *Every nonempty closed subset of X contains a maximum, and each nonempty chain has a least upper bound.*

Proof. See [9, Th. 1].

LEMMA 1.3. *Every nonempty subcontinuum of X contains a least element.*

Proof. See [1, Lemma 2].

LEMMA 1.4. *If $A \subset X$ is connected, if $A \cap M(y) \neq \emptyset$, and if $A \cap \cap (X \setminus M(y)) \neq \emptyset$, then $y \in A$.*

Proof. See [7, Lemma 4].

If $F\colon X \to X$ is a multifunction on a set X into itself, then F has a *fixed point* in case there exists an $x \in X$ such that $x \in F(x)$. If $F, G\colon X \to Y$ are two multifunctions on a set X into a set Y, then F, G have a coincidence point (or simply a coincidence) in case there exists an $x \in X$ such that $F(x) \cap G(x) \neq \emptyset$. A multifunction $F\colon X \to Y$ is *coincidence producing* for a family of multifunctions in case F has a coincidence with every member of the family.

2. Coincidences and fixed points of multifunctions. In [3] Schirmer proved the following theorem.

THEOREM A. *Any point closed* u.s.c. *multifunction $F\colon Z \to X$ from a compact, Hausdorff space Z onto a tree X is coincidence producing for all multifunctions $G\colon Z \to X$ which are either continuous or* u.s.c. *and point connected if it is either open or monotone.*

By permuting the hypotheses we see that there are four parts to Theorem A. We shall show that two of these can be derived from two well known fixed point theorems. The other two parts we shall obtain by giving a separate proof of Theorem 5.2 of [3] which was obtained from Theorem A. Then prove Theorem A by using this theorem. For this we shall need the following lemma whose proof is omitted (this lemma is similar to Lemma 5.1 in [3].)

LEMMA 2.1. *Let $F, G\colon X \to Y$ be two multifunctions on a set X into Y with G onto. Then F, G have a coincidence iff $F \circ G^{-1}\colon Y \to Y$ has a fixed point.*

Each multifunction $F\colon X \to Y$ can be considered to be the following subset of $X \times X$: $F = \{(x, y)\colon y \in F(x),\ x \in X\}$.

Using this interpretation of F we obtain the following lemma.

LEMMA 2.2. *Let $F\colon X \to Y$ be a point closed,* u.s.c. *multifunction on the space X into the regular space Y. Then F is a closed subset of $X \times Y$.*

Proof. Let $(x, y) \notin F$. Then since $F(x)$ is closed and Y is regular there exist disjoint open sets U, V in Y with $y \in U$, $F(x) \subset V$. Further, since F is u.s.c., there exists an open set W containing x such that $F(W) \subset V$. Thus $W \times U$ is an open set which contains (x, y) and such that $W \times U \cap F = \emptyset$.

Now we can prove the following lemma.

LEMMA 2.3. *Let $F\colon X \to Y$ be an* u.s.c., *point closed multifunction on a compact, T_1-space onto a regular space Y. Then F^{-1} is point closed and* u.s.c.

Proof. Let $A \subset X$ be closed and π_1, π_2 be projections on $X \times Y$ onto X and Y respectively. Then $(F^{-1})^{-1}(A) = F(A)$ and $F(A) = \pi_2\big(\pi_1^{-1}(A) \cap F\big)$ which is closed since the projection parallel to a compact space is closed.

COROLLARY 2.4. *If $F\colon X \to Y$ is onto and if both X, Y are compact, T_2-spaces, then F is* u.s.c. *and point closed iff F^{-1} is* u.s.c. *and point closed.*

It is also easy to show that the composition of point closed u.s.c. multifunctions on compact T_2-spaces is point closed and u.s.c. and that the composition of l.s.c. multifunctions is l.s.c.

For the remainder of this section assume that each multifunction is point closed, that Z is a compact T_2-space and that X is a tree.

In 1941 Wallace [12] showed that a tree had the fixed point property for point connected u.s.c. multifunctions. Now if $F\colon Z\to X$ is a u.s.c., monotone multifunction on the compact T_2-space Z onto the tree X, then F^{-1} is u.s.c. and connected, and if G is point closed, point connected and u.s.c., then $G\circ F^{-1}\colon X\to X$ is point closed, point connected and u.s.c. Thus $G\circ F^{-1}$ has a fixed point and hence, F, G have a coincidence.

In 1956 Plunkett [2] showed that trees have the fixed point property for continuous (point closed) multifunctions. Thus let $F\colon Z\to X$ be point closed, u.s.c. and open. Then F^{-1} is continuous and point closed, and if $G\colon Z\to X$ is continuous, then $G\circ F^{-1}\colon X\to X$ is continuous. Hence $G\circ F^{-1}$ has a fixed point and so F, G have a coincidence.

Now in order to prove the remaining two parts of the Theorem we prove the following two fixed point theorems.

THEOREM 2.5. *Let $F\colon X\to X$ be a multifunction on the tree X into itself and suppose there exists a compact T_2-space Z and multifunctions $F_1\colon X\to Z$ and $F_2\colon Z\to X$ such that F_1 is continuous, F_2 is point connected and* u.s.c. *and $F=F_2\circ F_1$. Then F has a fixed point.*

THEOREM 2.6. *In Theorem 2.5 let F_1 be point connected and* u.s.c. *and F_2 continuous. Then F has a fixed point.*

In order to prove these theorems we need the following two lemmas. The proof of Lemma 2.7 is routine and is omitted. Lemma 2.8 follows from the fact that the partial order $\leqslant$ is a closed subset of $X\times X$ (see Ward [10]) and from the fact that the graph of M is the graph of $\leqslant$ and that $M(x)$ is closed for each $x\in X$.

LEMMA 2.7. *If $F, G\colon X\to Y$ are two point closed,* u.s.c. *multifunctions on the space X into the normal space Y, then the set $\{x\colon F(x)\cap G(x)\neq\emptyset\}$ is closed.*

LEMMA 2.8. *Let X be a tree. Then the multifunction $M\colon X\to X$ defined by $M(x)=\{y\colon x\leqslant y\}$ is point closed and* u.s.c.

Proof of Theorem 2.5. The set $E=\{x\colon M(x)\cap F(x)\neq\emptyset\}$ is closed by Lemmas 2.7 and 2.8. Let $x_0\in E$ and assume that $x_0\notin F(x_0)$. We shall show that E contains an x_1 with $x_0<x_1$. Let $y_0\in F(x_0)\cap M(x_0)$ and let $z_0\in F_1(x_0)$ be such that $y_0\in F_2(z_0)$. Now $F_2(z_0)$ is closed and connected and so contains a minimal element y_1. Pick $x'\in X$ such that $x_0<x'<y_1$. Then $F_2(z_0)$ is contained in the open set $M(x')\setminus x'$. Thus there is an open set $U\subset Z$ such that $z_0\in U$ and $F_2(U)\subset M(x')\setminus x'$. Also, there is an open set $V\subset X$, $x_0\in V$, such that $F_1(x)\cap U\neq\emptyset$ for all $x\in V$. Thus there is an $x_1\in V$, $x_0<x_1<x'$, such that $F(x_1)\cap M(x')\setminus x'\neq\emptyset$ and so $x_1\in E$. Since E is closed, E contains a maximal element, and thus F has a fixed point.

Proof of Theorem 2.6. As above the set E is closed and let $x_0\in E$. Assume that $x_0\notin F(x_0)$, and let $y\in F(x_0)\cap M(x_0)$. Since $F(x_0)$ is closed there is an x_1, $x_0<x_1<y$, such that $\{x\colon x_0\leqslant x\leqslant x_1\}\cap F(x_0)=\emptyset$. Now $F_1(x_0)$ is a closed, connected subset of Z. Let

$$A=\{z\in F_1(x_0)\colon F_2(z)\cap M(x_1)\setminus x_1\neq\emptyset\}\,.$$

Since F_2 is l.s.c., A is a relatively open subset of $F_1(x_0)$, and since F_2 is u.s.c. and $x_1\notin F(x_0)$, A is a closed subset of $F_1(x_0)$. Thus $A=F_1(x_0)$. Hence, there is an open subset $U\subset Z$ such that $F_1(x_0)\subset U$ and $z\in U$ implies that $F_2(z)\cap M(x_1)\setminus x_1\neq\emptyset$. Thus there is an open set $V\subset X$, $x_0\in V$, such that $F_1(V)\subset U$. Thus there is an $x_2\in E$ and $x_0<x_2$, and as above F has a fixed point.

Now to complete the proof of Theorem A observe that if F is u.s.c. and open and G u.s.c. and point connected, then Theorem 2.5 applies to $G\circ F^{-1}$ and so by Lemma 2.1 F and G have a coincidence. Finally if F is u.s.c. and monotone and G is continuous, then Theorem 2.6 applies.

3. Biconnected multifunctions. In this section we prove some results about biconnected multifunctions on a tree which leave an endpoint fixed. We then use these results to prove the main theorem of [4]. We develop the proof of these results through a series of lemmas. In this section X denotes a tree and $e\in X$ is the minimal element for the partial order $\leqslant$ on X. If $x\leqslant y$, we set $[x,y]=\{z\colon x\leqslant z\leqslant y\}$.

LEMMA 3.1. *Let F be a point closed, connected multifunction on X. If $x<x_0$ implies $F(x)\subset M(x)$, and if $x_0\notin F(x_0)$, then $F(x_0)\subset M(x_0)$.*

Proof. Suppose $F(x_0)\cap\big(X\setminus M(x_0)\big)\neq\emptyset$ and let $y=\min F(x_0)$. Then set $z_0=\sup L(x_0)\cap L(y)$ and let $z_0<x_1<x_0$ ($z_0\neq x_0$ since $x_0\nleqslant y$). Since F is connected, $A=F([x_1,x_0])$ is connected and meets $M(x_1)$ and contains y. Thus if $z_0<x'\leqslant x_1$, $x'\in A$. But $x_1\leqslant x<x_0$ implies that $F(x)\subset M(x)\subset M(x_1)$ and so $x'\in F(x_0)$. This, in turn, implies that $x_1\in F(x_0)$ and so we conclude that $F(x_0)\subset M(x_0)$.

LEMMA 3.2. *Let $F\colon X\to X$ be a point closed, monotone, connected multifunction which does not have a fixed point. If $F(x_0)\subset M(x_0)$, then there exists a $x_1\in M(x_0)\setminus x_0$ such that $x_0\leqslant x\leqslant x_1$ implies $F(x)\subset M(x)$.*

Proof. Let $y_0=\min F(x_0)$ and let $x_0<y_1<y_0$. Then let $x_0<x_2<y_1$ with $F(x_2)\not\subset M(x_2)$. Then $F([x_0,x_2])$ is a connected set which meets $M(y_1)$ and its complement. Thus $F^{-1}(y_1)\cap[x_0,x_2]\neq\emptyset$. Similarly $F^{-1}(y_1)\cap[x_0,x]\neq\emptyset$ for all $x\in(x_0,x_2]$. Thus, since F is monotone, $x\in F^{-1}(y_1)$ for all $x\in(x_0,x_1]$ for some $x_1\in(x_0,x_2]$. But $x_1<y_1$ and so $F(x)\cap M(x)\neq\emptyset$ for all $x\in[x_0,x_1]$ and since F does not have a fixed point we conclude that $F(x)\subset M(x)$, $x_0\leqslant x\leqslant x_1$.

As a corollary to Lemmas 3.1 and 3.2 we get the following strengthened version of the main result of [7].

COROLLARY 3.3. *Each connected, point closed, monotone multifunction on a tree into itself has a fixed point.*

Before stating the main theorem of this section we need one more lemma.

LEMMA 3.3. *Let F: $X\to X$ be connected and monotone, and let $x_0\in X$ be such that $F(x_0)\subset M(x_0)$. If F does not have a fixed point in $[e, x_0]$, then $F(x)\subset M(x)$ for all $x\in L(x_0)=[e, x_0]$.*

Proof. Let $S=\{x\in L(x_0)\colon F(x)\not\subset M(x)\}$. If $S\neq\emptyset$, then lub $S=z_0$ exists in $L(x_0)$. First suppose that $F(z_0)\not\subset M(z_0)$. But then $F([z_0, x_0])$ is connected and meets both $M(z_0)$ and its complement and hence, contains z_0. But this implies that $z_0\in F(z_0)$. Thus $F(z_0)\subset M(z_0)$, and let $e<z<z_0$ (note $z_0\neq e$ as $S\neq\emptyset$) with $z\in S$. The set $F([z, z_0])$ is connected and meets $M(z_0)$ and its complement. Hence, $F^{-1}(z_0)\cap[z, z_0]\neq\emptyset$. Since F is monotone, there is an element $z'\in[z, z_0]$ such that $\{x\colon z'\leqslant x<z_0\}\subset F^{-1}(z_0)$. But then for $z'\leqslant x<z_0$, $F(x)\subset M(x)$ or $x\in F(x)$. This contradicts either the hypothesis of the lemma or the choice of $z_0=$ lub S. Thus $S=\emptyset$, and the lemma follows.

We are now ready to state the main theorem of this section. (In [4] Theorem 3.4 appeared as a corollary to the main theorem.)

THEOREM 3.4. *Let F: $X\to X$ be a point closed, point connected, monotone,* u.s.c. *multifunction on X onto X. If e is an endpoint, if $e\in F(e)$, and if $F(e)\neq X$, then F has a fixed point not e.*

Proof. First note that the hypotheses on F imply that F is biconnected, and that both F, F^{-1} are point closed. Next we assert that there is an $x\neq e$ in X such that either $F(x)\subset M(x)$ or $F^{-1}(x)\subset M(x)$ or F has a fixed point not e. For this let x_1 be a maximal element of $X\setminus F(e)$, and let $x_2\in X$ be such that $x_1\in F(x_2)$. Then since e is an endpoint, $L(x_1)\cap\cap L(x_2)\neq\{e\}$. Suppose that $x_1\neq x_2$, and let $x_3=\max L(x_1)\cap L(x_2)$. If $x_2=x_3$, we are done. Thus suppose $x_2\neq x_3$ and that $F(x_3)\not\subset M(x_3)$. Then $F([x_3, x_2])$ meets $M(x_3)$ and its complement. Hence, there is an $x\in[x_3, x_2]$ such that $x_3\in F(x)$ but then $x\in F^{-1}(x_3)$ and so either x_3 is the required fixed point or $F^{-1}(x_3)\subset M(x_3)$.

For convenience suppose that $F(x')\subset M(x')$ for some $x'\neq e$. Then by Lemma 3.3 if $e\leqslant x\leqslant x'$, $F(x)\subset M(x)$. Thus let $\mathcal{S}$ be a maximal chain of sets of the form $[e, x]$, $x\in X$, which satisfies (i) $[e, x']\in\mathcal{S}$, (ii) if $[e, x'']\in\mathcal{S}$ and $e\leqslant x\leqslant x''$, then $F(x)\subset M(x)$. Then set $x_0=$ lub $\bigcup\mathcal{S}$. By Lemma 3.1 $F(x_0)\subset M(x_0)$, and by Lemma 3.2 and the maximality of $\mathcal{S}$, x_0 is a fixed point, and $x_0\neq e$.

Remark. The properties of F used in the proof of Theorem 3.4 are that F is biconnected and both F, F^{-1} are point closed. However, Smithson showed in [6] that these conditions imply that F is u.s.c.

We now apply this Theorem 3.4 to obtain Schirmer's result [4].

THEOREM 3.5 (Schirmer). *Let F, G: $T\to X$ be two point closed monotone point connected,* u.s.c. *multifunctions on a tree T onto the tree X which have an endpoint e of T as a coincidence. If both $F(e)\neq X\neq G(e)$ and if either $F(e)$ or $G(e)$ is a singleton, then F and G have a coincidence distinct from e.*

Proof. Suppose that $F(e)=\{e'\}$ is a singleton. If $G^{-1}\circ F$: $T\to T$ satisfies, $G^{-1}\circ F(e)\neq T$, then Theorem 3.4 applies and $G^{-1}\circ F$ has a fixed point not e. Then F and G have a coincidence not e. Now suppose $G^{-1}\circ F(e)=T$. Then $G^{-1}(e')=T$ so $e'\in G(t)$ for all t, and if e' is not an endpoint of X, there exists a $t\in T$ such that $e'\in F(t)$ and $t\neq e$. Hence, in this case, F and G have a coincidence not e. Now assume that e' is an endpoint of X (and $G^{-1}\circ F(e)=T$). If $F^{-1}(e')$ is not a singleton, then there exists a $t\in T$, $t\neq e$, such that $e'\in F(t)$ and t is then a coincidence of F and G. Thus assume $\{e\}=F^{-1}(e')$. Then $G\circ F^{-1}(e')\neq X$, and Theorem 3.4 applies as above.

4. We conclude this paper by using the methods of the previous sections to prove two new coincidence theorems.

In [11] Ward proved that each u.s.c., continuum valued multifunction on a arcwise connected, hereditarily unicoherent, continuum into itself has a fixed point. From this we get:

THEOREM 4.1. *Let X be an hereditarily unicoherent arcwise connected metric continuum and let Z be a compact, T_2-space. If F: $Z\to X$ is a monotone, point closed,* u.s.c. *multifunction onto X, then F has a coincidence with every* u.s.c., *continuum valued multifunction G: $Z\to X$.*

Proof. The multifunction $G\circ F^{-1}$: $X\to X$ is u.s.c. and continuum valued and therefore has a fixed point. Hence, F and G have a coincidence.

Note that Theorem 4.1 implies Ward's theorem and so the fixed point theorem and the coincidence theorem are equivalent.

In [8] Smithson proved that every l.s.c. point connected multifunction on a tree into itself has a fixed point. Hence, we get:

THEOREM 4.2. *Let F: $X\to T$ be an open, monotone multifunction on the space X onto the tree T. Then F is coincidence producing for each* l.s.c., *point connected multifunction G: $X\to T$.*

Finally note that, as above, Theorem 4.2 is equivalent to the fixed point theorem used in its proof.

References

[1] C. E. Capel and W. L. Strother, *Multi-valued functions and partial order*, Portugal. Math. 17 (1958), pp. 41–47.

[2] R. L. Plunkett, *A fixed point theorem for continuous multivalued transformations*, Proc. Amer. Math. Soc. 7 (1956), pp. 160–163.

[3] H. Schirmer, *Coincidences and fixed points of multifunctions into trees*, Pac. J. Math. 34 (1970), pp. 759–767.
[4] — *Biconnected multifunctions of trees which have an endpoint as fixed point or coincidence*, Can. J. Math. 23 (1971), pp. 461–467.
[5] G. E. Schweigert, *Fixed elements and periodic types for homeomorphisms on s.l.c. continua*, Amer. J. Math. 66 (1944), pp. 229–244.
[6] R. E. Smithson, *Criteria for continuity*, Nieuw Archief voor Wiskunde 14 (1966), pp. 89–92.
[7] — *A fixed point theorem for connected multivalued functions*, Amer. Math. Monthly 73 (1966), pp. 351–355.
[8] — *Fixed point theorems for certain classes of multifunctions*, Proc. Amer. Math. Soc. 31 (1972), pp. 595–600.
[9] L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. 5 (1954), pp. 144–161.
[10] — *A note on dendrites and trees*, Proc. Amer. Math. Soc. 5 (1954), pp. 992–994.
[11] — *Characterization of the fixed point property for a class of set valued mappings*, Fund. Math. 50 (1961), pp. 159–164.
[12] A. D. Wallace, *A fixed point theorem for trees*, Bull. Amer. Math. Soc. 47 (1941), pp. 757–760.

Reçu par la Rédaction le 30. 10. 1972

Square bracket partition relations in L

by

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Abstract. We settle some questions about partition relations proposed by Erdös and Hajnal under the assumption that $V = L$: 1) $\varkappa \to [\varkappa]^2_\varkappa$ if and only if $\varkappa \to [\varkappa]^2_2$. 2) $\aleph_{\alpha+n} \not\to [\aleph_{\alpha+1}]^{n+1}_{\aleph_{\alpha+1}}$ for all ordinals α and integers n. 3) $\varkappa \not\to [\delta^+]^{<\omega}_{\delta^+}$ for all $\varkappa$, δ.

Introduction. Our goal in this paper will be to settle some of the open questions about partition relations proposed by Erdös and Hajnal [1] under the assumption that $V = L$. We will do this by combining some combinatorial lemmas with known results (particularly those of Jensen) on the fine structure of L. We begin by defining the square bracket partition relation $\varkappa \to [\gamma]^\delta_\alpha$ of Erdös, Hajnal and Rado [2] for $\varkappa$, γ, δ and α cardinals:

For every function $F\colon [\varkappa]^\delta \to \alpha$ (called a partition of $[\varkappa]^\delta$, the size δ subsets of $\varkappa$, into α pieces) there exists a set $C \subseteq \varkappa$ (called homogeneous for F) such that $\overline{\overline{C}} = \gamma$ and $F''[C]^\delta \neq \alpha$, i.e., some element of the range is omitted when F is restricted to the size δ subsets of C. If this is not the case we write $\varkappa \not\to [\gamma]^\delta_\alpha$.

These partition relations naturally split into two quite different types according to whether δ is infinite or finite. In the former case most of the relations are known to contradict the axiom of choice. Thus for example, AC implies that $\varkappa \not\to [\aleph_0]^{\aleph_0}_2$ for every $\varkappa$ (Erdös and Rado). The more important and interesting questions about these partition relations are therefore concerned with finite δ. In particular, the most fundamental question seems to be to determine the relationships among various apparent strengthenings and weakenings of the simplest non-trivial relation $\varkappa \to [\varkappa]^2_2$ (the well-known equivalent of weak compactness).

When one first considers this problem it seems that increasing the superscript should strengthen the relation while increasing the subscript should weaken it. At it turns out the strengthening achieved by increasing the superscript is illusory (as long as it remains finite). Indeed one can prove that $\varkappa \to [\varkappa]^2_2$ iff $\forall n < \omega$ $(\varkappa \to [\varkappa]^n_2)$ using an alternate formulation of $\varkappa \to [\varkappa]^2_2$ in terms of trees.

On the other hand, increasing the subscript does have some real effect. $\varkappa \to [\varkappa]^2_3$ is strictly weaker than $\varkappa \to [\varkappa]^2_2$ in that there are singular