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Square bracket partition relations in L

by

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Abstract. We settle some questions about partition relations proposed by Erdös and Hajnal under the assumption that $V = L$: 1) $\varkappa \to [\varkappa]^2_\varkappa$ if and only if $\varkappa \to [\varkappa]^2_2$. 2) $\aleph_{\alpha+n} \not\to [\aleph_{\alpha+1}]^{n+1}_{\aleph_{\alpha+1}}$ for all ordinals α and integers n. 3) $\varkappa \not\to [\delta^+]^{<\omega}_{\delta^+}$ for all $\varkappa$, δ.

Introduction. Our goal in this paper will be to settle some of the open questions about partition relations proposed by Erdös and Hajnal [1] under the assumption that $V = L$. We will do this by combining some combinatorial lemmas with known results (particularly those of Jensen) on the fine structure of L. We begin by defining the square bracket partition relation $\varkappa \to [\gamma]^\delta_\alpha$ of Erdös, Hajnal and Rado [2] for $\varkappa$, γ, δ and α cardinals:

For every function F: $[\varkappa]^\delta \to \alpha$ (called a partition of $[\varkappa]^\delta$, the size δ subsets of $\varkappa$, into α pieces) there exists a set $C \subseteq \varkappa$ (called homogeneous for F) such that $\overline{\overline{C}} = \gamma$ and $F''[C]^\delta \neq \alpha$, i.e., some element of the range is omitted when F is restricted to the size δ subsets of C. If this is not the case we write $\varkappa \not\to [\gamma]^\delta_\alpha$.

These partition relations naturally split into two quite different types according to whether δ is infinite or finite. In the former case most of the relations are known to contradict the axiom of choice. Thus for example, AC implies that $\varkappa \not\to [\aleph_0]^{\aleph_0}_2$ for every $\varkappa$ (Erdös and Rado). The more important and interesting questions about these partition relations are therefore concerned with finite δ. In particular, the most fundamental question seems to be to determine the relationships among various apparent strengthenings and weakenings of the simplest non-trivial relation $\varkappa \to [\varkappa]^2_2$ (the well-known equivalent of weak compactness).

When one first considers this problem it seems that increasing the superscript should strengthen the relation while increasing the subscript should weaken it. At it turns out the strengthening achieved by increasing the superscript is illusory (as long as it remains finite). Indeed one can prove that $\varkappa \to [\varkappa]^2_2$ iff $\forall n < \omega$ $(\varkappa \to [\varkappa]^n_2)$ using an alternate formulation of $\varkappa \to [\varkappa]^2_2$ in terms of trees.

On the other hand, increasing the subscript does have some real effect. $\varkappa \to [\varkappa]^2_3$ is strictly weaker than $\varkappa \to [\varkappa]^2_2$ in that there are singular strong limit cardinals such as $\beth_\omega$ which satisfy the former relation while the latter implies that $\varkappa$ is strongly inaccessible. If however, one requires that $\varkappa$ be regular as well, not too much is known even about $\varkappa\to[\varkappa]^2_3$. The best outright theorem seems to be that of Galvin and Shelah [3] that $\aleph_1\nrightarrow[\aleph_1]^2_4$. By assuming GCH Erdös, Hajnal and Rado [2] proved that, for regular $\varkappa$, $\varkappa\to[\varkappa]^2_3$ implies that $\varkappa$ is strongly inaccessible. Indeed, they show much more: $\varkappa^+\nrightarrow[\varkappa^+]^2_{\varkappa^+}$ for all $\varkappa$. Our first result here will settle the question entirely under the assumption that $V = L$. By extending a result of Martin we prove that (in ZF), $\varkappa\to[\varkappa]^2_\varkappa$ implies Souslin's hypothesis for (regular) $\varkappa$. Combining this with a theorem of Jensen, we then have that $\varkappa\to[\varkappa]^2_2$ iff $\varkappa\to[\varkappa]^2_\varkappa$ answering problem 16 of [2].

Remark. It is possible to give information about the relations obtained by increasing both the subscript and superscript without any extra set-theoretic assumptions. In particular, it is shown in [5] that for regular $\varkappa$, $\varkappa\to[\varkappa]^3_6$ implies $\varkappa\to[\varkappa]^2_2$ and more generally that $\varkappa\to[\varkappa]^m_{n^{m-2(2n-1)}}$ implies $\varkappa\to[\varkappa]^2_n$ ([1]).

We next consider (in L) the negative stepping up lemma proposed as problem 17, 17/A of [2]. To consider a simple case, Erdös and Hajnal ask if (assuming GCH) one can go from the fact that $\aleph_1\nrightarrow[\aleph_1]^2_{\aleph_1}$ to the conclusion that $\aleph_2\nrightarrow[\aleph_1]^3_{\aleph_1}$ (and so on to $\aleph_n\nrightarrow[\aleph_1]^{n+1}_{\aleph_1}$). Our second result establishes these negative relations in L (and indeed somewhat more). This however is the best possible result since, as is pointed out in an (unpublished) updated version of [2], $\aleph_2\nrightarrow[\aleph_1]^3_{\aleph_1}$ implies the negation of another partition relation known to be consistent with GCH. Our proof proceeds by combining a "strong" counterexample to $\aleph_1\to[\aleph_1]^2_{\aleph_1}$ with a Kurepa family of subsets of $\aleph_1$ to produce a counterexample to $\aleph_2\to[\aleph_1]^3_{\aleph_1}$. The existence of a Kurepa family is due to Solovay whose work has been considerably extended by Jensen and Kunen [7]. The general procedure can be continued to give, e.g. $\aleph_n\nrightarrow[\aleph_1]^{n+1}_{\aleph_1}$ and various other examples of the negative stepping-up lemma.

Finally, we will take a brief look at partition relations with δ infinite. Here the natural goal is to refute such relationships. Thus for example, problem 14A of [2] asks if $\varkappa\nrightarrow[\aleph_2]^{\aleph_0}_{\aleph_2}$ for every $\varkappa$. We have been informed that assuming $V = L$, Kunen has proven this. We improve the result by showing that one does not even need the full power of the infinite superscript. In particular we show that (assuming $V = L$), $\varkappa\nrightarrow[\delta^+]^{<\omega}_{\delta^+}$ for all $\varkappa$, δ. ($\varkappa\to[\gamma]^{<\omega}_\alpha$ means that for every $F\colon \bigcup_{n<\omega}[\varkappa]^n\to\alpha$ there is a homogeneous set $C\subset\varkappa$ of size γ such that $F''\bigcup_{n<\omega}[C]^n\neq\alpha$. This is clearly weaker than $\varkappa\to[\gamma]^{\aleph_0}_\alpha$.)

([1]) J. Schmerl has improved this last result to $\varkappa\to[\varkappa]^m_r$ implies $\varkappa\to[\varkappa]^2_n$ (for regular $\varkappa$) whose $r=\prod_{j<n}(j(m-1)+1)$.

The Theorems. We begin with some definitions concerning trees. A *tree* is a partially ordered set $(T,<)$ such that for every $x\in T$, $\{y|\ y<x\}$ is well ordered. The *rank* of x is the order type of this set. The *length* of T is the supremum of the ranks of the elements of T. A *chain* in T is a subset of T linearly ordered by $<$, while an *antichain* is a subset all of whose elements are incomparable with respect to $<$. For $\varkappa$ a regular cardinal we say that T is a *normal $\varkappa$-tree* iff

1) T has exactly one point of rank 0.
2) Every point has at least two immediate successors.
3) Every point has at least one successor at every level of T.
4) A chain of limit length has at most one immediate successor.
5) For every α, $\{x|\ \mathrm{rank}(x)=\alpha\}$ has size less than $\varkappa$.

Finally we say *Souslin's hypothesis* holds for (regular) $\varkappa$, $\mathrm{SH}(\varkappa)$, if every normal $\varkappa$-tree of length $\varkappa$ has a chain or an antichain of power $\varkappa$. Note that the stronger assumption that every such tree has a chain of power $\varkappa$ is equivalent (for strongly inaccessible $\varkappa$) to $\varkappa\to[\varkappa]^2_2$. On the other hand, Martin has shown that $\varkappa\to[\varkappa]^2_3$ implies Souslin's hypothesis for $\varkappa$. We now modify and extend his proof.

LEMMA 1. *For regular $\varkappa$, $\varkappa\to[\varkappa]^2_\varkappa$ implies that Souslin's hypothesis holds for $\varkappa$* ([2]).

Proof. It is clear from the normality conditions above that if there is a counterexample to $\mathrm{SH}(\varkappa)$ then there is one with underlying set $\varkappa$ for which every element of rank α has a least $\overline{\overline{\alpha}}$ many immediate successors $(\alpha<\varkappa)$. Let T be such a counterexample. For $\alpha<\varkappa$ and each x of rank α we label the immediate successors of x in a list of length at least α but less than $\varkappa$. We now define a function $F\colon[\varkappa]^2\to\varkappa\cup\{\infty\}$ as follows: If x, y are incomparable with respect to T we set $F\{x,y\}=\infty$. If they are comparable then say $x<y$. In this case there is precisely one immediate successor z of x such that $z\leqslant y$. We have labeled z with some ordinal $\delta<\varkappa$ and so can set $F\{x,y\}=\delta$.

Let $C\subseteq\varkappa$ be homogeneous for F, i.e. $\overline{\overline{C}}=\varkappa$ and $F''[C]^2\neq\varkappa\cup\{\infty\}$. If $\infty\notin F''[C]^2$ then by definition C is a chain in T of size $\varkappa$ — a contradiction. On the other hand, say $\delta<\varkappa$ is omitted from $F''[C]^2$. Consider then the set $D=\{y|\ (\exists x\in C)\ (y \text{ is the } \delta\text{th immediate successor of } x)\}$. As D clearly has size $\varkappa$ it suffices to establish that it is an antichain in T: Consider any $y_1, y_2\in D$ such that $y_1<y_2$. Let $x_1, x_2\in C$ be the elements guaranteed by the definition of D. Since $y_1<y_2$, it is clear that $x_1<y_1\leqslant x_2<y_2$. Thus x_2 is above the δth immediate successor of x_1, i.e. $F\{x_1,x_2\}=\delta$ but as $x_1, x_2\in C$ this is a contradiction. ■

COROLLARY 2. $(V = L)$. *For regular $\varkappa$, $\varkappa\to[\varkappa]^2_2$ iff $\varkappa\to[\varkappa]^2_\varkappa$.*

([2]) These two results have since been independently discovered by Jensen and Silver.

Proof. Jensen has shown [6] that for regular $\varkappa$, $\varkappa\to[\varkappa]^2_2$ iff SH($\varkappa$) but of course $\varkappa\to[\varkappa]^2_2$ implies $\varkappa\to[\varkappa]^2_\varkappa$. ■

We now turn to the negative stepping-up lemma: $\varkappa\not\to[\gamma]^n_\alpha$ implies $\varkappa^+\not\to[\gamma]^{n+1}_\alpha$. As a typical example of our results we will prove (in L) that $\aleph_2\not\to[\aleph_1]^3_{\aleph_1}$. Before beginning we need a few facts. First the proof (from GCH in [2]) that $\aleph_1\not\to[\aleph_1]^2_{\aleph_1}$ actually establishes much more. It constructs a function $G\colon[\aleph_1]^2\to\aleph_1$ such that

(*) for any countable subset C of $\aleph_1$ and any uncountable one U, $G''[C\times U]=\aleph_1$.

We will exploit this extra strength in our proof. Secondly we need Solovay's result that (in L) there is a Kurepa family for $\aleph_1$, i.e., there is a $B\subseteq 2^{\aleph_1}$ such that $\overline{\overline{B}}=\aleph_2$ but for each $x\subseteq\aleph_1$, $\overline{\overline{x}}<\aleph_1$ $\overline{\overline{B\restriction x}}\leqslant\overline{\overline{x}}$ (where $B\restriction x=\{y\cap x|\ y\in B\}$). We can now prove our sample result.

THEOREM 3 (V = L). $\aleph_2\not\to[\aleph_1]^3_{\aleph_1}$.

Proof. Let B and G be as above. We will define a partition $H\colon[B]^3\to\aleph_1$ such that for any $C\subseteq B$ of size $\aleph_1$, $H''[C]^3=\aleph_1$. To begin we define $F\colon[B]^2\to\aleph_1$ by $F\{x,y\}=\mu\alpha\big(x(\alpha)\neq y(\alpha)\big)$. (Recall that x and y are characteristic functions.) Note that for any triple $\{x,y,z\}\in[B]^3$ there are precisely two values that F takes on the three pairs that can be formed from x, y, and z. By abuse of notation we call this pair of values $F\{x,y,z\}$. It is of course a member of $[\aleph_1]^2$ and so we can define $H\{x,y,z\}=G(F\{x,y,z\})$. We claim that this is our desired counterexample to $\aleph_2\to[\aleph_1]^3_{\aleph_1}$.

Let C be any subset of B of size $\aleph_1$. We have two possibilities; either there is an $x\in C$ such that $A_x=\{F\{x,y\}|\ y\in C\}$ has cardinality $\aleph_1$ or not. In the former case we are done immediately. We simply fix such an x and note that as y and z range over C, $F\{x,y,z\}$ ranges (at least) over all pairs from A_x. Thus as G shows that $\aleph_1\not\to[\aleph_1]^2_{\aleph_1}$ and $\overline{\overline{A}}_x=\aleph_1$, $G''[A_x]^2=\aleph_1$ and so $H''[C]^3=\aleph_1$.

If on the other hand $\overline{\overline{A}}_x\leqslant\aleph_0$ for all $x\in C$, we proceed inductively to build a tree whose nodes are the elements of C. Associated with successive nodes x we will also have nested subsets E_x of C. We begin by letting x_0, the least element of C, be the bottom node and associating with it the set $E_{x_0}=C-\{x_0\}$. For each node x of the tree we consider the set $D_x=\{F\{x,y\}|\ y\in E_x\}$. For each $\alpha\in D_x$ we take the least $y\in E_x$ such that $F\{x,y\}=\alpha$. These elements are declared the immediate successors of x and for each such y we set $E_y=\{w\in E_x-\{y\}|\ F\{x,y\}=F\{x,w\}\}$. Finally, if $\{x_i\}_{i<\lambda}$ is a chain of limit length in the tree we take the least element of $\bigcap_{i<\lambda}E_{x_i}$ as the immediate successor of the chain and associate to it the rest of $\bigcap_{i<\lambda}E_{x_i}$. (Of course, if $\bigcap_{i<\lambda}E_{x_i}$ is empty the chain has no successor.) Since $\overline{\overline{A}}_x\leqslant\aleph_0$ for every $x\in C$, this tree is countably branching. Moreover, since B is a Kurepa family there can be only countably many nodes of any rank less than $\aleph_1$. In particular there is a node x of rank ω with E_x (which is also the set of its successors) of power $\aleph_1$. Since $E_x\subset B$, $\overline{\overline{F''[E_x^2]}}=\aleph_1$ as $\overline{\overline{E}}_x=\aleph_1$. Let $\{y_i\}_{i<\omega}$ be a chain of predecessors of x and let $\alpha_i=F\{y_i,x\}$. Note that by the construction of the tree $\alpha_i=F\{y_i,z\}$ for every $z\in E_x$, $i<\omega$. Moreover, we can get every pair from $\{\alpha_i|\ i<\omega\}\times F''[E_x]^2$ by applying F to the appropriate triple. Thus $\{\alpha_i,F\{z,\omega\}\}=F\{y_i,z,\omega\}$ by definition. By the property (*) of G we see that $H''[C]^3=\aleph_1$ since C contains E_x and y_i, $i<\omega$. ■

In the above proof we chose $\aleph_2\not\to[\aleph_1]^3_{\aleph_1}$ only as an example. As the strong form of $\aleph_1\not\to[\aleph_1]^2_{\aleph_1}$ holds for any successor cardinal (by GCH) as does Kurepa's hypothesis [7] the exact same proof shows that $\aleph_{\alpha+2}\not\to[\aleph_{\alpha+1}]^3_{\aleph_{\alpha+1}}$. Moreover by iterating the type of argument used above we could as well prove the more general

THEOREM 3A (V = L). $\aleph_{\alpha+n}\not\to[\aleph_{\alpha+1}]^{n+1}_{\aleph_{\alpha+1}}$.

For our final result we combine a combinatorial trick like those of [4] with a well-known theorem of Rowbottom [8]:

THEOREM 4 (V = L). $\varkappa\not\to[\delta^+]^{<\omega}_{\delta^+}$ *for all* $\varkappa,\delta$ (3).

Proof. If not, let $\varkappa,\delta$ be such that $\varkappa\to[\delta^+]^{<\omega}_{\delta^+}$. We will show that given any partition $F\colon[\varkappa]^{<\omega}\to\delta^+$, there is, in fact, a homogeneous $C\subseteq\varkappa$ of size δ^+ such that $\overline{\overline{F''[C]^{<\omega}}}\leqslant\delta$. (By a theorem of Rowbottom [8], this contradicts V = L and so suffices for our result.) If not, then let $F\colon[\varkappa]^{<\omega}\to\delta^+$ be such that $\overline{\overline{F''[C]^{<\omega}}}=\delta^+$ for every $C\subseteq\varkappa$ of power δ^+. Let $G\colon[\delta^+]^2\to\delta^+$ show that $\delta^+\not\to[\delta^+]^2_{\delta^+}$. Now define $H\colon[\varkappa]^{<\omega}\to\delta^+$ by setting

$$H\{x_1,x_2,\ldots,x_{2^i3^j}\}=G\{F\{x_1,\ldots,x_i\},F\{x_{i+1},\ldots,x_{i+j}\}\}.$$

If we now consider any $C\subseteq\varkappa$ of size δ^+ we know that $\overline{\overline{F''[C]^n}}=\delta^+$ for some $n<\omega$. We can thus easily extract δ^+ disjoint members of $[C]^n$: $\{x_{i1},\ldots,x_{in}\}_{i<\delta^+}$ such that $F''\{\{x_{1i},\ldots,x_{in}\}|\ i<\delta^+\}$ has cardinality δ^+. Given any two such elements $\{x_{i1},\ldots,x_{in}\}$, $\{x_{j1},\ldots,x_{jn}\}$ we note that if we arbitrarily extend them to a $2^n\cdot3^n$-triple $\{x_{i1},\ldots,x_{in},x_{j1},\ldots,x_{jn},\ldots,x_{2^n\cdot3^n}\}$ and apply H we get $G\{F\{x_{i1},\ldots,x_{in}\},F\{x_{j1},\ldots,x_{jn}\}\}$. Since we can get any pair from a set of size δ^+ in this way the choice of G shows that $H''[C]^{2n\cdot3n}=\delta^+$. ■

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On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces

by

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Abstract. In this paper the generalized base of countable order theory of H. H. Wicke and J. M. Worrell, Jr. is simplified. This theory allows to understand what the role of paracompactness is in the theories of metrizability and complete metrizability. We consider four classes of spaces and prove that they are closed under the action of perfect mappings and open and compact mappings. Furthermore we give a mapping characterization of these classes and show that in the class of paracompact spaces they are equal to the classes of metrizable spaces, completely metrizable spaces, paracompact p-spaces and paracompact Čech complete spaces. Besides, we construct an example of a locally completely metrizable metacompact space which is not Čech complete.

The aim of this paper is to introduce new characterizations of some classes of spaces investigated by H. H. Wicke and J. M. Worrell, Jr. Our characterizations allow us to simplify the theory of these authors.

We shall use the terminology and notation from [12]. By a *mapping* we always mean a continuous function. All spaces are assumed to be regular. If $\mathfrak{M}$ and $\mathfrak{N}$ are families of subsets of a certain space X and if for an arbitrary $N \in \mathfrak{N}$ there exists an $M \in \mathfrak{M}$ such that $M \subseteq N$, then we write that $\delta\mathfrak{M} < \mathfrak{N}$. The letter $\mathfrak{A}$ will always denote a centred family; centred families of closed sets will be denoted by $\mathfrak{F}$.

Let $\mathfrak{B} = \{\mathfrak{B}_n\}_{n=1}^{\infty}$ be a sequence of bases of a given space X. We recall that if for each sequence $\{B_n\}_{n=1}^{\infty}$ such that $B_n \in \mathfrak{B}_n$ one of the following conditions is satisfied:

(d) $\bigcap_{n=1}^{\infty} B_n \ni x$, then $\{B_n\}_{n=1}^{\infty}$ is a base at x,

(p) $\bigcap_{n=1}^{\infty} B_n \ni x$ and $\delta\mathfrak{F} < \{B_n\}_{n=1}^{\infty}$, then $\bigcap \mathfrak{F} \neq \emptyset$,

(c) $\delta\mathfrak{F} < \{B_n\}_{n=1}^{\infty}$, then $\bigcap \mathfrak{F} \neq \emptyset$,

then X is a Moore space or a p-space [17], or a Čech complete space [12, Theorem 3.8.2], respectively. If the conditions (d) and (c) are satisfied simultaneously, then X is a complete Moore space.