The equivalence of the Boolean prime ideal theorem and a theorem of functional analysis

by

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Abstract. Let Z be a partially ordered vector space with order unit 1, and let $P(Z)$ be the set of all positive linear functionals u on Z such that $u(1) = 1$. Let $E \subset Z$, then we can associate with E a quasi-order for $P(Z)$ by postulating that $u_1 \precsim_E u_2$ iff $u_1(z) \leqslant u_2(z)$ for all $z \in E$.

Dr. D. A. Edwards in a forthcoming paper [3] proves that the Axiom of Choice is effectively equivalent to the existence of an extreme point of $P(Z)$ which is minimal in $P(Z)$ for the quasi-ordering $\precsim_E$.

In this paper, I show that the weaker axiom viz. that every Boolean algebra has a prime ideal, is equivalent to the existence of an extreme point of $P(Z)$ for Z a real vector lattice.

1. Introduction. The aim of this paper is to show that the Boolean prime ideal theorem (PI) viz. that every Boolean algebra has a prime ideal, is equivalent to a statement (HBKML) which is a version of the Hahn-Banach theorem in conjunction with the Krein Milman theorem (see § 3). The axioms of set theory assumed in this paper may be taken to be those of the Zermelo-Fraenkel theory (see [1] & [8]) together with PI (in § 2 & § 3) or HBKML (in § 4). We denote the Axiom of Choice by AC.

The problem was suggested to me by my supervisor, Dr. D. A. Edwards, who has in effect proved in his forthcoming paper [3] that HBKML implies PI. I should like to thank him for his help and the Science Research Council for their financial support.

2. Preliminaries. Throughout this section we assume PI as an axiom. It has been shown (e. g. [5], [6], [7]) that it is possible to deduce the Hahn-Banach extension Theorem (HB) from PI.

HB. Let V be a linear subspace of a real vector space E and ϱ a sublinear functional on E (i.e. $\varrho(x+y) \leqslant \varrho(x)+\varrho(y)$ for all $x, y \in E$ and $\varrho(rx) = r\varrho(x)$ for all $0 \leqslant r \in \boldsymbol{R}$ and $x \in E$). If f is a linear functional defined on V such that $f(x) \leqslant \varrho(x)$ for all $x \in V$, then there exists a linear functional F on E such that $F = f$ on V and $F(x) \leqslant \varrho(x)$ for all $x \in E$.

We need the following separation theorem which can be deduced from HB without AC. The lemmas below bring together the material used in the proof of Theorem 1 given in [2] but are included here to facilitate the checking that AC is not used in the proofs.

THEOREM 1. *If K_1 and K_2 are disjoint closed convex subsets of a locally convex topological vector space E, and if K_1 is compact, then there exist constants c and ε, $\varepsilon > 0$ and a continuous linear functional f on E such that $f(K_2) \leqslant c - \varepsilon < c \leqslant f(K_1)$.*

LEMMA 1. *$K_1 - K_2$ is closed.*

Let $p \in \overline{K_1 - K_2}$, and for each neighbourhood U of p, let $K_U = \{k\colon k \in K_1,\ k \in U + K_2\}$. $p \in \overline{K_1 - K_2}$ so $K_U \neq \emptyset$. If $U_1 \subseteq U_2$ then $K_{U_1} \subseteq K_{U_2}$, and $\{\overline{K}_U\}_{U \text{ a nhd. of } p}$ have the finite intersection property.

K_1 is compact, so there exists $k_0 \in K_1$ and $k_0 \in \overline{K}_U$ for all neighbourhoods U of p. If N is any neighbourhood of the origin, then $(N + k_0) \cap (N + p + K_2) \neq \emptyset$. i.e. $(N - N + k_0) \cap (p + K_2) \neq \emptyset$. If M is any neighbourhood of the origin, then there exists a neighbourhood of the origin N with $N - N \subseteq M$. Thus any neighbourhood of k_0 intersects $p + K_2$ i.e. $p \in k_0 - K_2 \subseteq K_1 - K_2$.

LEMMA 2. *There exists a linear functional f on E and $c \in \boldsymbol{R}$ such that $f(U_0) \leqslant c$ and $f(K_1 - K_2) \geqslant c$, where U_0 is a convex neighbourhood of 0, disjoint from $K_1 - K_2$.*

Let $p \in K_1 - K_2$. 0 is an internal point of U_0 (in the convexity sense i.e. for all $x \in E$, there exists $\varepsilon > 0$ such that $\delta x \in U_0$ for $|\delta| \leqslant \varepsilon$). So $-p$ is an internal point of $U_0 - (K_1 - K_2) + p = K$ say.

Since $U_0 \cap (K_1 - K_2) = \emptyset$, $0 \notin U_0 - (K_1 - K_2)$ and hence $p \notin K$. Let m be the Minkowski functional of K. Then $m(p) \geqslant 1$.

Let $f_0(ap) = a(p)$ where $a \in \boldsymbol{R}$. Then f_0 is a linear functional defined on the 1-dimensional subspace of E consisting of the real multiples of p and $f_0(ap) \leqslant m(ap)$, $a \in \boldsymbol{R}$. By HB there exists a linear functional f on E such that $f(x) \leqslant m(x)$ for all $x \in E$. It follows that $f(K) \leqslant 1$ while $f(p) \geqslant 1$ and so there exists $c \in \boldsymbol{R}$ such that $f(U_0) \leqslant c$ and $f(K_1 - K_2) \geqslant c$.

LEMMA 3. *f is continuous.*

$f(U_0)$ is contained in a proper subinterval $[-a, \infty)$ or $(-\infty, a]$ of the real axis where $a > 0$. Let $M = U_0 \cap -U_0$. Then $M = -M$ and M is a neighbourhood of the origin such that $f(M) \subseteq [-a, a]$.

Given $\varepsilon > 0$, $f(\varepsilon a^{-1} M) \subseteq [-\varepsilon, +\varepsilon]$, so f is continuous at 0. Since f is linear then f is continuous everywhere.

Proof of Theorem 1. Now there exists a non-zero linear functional f and $d \in \boldsymbol{R}$ such that $f(K_1 - K_2) \geqslant d$ and $f(U_0) \leqslant d$. Since f is non-zero there exists $x \in E$ with $f(x) = 1$, and $f(\beta x) = \beta$ for all $\beta \in \boldsymbol{R}$. But $\beta x \in U_0$ for all β sufficiently small, so there exists $\varepsilon > 0$ such that $f(U_0)$ contains every scalar of modulus less than ε. Hence $f(K_1) - f(K_2) \geqslant d \geqslant \varepsilon$, i.e. every number in $f(K_1)$ is at least ε greater then any number in $f(K_2)$. Let $c = \inf f(K_1)$. Then $f(K_2) \leqslant c - \varepsilon < c \leqslant f(K_1)$.

3. Assuming PI as an axiom we wish to prove that the following statement holds:

HBKML. Let Z be a real vector lattice with order unit 1 and let $P = P(Z)$ be the set of all positive linear functionals u on Z such that $u(1) = 1$. Then P is a convex set which has at least one extreme point.

We use the notation of HBKML and denote by τ, the order unit seminorm topology on Z [4] and Z^* the continuous dual of Z. It is proved in [7] that PI $\Leftrightarrow$ Alaoglu's theorem.

If $f\colon P \to [-\infty, \infty)$ is bounded above, we define

$$\hat{f}(u) = \inf\{g(u)\colon g \in Z,\ g|_P \geqslant f\}$$

where $u \in P$ and $g|_P \geqslant f$ means that $g(u) \geqslant f(u)$ for all $u \in P$.

LEMMA 4. *If $f, g \in Z$ and $h\colon P \to [-\infty, \infty)$ is defined by $h(u) = \max[f(u), g(u)]$ for $u \in P$, then $\hat{h}(u) = f \vee g(u)$ where $\vee$ is the lattice operation in Z.*

Let ϱ denote the order-unit seminorm on Z. i.e.

$$\varrho(z) = \inf\{\beta > 0\colon -\beta 1 \leqslant z \leqslant \beta 1\}$$

for $z \in Z$ and let $\mathfrak{E} = \{z \in Z\colon \varrho(z) \leqslant 1\}$. Then $\mathfrak{E}$ is a τ-neighbourhood of the origin and so by Alaoglu's Theorem $\mathfrak{E}^0 = \{z \in Z^*\colon \sup_{Z \in \mathfrak{E}} |z^*(z)| \leqslant 1\}$ is w^*-compact.

Now $P \subseteq Z^*$ [4] and if $u \in P$ and $z \in \mathfrak{E}$ then $|u(z)| \leqslant 1$. So $P \subseteq \mathfrak{E}^0$. But P is w^*-closed and 1 is a w^*-continuous linear functional on Z^*, therefore P is w^*-compact.

If $f, g \in Z$ and for $u \in P$, $h(u) = \max[f(u), g(u)]$, then $f \vee g \in Z$ and $f \vee g(u) \geqslant h(u)$, $u \in P$. So $f \vee g(u) \geqslant \hat{h}(u)$, $u \in P$.

Conversely, if $k \in Z$ and $k|_P \geqslant h$ then for $u \in P$, $k(u) \geqslant f(u), g(u)$, and for $v \in Z^*$ $v \geqslant 0$, $v/v(1) \in P$. So $k(v) \geqslant f(v)$, $g(v)$ for all $v \in Z^*$, $v \geqslant 0$.

Since the positive cone in Z is τ-closed [4] $k \geqslant f, g$ as elements of Z, and so $k \geqslant f \vee g$. Therefore $\hat{h}(u) \geqslant f \vee g(u)$ for all $u \in P$ and from the above,

$$\hat{h}(u) = f \vee g(u) \quad \text{for} \quad u \in P.$$

Next we let $H = \{h\colon P \to \boldsymbol{R}\colon \exists n \in N \,\&\, f_1, f_2, \ldots, f_n \in Z$ with $h(u) = \max_{i \leqslant n}\{f_i(u)\}$ for all $u \in P\}$. Then by a similar argument to that used in Lemma 4 we have that if $h \in H$ and $h(u) = \max\{f_1(u), \ldots, f_n(u)\}$ for all $u \in P$ then $\hat{h}(u) = f_1 \vee f_2 \vee \ldots \vee f_n(u)$ for all $u \in P$. Now for $h \in H$, define $B_h = \{u \in P\colon \hat{h}(u) = h(u)\}$. Since $\hat{h}$ coincides with a w^*-continuous functional on P, B_h is w^*-closed.

LEMMA 5. *For $h \in H$ we have* i) *$B_h \neq \emptyset$ and* ii) *$\{\bigcap_{h \in H} B_h\} \neq \emptyset$.*

i) Suppose $h \in H$ is defined by $h(u) = \max[f(u), g(u)]$ where $u \in P$ and $f, g \in Z$ and suppose that $B_h = \emptyset$. (It is enough to consider the case when $h \in H$ is "defined" by two elements of Z.)

Then for all $u \in P$, $h(u) \neq \hat{h}(u)$, or equivalently by Lemma 4, $\max[f(u), g(u)] < f \vee g(u)$ for all $u \in P$. Let

$$A = \{(u, t) \in Z^* \times \boldsymbol{R}\colon\ u \in P, t \leqslant f(u)\}$$

and

$$B = \{(u, t) \in Z^* \times \boldsymbol{R}\colon\ u \in P, t \leqslant g(u)\}.$$

Then since f and g are bounded on P, A and B are convex, compact sets and if $K_1 = $ convex hull $(A \cup B)$ then K_1 is also convex and compact.

Let $K_2 = \{(u, t) \in Z^* \times \boldsymbol{R}\colon\ u \in P, t \geqslant f \vee g(u)\}$. Then K_2 is a closed convex set and since $f \vee g$ is affine on P we have $K_1 \cap K_2 = \emptyset$. By Theorem 1 there exists a w^*-continuous linear functional L on $Z^* \times \boldsymbol{R}$ and $\beta \in \boldsymbol{R}$ such that $\sup_{K_2} L(u, t) < \beta < \inf_{K_1} L(u, t)$.

Now $L(0, f(u) - f \vee g(u)) > 0$ and $f(u) - f \vee g(u) < 0$ for $u \in P$ and so $L(0, 1) < 0$.

Define $q\colon Z^* \to \boldsymbol{R}$ by $L(x, q(x)) = \beta$, i.e. $q(x) = -L(x, 0)/L(0, 1)$ then q is a w^*-continuous linear functional on Z^* and so $q \in Z$. (We refer the reader to the proof in [2] where AC is not used, that the w^*-continuous linear functionals on Z^* are the elements of Z).

Since $(u, f \vee g(u)) \in K_2$ for $u \in P$ we have $L(u, q(u)) - L(u, f \vee g(u)) > 0$. Therefore $[q(u) - f \vee g(u)]L(0, 1) > 0$ and so $q(u) - f \vee g(u) < 0$.

Also $(u, f(u)) \in K_1$ and so $f(u) - q(u) < 0$. Likewise $g(u) - q(u) < 0$.

So $\hat{h}(u) = f \vee g(u) > q(u) > g(u), f(u)$ for $u \in P$, contradicting the definition of $\hat{h}$. Therefore $B_h \neq \emptyset$.

ii) We now show that $\{\bigcap_{i=1}^{n} B_{h_i}\} \neq \emptyset$, for any finite collection $h_1, h_2, \ldots, h_n \in H$. For simplicity take $n = 2$ and $h_1(u) = \max[f_1(u), g_1(u)]$, $u \in P$, and $h_2(u) = \max[f_2(u), g_2(u)]$, $u \in P$, $f_i, g_i \in Z$, $i = 1, 2$.

Suppose that $B_{h_1} \cap B_{h_2} = \emptyset$, i.e. $B_{h_1}^c \cup B_{h_2}^c = P$. Then for all $u \in P$, either

$$\left.\begin{array}{c} \max[f_1(u), g_1(u)] < f_1 \vee g_1(u) \\ \text{or} \\ \max[f_2(u), g_2(u)] < f_2 \vee g_2(u) \end{array}\right\} \text{ or both}.$$

So for all $u \in P$,

$$\max[f_1(u), g_1(u)] + \max[f_2(u), g_2(u)] < f_1 \vee g_1(u) + f_2 \vee g_2(u).$$

i.e.

$$\max[f_1 + f_2(u), f_1 + g_2(u), g_1 + f_2(u), g_1 + g_2(u)] < [(f_1 + f_2) \vee (f_1 + g_2) \vee (g_1 + f_2) \vee (g_1 + g_2)(u)].$$

So letting

$$k(u) = \max[(f_1 + f_2)(u), (f_1 + g_2)(u), (g_1 + f_2)(u), (g_1 + g_2)(u)]$$

we have $k \in H$ and $k(u) < \hat{k}(u)$ for all $u \in P$, i.e. $B_k = \emptyset$ which contradicts i).

Therefore $B_{h_1} \cap B_{h_2} \neq \emptyset$. So $\{B_h\}_{h \in H}$ have the finite intersection property and since P is compact we have $\bigcap_{h \in H} B_h \neq \emptyset$.

THEOREM 2. *Axiom* PI $\Rightarrow$ HBKML.

Using the notation as above, $\emptyset \neq \bigcap_{h \in H} B_h \subseteq P$ by Lemma 5. We denote the extreme points of P by $\partial_e P$ and prove that if $u_0 \in \bigcap_{h \in H} B_h$ then $u_0 \in \partial_e P$.

For suppose that there exists $u_1, u_2 \in P$, $u_1 \neq u_2$ such that $u_0 = \frac{1}{2}u_1 + \frac{1}{2}u_2$. Then there exists $f \in Z$ such that $f(u_0) = 0$ and $f(u_2) > 0$. Then if $h(u) = \max[f(u), 0]$ for $u \in P$, we have $h \in H$ and $\hat{h}(u_1) = f \vee 0(u_1) \geqslant 0$, $\hat{h}(u_2) \geqslant f(u_2)$.

So $\hat{h}(u_0) = \frac{1}{2}\hat{h}[(u_1) + \hat{h}(u_2)] \geqslant \frac{1}{2}f(u_2) > 0$. But $h(u_0) = 0$. Therefore $u_0 \in B_h^c$ which contradicts the hypothesis.

We have $u_0 \in \{\bigcap_{h \in H} B_h\} \subseteq \partial_e P$ showing that $\partial_e P \neq \emptyset$.

4. Here we abandon PI as an axiom and seek to prove

THEOREM 3. *Axiom* HBKML $\Rightarrow$ PI.

In [3] Edwards proves that PI is implied by the following axiom:

HBKM. Let Z be a partially ordered vector space with order unit 1, and let $P = P(Z)$ be the set of all positive linear functionals u on Z such that $u(1) = 1$. Then P is a convex set which has at least one extreme point.

The main step is to prove

PROPOSITION 1. *If Ω is a non-empty set, β a Boolean subalgebra of $P(\Omega)$ and I a proper ideal of β, then I is contained in a maximal ideal M of β.*

This is proved by considering the linear algebra L over R of all real functions on Ω that are finite linear combinations of characteristic functions χ_a of elements a of β and letting L have the natural partial ordering. For each function $f\colon \Omega \to \boldsymbol{R}$, $S(f)$ is defined to be the set $\{w\colon f(w) \neq 0\}$ and $V = \{f \in L\colon\ S(f) \in I\}$. Then V is an order ideal of L containing $\{\chi_a\colon\ a \in I\}$. It follows from HBKM that the set $K = \{u \in P(L)\colon\ u \perp V\}$ has an extreme point u_0 which can be shown to be multiplicative and so $M = \{a \in \beta\colon\ u_0(\chi_a) = 0\}$ will suit our purposes.

We note that in the above L is in fact a vector lattice and V is a lattice ideal, so taking HBKML as an axiom rather than HBKM, it still follows that K has an extreme point u_0 and so Proposition 1 is true in our situation i.e. when HBKML is in force.

Using Proposition 1 and a theorem of Tarski's (see [3]) which states that if β is a Boolean algebra then there exists a Boolean algebra of sets B together with an epimorphism $b\colon B\to\beta$, it can be proved that HBKML implies PI.

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Quelques remarques sur les familles de fonctions de Baire de première classe

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Résumé. Soit R l'ensemble des nombres réels. Soit E un espace métrique, complet, séparable et parfait.

Dans ce travail j'introduis des classes de fonctions qui sont ponctuellement discontinues dans des ensembles de l'espace E et j'examine les relations entre ces classes, la classe des fonctions step-like et la classe des fonctions jouissant de la propriété (P). Les deux dernières classes ont été introduites par D. E. Peek dans son travail [4]. Dans le cas où $E=R$, j'examine les relations entre ces classes, la classe des fonctions jouissant de la propriété de Darboux, la classe des fonctions dérivées, et la classe des fonctions approximativement continues.

Dans le travail [4] D. E. Peek a introduit des sous-classes intéressantes de fonctions de Baire de classe 1: les fonctions step-like et les fonctions jouissant de la propriété (P). La définition de la fonction jouissant de la propriété (P) peut être obtenue en remplaçant dans la condition nécessaire et suffisante d'une fonction de Baire de classe 1 l'ensemble parfait par une somme dénombrable d'ensembles parfaits.

Dans ce travail j'introduis des fonctions semblables aux fonctions jouissant de la propriété (P) en remplaçant la somme d'ensembles parfaits par des ensembles jouissant d'autres propriétés. Je vais examiner les relations entre les classes des fonctions définies de cette manière et aussi les fonctions jouissant de la propriété de Darboux; parmi elles les fonctions dérivées et les fonctions approximativement continues.

Soit R l'ensemble des nombres réels.

Soit E un espace métrique, complet, séparable et parfait.

Soit μ une mesure définie sur un σ-corps K d'ensembles de l'espace E. Admettons, en outre, que μ soit σ-finie, sans-atomique et telle que tous les ensembles ouverts et non vides de l'espace E appartiennent à K et soient de mesure μ positive. Désignons respectivement par μ^* et $\overline{\mu}$ la mesure extérieure générée par la mesure μ, et le complété de la mesure μ.

J'introduis les notations suivantes:

$G_1=\{X\subset E;\ X\neq\emptyset\}$,

$G_2=\{X\subset E;\ X\neq\emptyset \text{ et } X \text{ est dénombrable}\}$,