that q_j isn't accessible in $D_j \cap S^- \subseteq f_j(N)$, so that q_j isn't an accessible point of S^-. On the other hand, if $x \in [-1, 1]-F$, then there is an infinite sequence $\{j_n|\ n > 0\}$ of non-zero integers such that x is in the closure of each D_{j_n}, for $\boldsymbol{j}_n = \{j_k|\ k \leqslant n\}$. In this case, $\alpha = \bigcup \{f_{j_n}(\alpha_{j_n})|\ n \in \omega\}$ is an arc from $\langle 0, 1\rangle$ to x contained in S. ∎

Acknowledgements. I wish to thank Professor John Gresser for suggesting the topic of the paper, and helping with its writing. I am also indebted to him for a number of the constructions, and for his valuable assistance in solving the problem.

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BOWLING GREEN STATE UNIVERSITY
Bowling Green, Ohio

Reçu par la Rédaction le 29. 9. 1972

Semigroups which admit few embeddings

by

Kenneth D. Magill, Jr. (Amherst, N. Y.)

Abstract. $S(X)$ is the semigroup, under composition, of all continuous selfmaps of the topological space X. Two classes of spaces are given such that if X is from the first and Y is from the second and φ is any isomorphism from $S(X)$ into $S(Y)$, then there is a unique idempotent v of $S(Y)$ and a unique homeomorphism h from the range of v onto X such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$. It follows from this that there is a fairly extensive class of spaces such that the semigroup of precisely three spaces from the class can be embedded in $S(I)$ and the semigroups of precisely five can be embedded in $S(R)$ where I and R denote respectively the closed unit interval and the space of real numbers.

1. Introduction. The symbol $S(X)$ is used to denote the semigroup, under composition, of all continuous selfmaps of the topological space X. It is well known that there exist semigroups $S(X)$ into which many other such semigroups may be embedded. In fact, given any collection of semigroups, one need only choose a set X whose cardinality is not less than that of any of the semigroups and then each semigroup of the collection can be embedded in $S(X)$ where X is given the discrete topology. In this case, $S(X)$ is, of course, simply the full transformation semigroup on X. The problem is made a bit more difficult by requiring that X satisfy various topological conditions and when we discuss some examples, we will see that for each collection of semigroups, one can produce an arcwise connected metric space X so that each semigroup of the collection can be embedded in $S(X)$. However, such semigroups are really not our main concern here. We are much more interested in semigroups at the other end of the spectrum, that is, in semigroups of continuous functions into which very few other such semigroups can be embedded.

The main theorem of the paper is proven in section 4 and it gives two classes of spaces such that if X is from the first and Y is from the second, then for each monomorphism φ from $S(X)$ into $S(Y)$, there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from X onto the range of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$. We then look at some special cases in more detail and to give some idea of the type of result we get, we mention the essential ingredients of a result on $S(I)$ and one on $S(R)$ where I is the closed unit interval and

R is the space of real numbers. Basically, we produce a fairly extensive class of spaces such that for any space X in the class, $S(X)$ can be embedded in $S(I)$ if and only if X is homeomorphic to either I, the two-point discrete space or the one-point space. Moreover, $S(X)$ can be embedded in $S(R)$ if and only if X is homeomorphic to either R, I, a half-open interval, the two-point discrete space or the one-point space. In other words, there exists a fairly extensive class of semigroups of continuous selfmaps such that from this entire class, precisely three of the semigroups can be embedded in $S(I)$ and precisely five can be embedded in $S(R)$.

2. Some definitions and preliminary results.

DEFINITION (2.1). A topological space X is an S^**-space* if it is T_1 and for each closed subset H of X and each point $p \in X-H$, there is a continuous selfmap f of X and a point $q \in X$ such that $f(x) = q$ for $x \in H$ and $f(p) \neq q$.

DEFINITION (2.2). A space X is a *strong* S^**-space* if it is T_1 and for a pair of disjoint closed subsets A and B of X, there exists a continuous selfmap f of X and distinct points p and q of X such that $f(x) = p$ for $x \in A$ and $f(x) = q$ for $x \in B$.

S^*-spaces were introduced in [4] and strong S^*-spaces in [5]. It is observed in these papers that any completely regular Hausdorff space which contains an arc is an S^*-space and a normal Hausdorff space which contains an arc is a strong S^*-space.

In this paper, when we say a space is 0*-dimensional*, we mean simply that it is Hausdorff and has a basis of sets which are both closed and open. By a Lebesgue 0-dimensional space, we mean a Hausdorff space in which every finite open cover has a refinement by a partition of the space onto open sets. This agrees with the definition given in [1, p. 246] if one considers only normal spaces. It is well known that a Lebesgue 0-dimensional space is 0-dimensional but that a 0-dimensional space need not be a Lebesgue 0-dimensional. However, a 0-dimensional Lindelöf space is Lebesgue 0-dimensional [1, Theorem 16.17, p. 247].

Now, we observed in [4] that any 0-dimensional space is an S^*-space. As for strong S^*-spaces, we prove

PROPOSITION (2.3). *Every Lebesgue* 0*-dimensional space is a strong* S^**-space.*

Proof. Let A and B be disjoint closed subsets of the Lebesgue 0-dimensional space X. Then $\{(X-A), (X-B)\}$ is an open cover of X and hence there is a partition $\{V_i\}_{i=1}^N$ which refines it. Define

$$W = \bigcup \{V_i\colon V_i \subset X-A\}.$$

The W is both closed and open, $A \subset X-W$ and $B \subset W$. Select two distinct points p and q of X and define a selfmap f of X by $f(x) = p$ for $x \in W$ and $f(x) = q$ for $x \in X-W$. The function f is continuous and it follows that X is a strong S^*-space.

DEFINITION (2.4). A topological space X is said to be *strongly conformable* if it is a first countable strong S^*-space and for each pair of compact, countable subspaces A and B each having exactly one limit point, there exists a continuous selfmap f of X mapping A into B such that $B-f(A)$ is finite.

If, in the latter definition, one replaces the requirement that X be a strong S^*-space by the requirement that X be merely an S^*-space, then one has the definition of a *conformable* space which was introduced in [6, Definition (3.3)]. The proof of the following result is essentially a combination of the techniques used in the proofs of Theorems (3.4) and (3.5) of [6] and will not be given.

PROPOSITION (2.5). *All locally Euclidean normal spaces and all Lebesgue* 0*-dimensional metric spaces are strongly conformable.*

As in [6], we regard a space as being locally Euclidean if it is Hausdorff and each point has a neighborhood which is homeomorphic to some Euclidean N-space and there is no requirement that all of these neighborhoods have the same dimension.

Now let us recall that a space X is said to be *homogeneous* if for each pair of points p and q in X, there exists a homeomorphism h from X onto X such that $h(p) = q$. We weaken this requirement considerably in our next definition.

DEFINITION (2.6). A topological space X is said to be *quasi-homogeneous* if for each nonempty open set G of X and each point p in X, there exist continuous selfmaps f and g of X such that $g(p) \in G$ and $f \circ g$ is the identity on X.

The next result gives a simple sufficient condition that a space be quasi-homogeneous. Before stating it, let us define a space X to be an *absolute retract* if it is Hausdorff normal and for each closed subset A of a normal space Y, each continuous map from A into X can be extended to a continuous map from Y into X.

PROPOSITION (2.7). *Let X be any absolute retract with the property that each nonempty open subset contains a copy of X which is closed in X. Then X is quasi-homogeneous.*

Proof. Let G be any nonempty open subset of X and let p be any point of X. By hypothesis, there exists a homeomorphism g from X onto a subspace H of G which is closed in X. Now any absolute retract is necessarily normal so the function g^{-1} which maps H continuously into X

has a continuous extension f which maps X into X. Evidently, both f and g belong to $S(X)$ and $f \circ g$ is the identity on X.

Since for each positive integer N, the closed unit ball B^N in Euclidean N-space E^N satisfies the hypothesis of the latter result, we immediately get

COROLLARY (2.8). *Each closed unit ball B^N is quasi-homogeneous.*

Of course each E^N is also quasi-homogeneous since it is homogeneous. The spaces B^N are examples of quasi-homogeneous spaces which are not homogeneous.

PROPOSITION (2.9). *Any product of quasi-homogeneous spaces is quasi-homogeneous.*

Proof. Let $X = \prod \{Y_a\colon a \in A\}$ where each Y_a is quasi-homogeneous. Let p be any point of X and G any nonempty open subset of X. Then G contains a basic open set of the form

$$P_1^{-1}(H_1) \cap P_2^{-1}(H_2) \cap \ldots \cap P_N^{-1}(H_N)$$

where P_j is the projection map from X onto Y_j and H_j is a nonempty open subset of Y_j. Then there are continuous selfmaps f_j and g_j of Y_j such that $g_j(p_j) = g_j(P_j(p)) \in H_j$ and $f_j \circ g_j = i_j$, the identity map on Y_j. Now define selfmaps f and g of X by

$$\begin{aligned} (f(t))_j &= f_j(t_j) && \text{for} \quad j = 1, 2, \ldots, N, \\ (f(t))_a &= t_a && \text{for} \quad a \neq 1, 2, \ldots, N, \\ (g(t))_j &= g_j(t_j) && \text{for} \quad j = 1, 2, \ldots, N, \\ (g(t))_a &= t_a && \text{for} \quad a \neq 1, 2, \ldots, N. \end{aligned}$$

Then f and g are continuous selfmaps of X such that $f \circ g$ is the identity on X and $g(p) \in G$. Consequently, X is quasi-homogeneous.

By a *retract* of a space X, we simply mean any subspace V of X which is the range of an idempotent continuous selfmap of X.

DEFINITION (2.10). A topological space X is a *spray* if it is Hausdorff, connected, first countable and, in addition satisfies the following three conditions:

(2.10.1) A discrete subspace can be at most countable.

(2.10.2) Each nondegenerate connected subset has nonempty interior.

(2.10.3) Let $\{A_\delta\colon \delta \in \Delta\}$ be any uncountable collection of retracts of X such that each has more than one point. Then there is at least one whose boundary intersects the interior (with respect to X) of another.

Conditions (2.10.2) and (2.10.3) are really quite stringent. For example, if X and Y are two connected T_1 spaces each with more than one point, then $X \times Y$ will not satisfy (2.10.2). Merely choose any $a \in X$ and let

$$A_a = \{(a, y)\colon y \in Y\}.$$

Then A_a is a connected subset of $X \times Y$ but it has empty interior. As for (2.10.3) suppose that X is Hausdorff and has uncountably many points and that Y is also Hausdorff and has at least two points. Now A_a as defined above is a retract of $X \times Y$ and since $X \times Y$ is Hausdorff, it is closed. Thus, it contains its boundary. Since $\{A_a\colon a \in X\}$ is mutually disjoint and uncountable, it follows that (2.10.3) is not satisfied.

Thus, it follows for one reason or another that very few products of spaces are sprays. In the next section, however, we will see a number of examples of sprays. In fact, we will look at a method for constructing various examples which will be useful to us.

3. A method for constructing examples. Let $\{(Y_a, p_a)\colon a \in A\}$ be a collection of ordered pairs where, for each $a \in A$, Y_a is a metric space with metric d_a and p_a is a point of Y_a. For purposes of discussion, it will be convenient to assume that all of these spaces are mutually disjoint although this is not really necessary and one will easily see the appropriate modifications to make when they are not mutually disjoint. Choose some point q which is not in any of the Y_a and let

$$X = [\bigcup \{Y_a - \{p_a\}\colon a \in A\}] \cup \{q\}.$$

Define a metric d on X as follows:

$$\begin{aligned} d(q, y) &= d_a(p_a, y) && \text{when} \quad y \in Y_a, \\ d(x, y) &= d_a(x, y) && \text{when} \quad x, y \in Y_a, \\ d(x, y) &= d_a(x, p_a) + d_\beta(p_\beta, y) && \text{when} \quad x \in Y_a \text{ and } y \in Y_\beta. \end{aligned}$$

One shows in a routine manner that d actually is a metric on X.

DEFINITION (3.1). The space X with the metric d is referred to as the *bonded union* of the pairs $\{(Y_a, p_a)\colon a \in A\}$. For each a, the point p_a is referred to as the *bonding point* of Y_a and the point q in X is referred to as the *exceptional point* of X.

If the family happens to be finite in number, the bonded union is nothing more than the quotient space obtained by first taking the free union of the spaces in the family and then identifying the bonding points. This is not necessarily true, however, when the family is infinite. For example, let

$$L_n = \{(x, y) \in R \times R\colon y = x/n \text{ and } 0 \leqslant x \leqslant 1\}$$

where n is a positive integer and let

$$I_{\aleph_0} = \bigcup \{L_n\}_{n=1}^{\infty}$$

where the topology on $I_{\aleph_0}$ is that which it inherits from the Euclidean plane. One shows easily that $I_{\aleph_0}$ is homeomorphic to the bonded union of a countably infinite number of copies of the closed unit interval I where for each copy, the bonding point is one of the endpoints. This differs from the space I_* which is the free union of the spaces L_n with the origins indentified, for the set

$$\bigcup \{(x, y)\colon\ y = x/n \text{ and } 0 \leqslant x < 1/n\}_{n=1}^{\infty}$$

is open in I_* but not in $I_{\aleph_0}$.

The construction of $I_{\aleph_0}$ can be generalized by letting α be any cardinal number and taking the bonded union of α copies of I where for each copy, the bonding point is one of the endpoints. We do the same sort of thing with the space J of non-negative real numbers and the unit circle C of the Euclidean plane. We gather all this together in the following

DEFINITION (3.2). Let α be any cardinal number. Then

(3.2.1) I_α denotes the bonded union of α copies of the closed unit interval I where the bonding point in each copy is one of the endpoints.

(3.2.2) J_α denotes the bonded union of α copies of J where the bonding point in each copy is the unique endpoint.

(3.2.3) C_α denotes the bonded union of α copies of C where the bonding points in each copy is arbitrary.

The following result summarizes some properties of these spaces.

PROPOSITION (3.3).

(3.3.1) *I_1 and I_2 are homeomorphic and this is the only case where I_α and I_β are homeomorphic with $\alpha \neq \beta$.*

(3.3.2) *J_2 is homeomorphic to the space R of real numbers.*

(3.3.3) *I_α, J_α and C_α are all arcwise connected for each cardinal number α.*

(3.3.4) *The spaces $I_\alpha, J_\alpha, C_\alpha$ are all sprays, if and only if α is a finite cardinal.*

Proof. With the possible exception of (3.3.4) the statements are rather evident. We verify (3.3.4) for the space I_N, the remaining cases being similar. First of all, suppose α is a positive integer N. We may take I_N to be the space $\bigcup \{L_n\}_{n=1}^{N}$ where

$$L_n = \{(x, y) \in R \times R\colon\ y = x/n \text{ and } 0 \leqslant x \leqslant 1\}.$$

Let $\{A_\delta\colon\ \delta \in \Delta\}$ be an uncountable collection of retracts of I_N, each having more than one point. We consider two cases.

Case 1. An infinite number of the A_δ contain the origin q.

We assume that no boundary of any of the retracts intersects the interior, with respect to I_N, of any other retract and we obtain a contradiction. Let $\Delta' = \{\delta \in \Delta\colon\ q \in A_\delta\}$ and we consider the subspace L_1 of I_N. There are two possibilities:

(1a) $L_1 \cap A_\gamma$ is a nondegenerate subinterval of L_1 for some $\gamma \in \Delta'$.

(1b) $$L_1 \cap A_\delta = \{q\} \quad \text{for each } \delta \in \Delta'.$$

If (1a) holds, then each $L_1 \cap A_\delta$ must be either $L_1 \cap A_\gamma$ or $\{q\}$, otherwise the boundary of A_γ would intersect the interior, with respect to I_N of some A_δ, or conversely. It follows that regardless of which one of (1a) and (1b) holds, we may conclude than an infinite number of the A_δ all intersect L_1 in an identical fashion. Denote the indices of all these A_δ by Δ_1'. In a similar manner, there is an infinite subset Δ_2' of Δ_1' such that all of the retracts with indices in Δ_2' intersect L_2 (and of course, L_1 also) in exactly the same way. Continuing in this manner, we conclude the existence of an infinite subset Δ_N' of indices such that any two retracts with indices in Δ_N' intersect L_n, $1 \leqslant n \leqslant N$, in exactly the same way. This implies that all these retracts are identical which is the desired contradiction since they are, in fact, all distinct.

Case 2. Only a finite number of the A_δ contain the origin q.

Let $\Delta' = \{\delta \in \Delta\colon\ q \notin A_\delta\}$ and note that Δ' is uncountable. Then for each $\delta \in \Delta'$, the retract A_δ is contained in $I_N - \{q\}$ which is just the free union of N copies of a half-open interval. Since A_δ is connected it must be contained in one of these half-open intervals. It readily follows that some $L_n - \{q\}$ contains an uncountable number of these A_δ. Now each of these A_δ is a nondegenerate subinterval of the half-open interval $L_n - \{q\}$ and it follows easily that the boundary of one of the A_δ must intersect the interior (with respect to I_N) of another. Thus, condition (2.10.3) is satisfied and since the remaining conditions in Definition (2.10) are also satisfied, we conclude that I_N is a spray.

Now suppose that α is an infinite cardinal. We show that I_α is not a spray. Let Δ be any index set whose cardinal number is α and for each $\delta \in \Delta$, let h_δ be a homeomorphism from the closed unit interval I onto a space Y_δ. Then I_α is topologically the bonded union of the pairs $\{(Y_\alpha, h_\delta(0))\colon\ \delta \in \Delta\}$. Now for each proper nonvoid subset Ω of Δ, we associate a retract of I_α as follows: let

$$A_\Omega = \bigcup \{Y_\delta\colon\ \delta \in \Omega\}.$$

It is evident that there are uncountably many such subspaces. To see that they are retracts, choose any $\gamma \in \Omega$ and define a function f as follows:

$$f(x) = x \qquad \text{for} \quad x \in A_\Omega ,$$
$$f(x) = h_\gamma(h_\delta^{-1}(x)) \qquad \text{for} \quad x \in Y_\delta,\ \delta \notin \Omega .$$

The function f is continuous and it is the identity on its range which is A_Ω. Thus, f is idempotent and, consequently, A_Ω is a retract. Now in each of these A_Ω the boundary is the exceptional point which we denote by q, so if the boundary of one of these retracts was to intersect the interior with respect to I_α, of another, say A_Ω, then q would be contained in some open subset of I_α which would be contained in A_Ω. This, however, is impossible since such an open subset would intersect each Y_α in more than one point while A_Ω intersects at least one Y_δ at only one point, the exceptional point q. This follows since Ω is a proper subset of Δ. Thus, we have produced an uncountable family of nondegenerate retracts with the property that given any one, its boundary will not intersect the interior of any other. Hence condition (2.10.3) is not satisfied and, consequently, I_α cannot be a spray when α is infinite.

Remark. $I_{\aleph_0}$ satisfies every condition of Definition (2.10) with the single exception of (2.10.3) so it was necessary to work with condition (2.10.3) to show that I_α is not a spray when α is an infinite cardinal.

Now we are in a position to verify a statement which we made in the introduction.

PROPOSITION (3.4). *Given any collection of semigroups, there exists an arcwise connected metric space X such that each semigroup in the collection is isomorphic to a subsemigroup of $S(X)$.*

Proof. Since each semigroup in such a collection can be embedded in $S(\Delta)$ where Δ is a sufficiently large discrete space, we need only show that for any discrete space Δ, there exists an arcwise connected metric space X such that $S(\Delta)$ can be embedded in $S(X)$. We use the same notation as in the proof of Proposition (3.3). The symbol Δ represents an index set of cardinality α and we show that $S(\Delta)$ (where Δ is given the discrete topology) can be embedded in $S(I_\alpha)$ where, as in the previous proof, I_α is taken to be the bonded union of the pairs $\{(Y_\delta, h_\delta(0)) \colon \delta \in \Delta\}$. For any $f \in S(\Delta)$, we define a function $\varphi(f)$ on I_α as follows:

$$\big(\varphi(f)\big)(x) = h_{f(\delta)}\big(h_\delta^{-1}(x)\big) \qquad \text{for} \quad x \in Y_\delta - h_\delta(0) ,$$
$$\big(\varphi(f)\big)(q) = q \qquad \text{where } q \text{ is the exceptional point} .$$

One shows rather easily that $\varphi(f)$ is a continuous selfmap of $S(I_\alpha)$ and just as easily that φ is a monomorphism from $S(\Delta)$ into $S(I_\alpha)$.

PROPOSITION (3.5). *Given any collection of semigroups, there exists a compact, connected, Hausdorff strong S^*-space X such that each semigroup in the collection is isomorphic to a subsemigroup of $S(X)$.*

Proof. We use the notation of Proposition (3.4). It is sufficient to show that $S(I_\alpha)$ can be embedded in some $S(X)$ where X is a compact, connected, Hausdorff, strong S^*-space. The candidate for this is βI_α, the Stone-Čech compactification of I_α. Since I_α is connected, βI_α is also and since βI_α contains an arc (I_α is arcwise connected) it follows that βI_α is a strong S^*-space [5, Theorem (2.6), p. 327]. The embedding is obtained by mapping f in $S(I_\alpha)$ into its Stone-Čech extension in $S(\beta I_\alpha)$.

4. The embedding theorem. In this section, we prove the main result of the paper.

MAIN THEOREM. *Let X be a strongly conformable, quasi-homogeneous, completely regular space which is not totally disconnected and let Y be a spray. Then for each monomorphism φ from $S(X)$ into $S(Y)$, there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from X onto the range of v such that*

$$\varphi(f) = h \circ f \circ h^{-1} \circ v$$

for each f in $S(X)$.

Proof. Let φ be a monomorphism from $S(X)$ into $S(Y)$ and let L denote the collection of all constant functions on X. Then L is a left zero semigroup. Moreover, since X is completely regular and T_1 and is not totally disconnected, it must have uncountably many elements. Thus, L is an uncountable left zero semigroup and consequently, $L^* = \varphi(L)$ is also. We first show, by contradiction, that at least one function in L^* is constant. Assume that the contrary is true. Since each function in L^* is idempotent, the range of each such function is a nondegenerate retract of Y and it now follows from (2.10.3) that there exist two functions v and w of L^* such that

$$\mathrm{bd} V \cap \mathrm{int} W \neq \emptyset \tag{1}$$

where V is the range of v, W is the range of w, $\mathrm{bd} V$ is the boundary of V and $\mathrm{int} W$ is the interior (both with respect to Y) of W. Let r be any point in $\mathrm{bd} V \cap \mathrm{int} W$. Since retracts are closed in Hausdorff spaces, $r \in V$ and hence $v(r) = r$. It follows that

$$r \in \mathrm{int} W \cap v^{-1}(\mathrm{int} W) \cap \mathrm{cl}(Y - V) . \tag{2}$$

Consequently,

$$\mathrm{int} W \cap v^{-1}(\mathrm{int} W) \cap (Y - V) \neq \emptyset \tag{3}$$

and we choose any point t which belongs to the latter set. Then,

$$t \in W - V \tag{4}$$

and

$$v(t) \in W. \tag{5}$$

Since w is the identity on its range, it follows from (5) that $w(v(t)) = v(t)$. However, it follows from (4) that $v(t) \neq t = w(t)$. All this implies that $w \circ v \neq w$ which is a contradiction since L^* is a left zero semigroup. Therefore L^* must contain a function which maps every point of Y into some point p in Y. We denote this function by $\langle p\rangle$. But then for any $f \in L^*$, we have $f = f \circ \langle p\rangle = \langle f(p)\rangle$. That is, L^* consists entirely of constant functions. This fact allows us to define a mapping h from X into Y. Let $x \in X$ be given. Then $\langle x\rangle \in L$ and $\varphi\langle x\rangle \in L^*$ is a constant function. Thus, $\varphi\langle x\rangle = \langle y\rangle$ for some $y \in Y$. We define $h(x) = y$ and we note that

$$\varphi\langle x\rangle = \langle h(x)\rangle \quad \text{for each } x \in X. \tag{6}$$

We use this latter fact to get

$$\begin{aligned}\langle\varphi(f)(h(x))\rangle &= \varphi(f) \circ \langle h(x)\rangle = \varphi(f) \circ \varphi\langle x\rangle \\ &= \varphi(f \circ \langle x\rangle) = \varphi\langle f(x)\rangle = \langle h(f(x))\rangle\end{aligned}$$

which implies that

$$\varphi(f) \circ h = h \circ f \quad \text{for each } f \in S(X). \tag{7}$$

One uses this fact to prove that

$$h(f^{-1}(x)) = h(X) \cap (\varphi(f))^{-1}(h(x)) \tag{8}$$

for each $x \in X$ and $f \in S(X)$. We verify only one inclusion since the other follows in a similar manner. Suppose that

$$y \in h(X) \cap (\varphi(f))^{-1}(h(x)).$$

Then $y = h(a)$ for some $a \in X$ and $(\varphi(f))(h(a)) = h(x)$ or, equivalently

$$\varphi(f) \circ \varphi\langle a\rangle = \varphi\langle x\rangle$$

which is the same as

$$\varphi\langle f(a)\rangle = \varphi\langle x\rangle.$$

Thus, $\langle f(a)\rangle = \langle x\rangle$ since φ is injective and this implies that $a \in f^{-1}(x)$. This, in turn, implies that $y = h(a) \in h(f^{-1}(x))$.

Now, since X is an S^*-space, the sets of the form $f^{-1}(x)$, $x \in X$ and $f \in S(X)$ form a basis for the closed subsets of X and it is now immediate

from (8) that h^{-1} is a continuous map from $h(X)$ onto X. Next, we want to show that h is also continuous and since X is first countable, we can use sequences to do this. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of distinct points of X which converges to a point p in X which is distinct from all the points in the sequence. We must show that the sequence $\{h(x_n)\}_{n=1}^{\infty}$ converges to $h(p)$. Since X is uncountable and h is injective, $h(X)$ is also uncountable and, in view of (2.10.1), has a nonisolated point q. Moreover, $h(X)$ is first countable since Y is, so there exists an infinite sequence of distinct points of $h(X)$ which converges to q. Denote the set consisting of the points of the sequence together with the limit point q by A. Then A is a compact, countable subset of $h(X)$ which has exactly one limit point. Now h^{-1} is injective and continuous and since X is Hausdorff, the restriction of h^{-1} to A is a homeomorphism. Consequently, $h^{-1}(A)$ is a compact countable subset of X with exactly one limit point. Now let

$$B = [\bigcup \{x_n\}_{n=1}^{\infty}] \cup \{p\}.$$

Then B is also a compact, countable subset of X and since X is strongly conformable there exists a continuous selfmap f of X which maps $h^{-1}(A)$ into B in such a way that $B - f(h^{-1}(A))$ is finite. Thus, there is a positive integer N such that x_n belongs to $f(h^{-1}(A))$ for $n \geqslant N$. It also follows that

$$f(h^{-1}(q)) = p. \tag{9}$$

Now for each $n \geqslant N$, choose a point in A which $f \circ h^{-1}$ maps into x_n and denote the point by y_n. In this way, we get a sequence $\{y_n\}_{n=N}^{\infty}$ of distinct points in A which must necessarily converge to the point q. Then, for $n \geqslant N$, it follows from (7) that

$$h(x_n) = h(f(h^{-1}(y_n))) = \varphi(f)(h(h^{-1}(y_n))) = \varphi(f)(y_n)$$

and from (7) and (9), we get

$$h(p) = h(f(h^{-1}(q))) = \varphi(f)(h(h^{-1}(q))) = \varphi(f)(q).$$

Since $\varphi(f)$ is continuous on Y and $\lim y_n = q$, it now becomes apparent that $\lim h(x_n) = h(p)$. Thus, h is continuous and we have now established that h is a homeomorphism from X *into* Y.

Next, we show that $h(X)$ is a closed subset of Y. Suppose, to the contrary, that this is not so. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of distinct points of $h(X)$ which converges to some point $r \in Y - h(X)$. The set $\{a_n\}_{n=1}^{\infty}$ has no limit points in $h(X)$ and since h is a homeomorphism, the set $\{h^{-1}(a_n)\}_{n=1}^{\infty}$ has no limit points in X. Thus $\{h^{-1}(a_{2n-1})\}_{n=1}^{\infty}$ and $\{h^{-1}(a_{2n})\}_{n=1}^{\infty}$ are disjoint closed subsets of X and since X is a strong

S^*-space, there exist distinct points p and q in X and a continuous selfmap f of X such that

$$f(h^{-1}(a_{2n-1})) = p \quad \text{and} \quad f(h^{-1}(a_{2n})) = q$$

for each positive integer n. We again appeal to (7) to conclude that

$$\varphi(f)(a_{2n-1}) = h(p) \quad \text{and} \quad \varphi(f)(a_{2n}) = h(q)\,.$$

Since $h(p)$ and $h(q)$ are distinct, the sequence $\{\varphi(f)(a_n)\}_{n=1}^{\infty}$ does not converge. This, however, is a contradiction since $\{a_n\}_{n=1}^{\infty}$ does converge. This establishes the fact that $h(X)$ is a closed subset of Y.

Now let i_X denote the identity mapping on X. Then $\varphi(i_X)$ is an idempotent of $S(Y)$. We denote it by v and its range by V. We next want to show that

$$(10) \qquad h(X) = V\,.$$

One inclusion is rather easy to get. For any $x \in X$, we have

$$\langle v(h(x))\rangle = v \circ \langle h(x)\rangle = \varphi(i_X) \circ \varphi\langle x\rangle = \varphi(i_X \circ \langle x\rangle) = \varphi\langle x\rangle = \langle h(x)\rangle\,.$$

Thus, $v(h(x)) = h(x)$ which implies that $h(X) \subset V$. Suppose, however, that (10) does not hold. Then $h(X)$ is a proper closed subset of V which is connected since it is a continuous image of the connected space Y. It follows that there exists a point p such that

$$(11) \qquad p \in h(X) \cap \mathrm{cl}_V(V - h(X))\,.$$

Since X is not totally disconnected, neither is $h(X)$ and condition (2.10.2) assures the existence of a nonempty open subset G of Y such that $G \subset h(X)$. Then $h^{-1}(G)$ is a nonempty open subset of X and since X is quasi-homogeneous there exist continuous selfmaps f and g of X such that $g(h^{-1}(p)) \in h^{-1}(G)$ and $f \circ g = i_X$. It follows from this and (7) that

$$\varphi(g)(p) = \varphi(g)\big(h(h^{-1}(p))\big) = h\big(g(h^{-1}(p))\big) \in G$$

and, hence, there exists an open subset H of Y such that

$$(12) \qquad p \in H \quad \text{and} \quad \varphi(g)(H) \subset G\,.$$

From (11), we conclude that there is a point q such that

$$(13) \qquad q \in H \cap (V - h(X))\,.$$

The statements (12) and (13) together with the fact that $G \subset h(X)$ result in

$$(14) \qquad \varphi(g)(q) \in h(X)\,.$$

We use this and the fact that $q \in V$ to get

$$q = v(q) = \varphi(i_X)(q) = \varphi(f \circ g)(q) = \varphi(f)\big(\varphi(g)(q)\big) \in \varphi(f)\big(h(X)\big)\,.$$

That is, $q = \varphi(f)(h(x))$ for some $x \in X$. But (7) implies that $\varphi(f)(h(x)) = h(f(x))$ which, in turn, implies that $q \in h(X)$. This, of course, contradicts (13) so we conclude that statement (10) is, indeed, valid. Because of (10), we have $v = h \circ h^{-1} \circ v$. We use this and (7) to show that

$$(15) \qquad \varphi(f) = h \circ f \circ h^{-1} \circ v \quad \text{for each } f \in S(X)\,.$$

Let any $f \in S(X)$ be given. Then,

$$\begin{aligned} \varphi(f) &= \varphi(f \circ i_X) = \varphi(f) \circ \varphi(i_X) = \varphi(f) \circ v \\ &= \varphi(f) \circ h \circ h^{-1} \circ v = h \circ f \circ h^{-1} \circ v \end{aligned}$$

which verifies (15).

In order to complete the proof of the theorem we need only to show that the function h and v are unique. Suppose that there also exist functions k and w such that

$$(16) \qquad \varphi(f) = k \circ f \circ k^{-1} \circ w \quad \text{for each } f \in S(X)\,.$$

Then (15) and (16) together yield

$$v = \varphi(i_X) = w$$

and for any $x \in X$,

$$\langle h(x)\rangle = \varphi\langle x\rangle = \langle k(x)\rangle$$

which implies that $h = k$.

5. Applications of the main theorem. The results in this section show that, in a certain sense, the semigroups of continuous selfmaps on a number of sprays contain very few subsemigroups which are isomorphic to semigroups of continuous selfmaps.

THEOREM (5.1). *Let X be a strongly conformable, quasi-homogeneous, completely regular space which is not totally disconnected. Then for each positive integer N, $S(X)$ can be embedded in $S(I_N)$ if and only if X is homeomorphic to the closed unit interval I.*

Proof. We use the same notation as in the proof of Proposition (3.3). In particular, we take I_N to be $\bigcup \{L_n\}_{n=1}^{N}$ where

$$L_n = \{(x, y) \in R \times R\colon\ y = x/n \text{ and } 0 \leqslant x \leqslant 1\}\,.$$

First, suppose that X is homeomorphic to I. Then there exists a homeomorphism h from X onto L_1 and an idempotent v of $S(I_N)$ whose range is L_1. The mapping which sends f in $S(X)$ into $h\circ f\circ h^{-1}\circ v$ is an embedding of $S(X)$ into $S(I_N)$.

Conversely, suppose $S(X)$ can be embedded into $S(I_N)$ with some monomorphism φ. Then, according to the main theorem, there exists an idempotent v of $S(I_N)$ and a homeomorphism h from X onto the range V of v such that $\varphi(f)=h\circ f\circ h^{-1}\circ v$ for each $f\in S(X)$. Thus, X is homeomorphic to the retract V. We must show that V is homeomorphic to I. First of all V has more than one point since X does. Furthermore, any nondegenerate retract of I_N is homeomorphic to some I_M where $1\leqslant M\leqslant N$ so V is homeomorphic to such an I_M. We only need to show that M cannot exceed two since I_1 and I_2 are both homeomorphic to I. Since V is homeomorphic to X, it must be quasi-homogeneous. This prohibits M from exceeding two since otherwise there could not possibly exist two continuous selfmaps f and g of I_M so that $f\circ g$ would be the identity on I_M while g would map the origin into the open subset

$$\{(x,y)\in R\times R\colon\ y=x \text{ and } \tfrac{1}{3}\leqslant x\leqslant \tfrac{2}{3}\}.$$

This latter theorem was essentially proven by appealing to the main theorem and then determining which retracts of I_N were quasi-homogeneous. The same technique yields analogous results for the spaces J_N and C_N and we state these results without proof.

THEOREM (5.2). *Let X be a strongly conformable quasi-homogeneous completely regular space which is not totally disconnected. Then, for each positive integer N, $S(X)$ can be embedded in $S(J_N)$ $(N\geqslant 2)$ if and only if X is homeomorphic to either I, R or a half-open interval.*

THEOREM (5.3). *Let X be a strongly conformable quasi-homogeneous, completely regular space which is not totally disconnected. Then for each positive integer N, $S(X)$ can be embedded in $S(C_N)$ if and only if X is homeomorphic to either I or the unit circle C.*

In each of the previous three theorems, we required that X not be totally disconnected. It turns out that when one is considering embedding $S(X)$ into either $S(I)$ or $S(R)$, this requirement can be dropped without significantly increasing the number of possibilities for X. The precise statements are given in the next two results.

THEOREM (5.4). *Let X be a strongly conformable, quasi-homogeneous, completely regular space. Then $S(X)$ can be embedded into $S(I)$ if and only if X is homeomorphic to either I, the two-point discrete space or the one-point space.*

Proof. (Necessity). Suppose that $S(X)$ can be embedded in $S(I)$. It is immediate from Theorem (5.1) that if X is not totally disconnected, then X is homeomorphic to I. We consider the case where X is totally disconnected and we show that X cannot have more than two points.

Assume to the contrary that it does. Since it is not connected, it is the union of two nonempty disjoint clopen (both closed and open) subsets A and A'. One of these, say A', has more than one points and since it is not connected, it must be the union of two nonempty disjoint clopen subsets B and C. Then A, B and C are mutually disjoint nonempty clopen subsets of X whose union is all of X. Choose points $a\in A$, $b\in B$ and $c\in C$ and define a function f by

$$f(x)=b \quad\text{for}\quad x\in A,$$
$$f(x)=c \quad\text{for}\quad x\in B,$$
$$f(x)=a \quad\text{for}\quad x\in C.$$

Then f is continuous and $\{f,f^2,f^3\}$ is a subset of $S(X)$ with three elements. This implies that $S(I)$ has a subgroup of order three and we have reached a contradiction since Theorem (5.6) of [3, p. 145] assures us that the only finite subgroups of $S(I)$ (and $S(R)$ as well) have order either one or two. Thus X, in this case, has no more than two points.

(Sufficiency). It is immediately evident that if X is homeomorphic to either I or the one-point space, then $S(X)$ can be embedded in $S(I)$. It is slightly less immediate when X is the two-point discrete space so we give an argument for this case. Choose any continuous selfmap of I so that $f\neq i_I$, the identity map on I, but $f\circ f=i_I$. For example, the map which sends a point x into $1-x$ will do. Then choose two points a and b such that $f(a)=b$ and $f(b)=a$. One readily shows that $S(X)$ is isomorphic to the subsemigroup of $S(I)$ consisting of the functions $\langle a\rangle$, $\langle b\rangle$, f and i_I.

The analogous theorem about $S(R)$ is proven in much the same way so we will be content with merely stating it.

THEOREM (5.5). *Let X be a strongly conformable, quasi-homogeneous, completely regular space. Then $S(X)$ can be embedded in $S(R)$ if and only if X is homeomorphic to either R, I, a half-open interval, the two-point discrete space or the one-point space.*

Now let us observe that there are many monomorphisms from $S(I)$ into $S(I)$ which are not automorphisms. We need only choose an idempotent v different from the identity map and a homeomorphism h from I onto the range of v and define

$$\varphi(f)=h\circ f\circ h^{-1}\circ v.$$

Then φ is a monomorphism from $S(I)$ into $S(I)$ which is not an automorphism. The concluding result of this paper shows that the situation is quite different for $S(R)$.

THEOREM (5.6). *Every monomorphism from $S(R)$ into $S(R)$ is, in fact, an automorphism.*

Proof. Let φ be a monomorphism from $S(R)$ into $S(R)$. By the main theorem, there exists an idempotent v of $S(R)$ and a homeomorphism h from R onto the range V of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(R)$. Now V is closed in R and since it is homeomorphic to R, we must have $V = R$. This forces v to be the identity map which means that φ is an automorphism.

6. Some concluding remarks. Let S be a semigroup with identity i and let T be an arbitrary semigroup. The notion of embedding S into T with an α-monomorphism was introduced in [2] and we recall the definition now. A monomorphism φ from S into T is an α-monomorphism if for each left zero $z \in T, \varphi(i)z = z$ implies $z \in \varphi(S)$. Now for any two spaces X and Y, if one chooses an idempotent continuous selfmap v of Y and if there exists a homeomorphism h from X onto the range of v, then the mapping φ defined by

$$(*) \qquad \varphi(f) = h \circ f \circ h^{-1} \circ v \quad \text{for each } f \in S(X)$$

is an α-monomorphism from $S(X)$ into $S(Y)$. It was shown in [6, Theorems (5.6) and (5.7)] that for a great many spaces X and Y all the α-monomorphisms are obtained in exactly this manner. For example, it follows from Theorem (5.6) of [6] that if X is any Hausdorff S^*-space then any α-monomorphism from $S(X)$ into $S(\beta I_\gamma)$ (we recall that βI_γ is the Stone-Čech compactification of the bonded union I_γ) must take the form $(*)$. This places a considerable restriction on X since it must then be homeomorphic to a retract of βI_γ which, among other things, forces it to be compact, connected and to contain a dense arcwise connected subspace. And yet, we observe in the proof of Proposition (3.5) of this paper that any semigroup can be embedded in $S(\beta I_\gamma)$ if one chooses the cardinal number γ to be sufficiently large. This means that for a large cardinal number γ, there are many monomorphisms from semigroups of continuous selfmaps into $S(\beta I_\gamma)$ which are *not* α-monomorphisms. The main theorem of this paper shows, among other things, that quite the opposite is true about sprays. That is, for a great many X. If there is a monomorphism from $S(X)$ into $S(Y)$, Y a spray, then it must be an α-monomorphism.

We conclude this paper with one more observation and that is that various semigroups of relations are a great deal more lenient in allowing embeddings than are the corresponding semigroups of continuous selfmaps. For a specific example, let X be any Hausdorff space and $\mathfrak{S}[X]$ denote the semigroup of all compact relations (compact subsets of $X \times X$) under ordinary composition of relations. If X happens to be compact,

then $S(X)$ is a subsemigroup of $\mathfrak{S}[X]$. At any rate, it follows from Theorem (5.2) of [7, p. 72] that if X is any compact metric space, then $\mathfrak{S}[X]$ can be embedded in $\mathfrak{S}[I]$ while Theorem (5.4) of this paper tell us that among all the semigroups $S(X)$ (X strongly conformable, quasi-homogeneous and completely regular) only three can be embedded in $S(I)$. In particular, it follows that for the closed unit ball B^N in Euclidean N-space, $N > 1$, there is a monomorphism from $\mathfrak{S}[B^N]$ into $\mathfrak{S}[I]$ (in fact, there are many) but it must map some functions in $S(B^N)$ into closed relations on I which are not continuous functions on I.

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STATE UNIVERSITY OF NEW YORK AT BUFFALO

Reçu par la Rédaction le 6. 11. 1972