Absolute Z-sets

by

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Abstract. A closed subset K of a separable metric space X is said to be a *Z-set* in X provided that for each nonempty homotopically trivial open set U in X, $U \setminus K$ is nonempty and homotopically trivial. We define an *absolute Z-set* to be a topologically, complete separable metric space X such that for each closed embedding f of X into Hilbert space l_2, $f(X)$ is a Z-set in l_2. Then main result in this paper is the following characterization of absolute Z-sets. Let X be a topologically complete separable metric space. Then X is an absolute Z-set if and only if X is σ-compact.

1. Introduction. A closed subset K of a separable metric space X is said to be a *Z-set* in X provided that for each nonempty homotopically trivial open set U in X, $U \setminus K$ is nonempty and homotopically trivial. The concept of a Z-set was introduced by Anderson in [2] as a means of giving a topological characterization of infinite deficiency. Since that time Z-sets have been widely used in the study of the topology of certain infinite dimensional spaces and manifolds.

It is easily seen that each topologically complete (or compact) separable metric space can be embedded in separable Hilbert space, l_2 (or the Hilbert cube, Q) so that the image has infinite deficiency and thus is a Z-set in l_2 (or Q). The question then arises as to which spaces admit closed embeddings only as Z-sets. We define an *absolute Z-set* to be a topologically complete separable metric space X such that for each closed embedding f of X into l_2, $f(X)$ is a Z-set in l_2. Also, an *absolute compact Z-set* is a compact separable metric space X such that for each embedding f of X into Q, $f(X)$ is a Z-set in Q.

The main result in this paper is the following characterization of absolute Z-sets.

THEOREM 1. *Let X be a topologically complete separable metric space. Then X is an absolute Z-set if and only if X is σ-compact* (i.e., *the countable union of compact sets*).

The next theorem provides an alternate characterization of absolute Z-sets.

THEOREM 2. *Let X be a topologically complete separable metric space. Then X is an absolute Z-set if and only if X does not contain a closed set which is homeomorphic to the space of irrationals.*

The equivalence of these two characterizations is given by the following result which is stated in [5] and whose proof follows from Theorem 7 of [6].

PROPOSITION 1. *Let X be a topologically complete separable metric space. Then X is σ-compact if and only if X does not contain a closed copy of the space of irrationals.*

The corresponding characterizations of absolute compact Z-sets, given by the following theorems, are more easily obtained.

THEOREM 3. *Let X be a compact separable metric space. Then X is an absolute compact Z-set if and only if X is countable.*

Since a compact separable metric space is countable if and only if it does not contain a topological Cantor set, the next theorem is obviously equivalent to Theorem 3.

THEOREM 4. *Let X be a compact separable metric space. Then X is an absolute compact Z-set if and only if X does not contain a topological Cantor set.*

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2. Preliminaries. Let s denote the countable infinite product of the open intervals $(-1,1)$ and regard the Hilbert cube, Q, as the countable infinite product of the closed intervals $[-1,1]$. If l_2 denotes the space of square summable sequences of reals with the norm topology, then by [1] s and l_2 are homeomorphic. By an l_2-*manifold* we mean a separable metric space having an open cover, each element of which is homeomorphic to l_2. Also, a Q-*manifold* is a separable metric space which has an open cover of elements homeomorphic to open subsets of Q. For each $i>0$, let τ_i be the projection function of l_2, s or Q onto its ith coordinate space; that is, $\tau_i\big((x_n)_{n>0}\big)=x_i$. A subset K of l_2, s or Q is said to have *infinite deficiency* if for infinitely many i, $\tau_i(K)$ is a point.

Several well known and standard properties of Z-sets are used in proofs throughout this paper and are now listed for easy reference.

PROPERTY 1. Any compact subset of s, l_2 or an l_2-manifold is a Z-set.

PROPERTY 2. Let X be an l_2-manifold. Then $X\times(0,1]$ is an l_2-manifold and any closed subset of $X\times\{1\}$ is a Z-set in $X\times(0,1]$.

Let X be s, l_2, Q or an l_2-manifold.

PROPERTY 3. If K is a Z-set in X and K' is a closed subset of K, then K' is a Z-set in X.

PROPERTY 4. If K is a closed subset of X which is the countable union of Z-sets in X, then K is a Z-set in X.

Let $\mathcal{H}(X)$ denote the set of homeomorphisms of a space X onto itself. The following two theorems were the first important results involving Z-sets.

PROPOSITION 2. *A closed subset K of X ($X=s$, l_2 or Q) is a Z-set in X if and only if there exists $h\in\mathcal{H}(X)$ such that $h(K)$ has infinite deficiency in X.*

PROPOSITION 3. [2] *Let K be a Z-set in $X=s$, l_2 or Q and let h be a homeomorphism of K into X. Then there exists $h^*\in\mathcal{H}(X)$ such that $h^*|K=h$ if and only if $h(K)$ is a Z-set in X.*

These results have been generalized to l_2-manifolds in [4] and to Q-manifolds in [3]. We state the homeomorphism extension theorem of [4] as it is used later. Let $h\in\mathcal{H}(X)$ and let $\mathcal{U}$ be an open cover of X. Then h is said to be *limited* by $\mathcal{U}$ if for each $x\in X$, there exists $U\in\mathcal{U}$ such that both x and $h(x)$ are in U. A *homotopy* H of a subset K of X into X is a map of $K\times I$ into X such that $H|K\times\{0\}$ is the inclusion map. If H is a homotopy of K into X and $\mathcal{U}$ is an open cover of X, then H is *pathwise limited* by $\mathcal{U}$ if for each $x\in K$, there exists $U\in\mathcal{U}$ such that $H(\{x\}\times I)\subset U$. If V is a subset of X and if $\mathcal{U}$ is a collection of subsets of X, define

$$\mathrm{st}(V,\mathcal{U})=\bigcup\{U\in\mathcal{U}\colon\ U\cap V\neq\emptyset\}$$

and let $\mathrm{st}^1(\mathcal{U})=\mathrm{st}(\mathcal{U})=\{\mathrm{st}(U,\mathcal{U})\colon\ U\in\mathcal{U}\}$. Also, for each integer $n>1$, let $\mathrm{st}^n(\mathcal{U})=\{\mathrm{st}\big(U,\mathrm{st}^{n-1}(\mathcal{U})\big)\colon\ U\in\mathcal{U}\}$.

PROPOSITION 4. [4] *Let X be an l_2-manifold, let K_1 and K_2 be Z-sets in X, let $\mathcal{U}$ be an open cover of X and let h be a homeomorphism of K_1 onto K_2. If there is a homotopy H of K_1 into X such that $H|K_1\times\{1\}=h$ and H is pathwise limited by $\mathcal{U}$, then h can be extended to $h^*\in\mathcal{H}(X)$ where h^* is limited by $\mathrm{st}^4(\mathcal{U})$.*

3. A characterization of absolute Z-sets. In this section we obtain the characterizations of absolute Z-sets based on a theorem whose proof is briefly sketched in this section and presented in detail in the remaining sections.

First we observe that since the image of a closed embedding of a σ-compact space is closed and σ-compact and since each closed σ-compact subset of l_2 is a Z-set in l_2 (Properties 1 and 4), it is clear that every σ-compact topologically complete separable metric space is an absolute Z-set. To complete the characterization we suppose X is a topologically complete separable metric space which contains a closed copy E of the space of irrationals and show that X is not an absolute Z-set. To exhibit a closed embedding of X into l_2 such that the image is not a Z-set in l_2, we use the following results. The proof of the first is contained in this paper and the second can be found in [7].

A subset K of a space X is said to be *locally bicollared* if for each $p \in K$, there exists a relative open set U in K containing p and an open embedding h of $U \times (-1, 1)$ into X such that for each $x \in U$, $h(x, 0) = x$. Let $M = \{(x_i)_{i>0} \in l_2 \colon x_1 = 0\}$ and let F be a closed copy of the space of irrationals in $Y = \{(x_i)_{i>0} \in l_2 \colon x_1 \geqslant 2\}$.

THEOREM 5. *There exists a closed embedding H of M into l_2 such that $F \subset H(M)$ and $H^{-1}(F)$ is a Z-set in M. Moreover, we may require that $H(M) \setminus F$ be locally bicollared.*

PROPOSITION 5. [7] *There exists a closed copy of the space of irrationals in l_2 which is not a Z-set in l_2.*

By Proposition 5, let F' be a closed copy of the space of irrationals in l_2 which is not a Z-set in l_2, let φ be a homeomorphism of Y onto l_2 and let $F = \varphi^{-1}(F')$. Then F is a closed copy of the space of irrationals in Y. By Theorem 5, let H be a closed embedding of M into l_2 such that $F \subset H(M)$ and $H^{-1}(F)$ is a Z-set in M. Since X is a topologically complete separable metric space, there is an embedding f of X into M such that $f(X)$ is a Z-set in M so that Property 3, $f(E)$ is also a Z-set in M. Thus, by Proposition 3, there exists $g \in \mathcal{H}(M)$ where $g(f(E)) = H^{-1}(F)$. Let $h = \varphi \circ H \circ g \circ f$. Then h is a closed embedding of X into l_2 such that $F' \subset h(X)$. Note that $h(X)$ is not a Z-set in l_2 since if it were, F' would be also (Property 3). Therefore X is not an absolute Z-set.

The characterizations of absolute compact Z-sets are obtained in a similar manner. As each compact countable subset of Q is a Z-set in Q, a compact countable metric space is an absolute compact Z-set. Now suppose X is a compact separable metric space which contains a topological Cantor set. To exhibit an embedding h of X into Q such that $h(X)$ is not a Z-set in Q, we use the following results and exactly the same procedure as in the case of absolute Z-sets.

Let $M' = \{(x_i)_{i>0} \in Q \colon x_1 = 0\}$ and let C be a topological Cantor set in $Y' = \{(x_i)_{i>0} \in Q \colon x_1 \geqslant \frac{1}{2}\}$.

THEOREM 6. *There exists an embedding H of M' into Q such that $C \subset H(M')$ and $H^{-1}(C)$ is a Z-set in M'. Moreover, we may require that $H(M') \setminus C$ be locally bicollared.*

PROPOSITION 6. [7] *There exists a topological Cantor set in Q which is not a Z-set in Q.*

We now give a brief intuitive argument which if done rigorously would yield a proof of Theorem 6 as well as the known result which states that if C is any topological Cantor set in E^n (n-dimensional Euclidean space) then there is an embedding h of E^{n-1} into E^n such that $C \subset h(E^{n-1})$. First, cover C with a finite number of small pairwise disjoint connected open sets and then push "fingers" up from M', one to each element of the cover. Refine the first cover of C with another finite cover of pairwise disjoint connected open sets of even smaller diameter and from the tips of the original "fingers", push smaller "fingers" toward the elements of the new cover and inside the elements of the first cover. Continuing in this manner, we may obtain as a limit of this process an embedding h of M' such that $C \subset h(M')$ and for each point p of $M' \setminus h^{-1}(C)$, a neighborhood of $h(p)$ is determined after finitely many steps of the process. Thus we may require that $h(M') \setminus C$ be locally bicollared.

The intuition behind the proof of Theorem 5 is essentially the same as the above except instead of finite covers of the Cantor set and a finite number of "fingers" we use countable covers of the irrationals and a countable number of "fingers" while requiring at each stage that the closure of the union of any collection of "fingers" be the union of that collection.

The details of the proof of Theorem 6 are not given. However, the proof of Theorem 5 is presented in full in the remainder of this paper and with suitable simplifications, such as using finite instead of infinite collections, the lemmas to Theorem 5 — indeed, the proof of the theorem itself — can be easily modified to yield the proof of Theorem 6.

4. Some technical lemmas. In this section we develop some notation, definitions and lemmas which will be used in the rest of this paper. If $\mathcal{U}$ is a collection of subsets of X, then mesh $\mathcal{U} = \sup\{\text{diameter } U \colon U \in \mathcal{U}\}$. Suppose $\mathcal{U}$ and $\mathcal{V}$ are collections of open subsets of X. Then $\mathcal{U}$ is a *refinement* of $\mathcal{V}$ if for each $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $U \subset V$. A collection $\mathcal{B} = \{B_i\}_{i>0}$ of sets in a space X is said to be *discrete* if (1) $i \neq j$ implies $B_i \cap B_j = \emptyset$ and (2) for each subset α of N, $\bigcup_{i \in \alpha} \overline{B_i}$ is closed. By N we mean the set of positive integers and $\overline{A}$ or $\mathrm{cl}(A)$ denotes the closure of A.

The following lemmas are used several times in later proofs.

LEMMA 1. *Let K be a closed subset of a separable metric space X and let $\{A_i\}_{i>0}$ be a discrete collection of closed sets in X such that for each $i > 0$, $\operatorname{diam} A_i < \varepsilon_i$ and $A_i \cap K = \emptyset$. Then there exists a discrete collection $\{U_i\}_{i>0}$ of open sets in X such that for each $i > 0$, $A_i \subset U_i$, $U_i \cap K = \emptyset$ and $\operatorname{diam} U_i < 2\varepsilon_i$.*

The proof is elementary, first using the normality of X to obtain a pairwise disjoint collection $\{W_i\}_{i>0}$ of open sets in X such that for each $i > 0$, $A_i \subset W_i$ and $W_i \cap K = \emptyset$, and then letting V_i be an ε_i/i-neighborhood of A_i and $U_i = V_i \cap W_i$.

The proof of the next lemma is routine and is not given explicitly; the proof uses the previous lemma and the fact that if F is a copy of the irrationals, then there is an infinite discrete cover of F by open and closed sets in F.

Lemma 2. *Let F and K be closed subsets of a separable metric space X such that F is a copy of the irrationals and $F \cap K = \emptyset$. Then for each $\varepsilon > 0$, there exists an infinite discrete collection $\mathcal{V} = \{V_i\}_{i>0}$ of open sets in X such that:*

(1) *for each $i > 0$, $V_i \cap F \neq \emptyset$;*

(2) *$F \subset \bigcup_{i>0} V_i$, $K \cap (\bigcup \overline{V}_i) = \emptyset$; and*

(3) *$\operatorname{mesh}\mathcal{V} < \varepsilon$.*

Throughout the sequel we let $M = \{(x_i)_{i>0} \in l_2\colon\ x_1 = 0\}$ and $l_2^+ = \{(x_i)_{i>0} \in l_2\colon\ x_1 > 0\}$. A repeated use of Lemma 2 provides a straightforward inductive proof of the following.

Lemma 3. *Let F be a copy of the irrationals in l_2^+ which is closed in l_2. Then there exists a sequence $(\mathcal{V}_n)_{n>0}$ of discrete covers of F such that:*

(1) *for each $n > 0$, $V \in \mathcal{V}_n$ implies that V is open in l_2, $\overline{V} \subset l_2^+$ and $V \cap F \neq \emptyset$;*

(2) *for each $n > 0$, $\mathcal{V}_n$ is countably infinite and $\operatorname{mesh}\mathcal{V}_n < 1/2^n$;*

(3) *for each $n > 0$, $\mathcal{V}_{n+1}$ is a refinement of $\mathcal{V}_n$; and*

(4) *each element of $\mathcal{V}_n$ contains infinitely many elements of $\mathcal{V}_{n+1}$.*

5. Construction of an "ω-stage starset system". Here, we construct what we call an ω-stage starset system. This system, which is a specific union of arcs in l_2 converging to a closed copy of the irrationals, is used in obtaining the embedding of the next section.

By e_i we mean the point in l_2 with ith coordinate one and all other coordinates zero and Q_i is $\{te_i\colon\ 0 \leqslant t \leqslant 1\}$. We define a subset S of l_2 to be a *starset at x* if there exists an infinite subset α of N and $h \in \mathcal{H}(l_2)$ such that $h(\theta) = x$ and $h(\bigcup_{i\in\alpha} Q_i) = S$ where θ is the origin of l_2. We call x the *base point* of S and for each $i \in \alpha$, we call $h(Q_i)$ a *basic arc* of S and $h(e_i)$ a *tip* of S. If, in a union of starsets, a tip of one starset is not the base point of another, it is said to be a *free tip*.

A *1-stage starset system*, S^1, in l_2 is the union of a discrete collection $\{S_i^1\}_{i>0}$ of starsets in l_2. Inductively, an *n-stage starset system*, S^n, in l_2 is the union of an $(n-1)$-stage starset system, S^{n-1}, and a set T, where T is the union of a discrete collection, $\{S_i^n\}_{i>0}$, of starsets in l_2 such that:

(1) for each $i > 0$, $S_i^n \cap S^{n-1} = \{x\}$ where x is the base point of S_i^n and a free tip of a starset in S^{n-1}; and

(2) for each free tip x in S^{n-1}, there is a unique $i > 0$ such that x is the base point of S_i^n.

In this case we say that S^n is derived from S^{n-1}. Let $(S^n)_{n>0}$ be a sequence of n-stage starset systems such that for each $n > 1$, S^n is derived from S^{n-1}. Then $S^* = \bigcup_{n>0} S^n$ is said to be the *ω-stage starset system* derived from $(S^n)_{n>0}$. If, in addition, $\{A_i\}_{i>0}$ is a discrete collection of arcs in l_2 such that for each $i > 0$, $A_i \cap S^* = \{x_i\}$, where x_i is the base point of the starset S_i^1 in S^* and is an endpoint of A_i, then S^* is said to be based on $A = \bigcup_{i>0} A_i$ at $\{x_i\}_{i>0}$. The following lemma provides a method of constructing an n-stage starset system from a given $(n-1)$-stage starset system.

Lemma 4. *Let $x \in l_2$ and let U be an open set in l_2 containing x. If K is a Z-set in l_2 and $x \in K$, then there is a starset, S, at x such that $S \subset U$ and $S \cap K = \{x\}$.*

Proof. Let $\alpha \subset N$ such that both α and $N \setminus \alpha$ are infinite. Since K is a Z-set in l_2, there exists $g \in \mathcal{H}(l_2)$ such that $g(x) = \theta$ and for each $i \in \alpha$, $\tau_i(g(K)) = 0$ (Proposition 1). Let $\varepsilon > 0$ be such that $B_\varepsilon(\theta) \subset g(U)$ and define $f \in \mathcal{H}(l_2)$ by $f((x_i)_{i>0}) = (\varepsilon x_i)_{i>0}$($B_\varepsilon(\theta)$ is the ε-ball about the origin). Then $S = g^{-1} \circ f(\bigcup_{i\in\alpha} Q_i)$ is a starset at x such that $S \subset U$ and $S \cap K = \{x\}$.

Let S^* be an ω-stage starset system derived from $(S^n)_{n>0}$. Then $J = \bigcup_{n>0} J_n$ is a *basic union of arcs in* S^* if for each $n > 0$, J_n is a basic arc of a starset in $\operatorname{cl}(S^n \setminus S^{n>1})$ and $J_m \cap J_n \neq \emptyset$ if and only if $|m-n| = 1$.

The next lemma provides the desired ω-stage starset system.

Lemma 5. *Let F be a copy of the irrationals in l_2^+ which is closed in l_2 and let $(\mathcal{V}^n)_{n>0}$ be a sequence of covers of F as described in Lemma 3. If A is the union of a discrete collection $\{A_i\}_{i>0}$ of arcs in $M \cap l_2^+$ such that $A \cap F = \emptyset$ and for each $i > 0$, A_i has an endpoint $p_i \in V_i^1$ where $\mathcal{V}_1 = \{V_i^1\}_{i>0}$, then there exists an ω-stage starset system $S^* = \bigcup_{n>0} S^n$ based on A at $\{p_i\}_{i>0}$ and having the following properties:*

(1) *$\overline{S^*} = S^* \cup F$, $S^* \cap F = \emptyset$ and $S^* \cap M = \emptyset$;*

(2) *if $J = \bigcup_{n>0} J_n$ is a basic union of arcs in S^*, then $\operatorname{diam} J_n < 18/2^n$;*

(3) *if J and J' are distinct basic unions of arcs in S^*, then $\overline{J} \setminus J \neq \overline{J'} \setminus J'$; and*

(4) *for each $n > 0$, $\mathcal{V}_n = \{V_i^n\}$ and the free tips $\{s_i^n\}_{i>0}$ of S^n can be ordered so that $s_i^n \in V_i^n$.*

Proof. We will inductively construct a sequence, $(S^n)_{n>0}$, of n-stage starset systems such that the derived ω-stage starset system, S^*, has the desired properties.

Since A is a closed σ-compact subset of l_2, A is a Z-set in l_2 (Properties 1 and 4). Thus, by Lemma 4, for each $i > 0$ let $T_i = \bigcup_{j>0} J_{ij}$ be a starset at p_i such that $T_i \subset V_i^1 \setminus F$ and $T_i \cap A = \{p_i\}$. Denote the free tip of J_{ij} by s_{ij}. By Lemma 3, each V_i^1 contains infinitely many elements of $\mathcal{V}_2$. Denote this set of elements by $\{V_{ij}^2\}_{j>0}$ and let $t_{ij} \in V_{ij}^2 \setminus (F \cup T_i \cup A)$. Then both s_{ij} and t_{ij} are in V_i^1 and thus $d(s_{ij}, t_{ij}) < \frac{1}{2}$. As A, $\bigcup_{i>0, j>0} \{s_{ij}\}$ and $\bigcup_{i>0, j>0} \{t_{ij}\}$ are closed σ-compact subsets of the l_2-manifold

$(M\cup l_2^+)\setminus F$, they are all Z-sets in $(M\cup l_2^+)\setminus F$. Property 2 implies M is also a Z-set in $(M\cup l_2^+)\setminus F$. Since $\{s_{ij}\}_{i>0,j>0}$ and $\{t_{ij}\}_{i>0,j>0}$ are discrete collections, f_1, defined by $f_1(s_{ij})=t_{ij}$ and $f_1|(A\cup M)=\mathrm{id}$, is a homeomorphism. Let

$$\mathcal{B}=\{B_{1/2}(x)\colon\ x\in(M\cup l_2^+)\setminus F\}$$

where

$$B_{1/2}(x)=\{y\in(M\cup l_2^+)\setminus F\colon\ d(x,y)<\tfrac{1}{2}\}.$$

Since for each $i>0$ and $j>0$, $t_{ij}\in B_{1/2}(s_{ij})$ and $B_{1/2}(s_{ij})$ is arcwise connected, f_1 is clearly homotopic to the identity by a homotopy which is pathwise limited by $\mathcal{B}$. By the Homeomorphism Extension Theorem (Proposition 4), there exists $f_1^*\in\mathcal{H}(M\cup l_2^+)\setminus F)$ such that f_1^* extends f_1 and f_1^* is limited by $\mathrm{st}^4(\mathcal{B})$. Note that f_1^* limited by $\mathrm{st}^4(\mathcal{B})$ and $J_{ij}\subset B_{1/2}(s_{ij})$ implies $\operatorname{diam}\big(f_1^*(J_{ij})\big)<9$. For each $i>0$, let $S_i^1=\bigcup\limits_{j>0}f_1^*(J_{ij})$ and let $S^1=\bigcup\limits_{i>0}S_i^1$. Then S_i^1 is a starset at p_i and S^1 is a 1-stage starset system.

Now assume an $(n-1)$-stage starset system, S^{n-1}, has been defined so that the following conditions are satisfied:

(1) $S^1\subset S^{n-1}$, $S^{n-1}\cap(M\cup F)=\emptyset$ and $S^{n-1}\cap A=\bigcup\limits_{i>0}\{p_i\}$;

(2) if $\{s_i^{n-1}\}_{i>0}$ denotes the set of free tips in S^{n-1} and if $\mathcal{V}_n=\{V_i^n\}_{i>0}$, then $s_i^{n-1}\in V_i^n$; and

(3) if $s_i^{n-1}\in V_i^n$ is a free tip of a basic arc J of a starset whose base point is in $V\in\mathcal{V}_{n-1}$, then $V_i^n\subset V$ and $\operatorname{diam}J<18/2^{n-1}$.

Since $S^{n-1}\cup A$ is a closed σ-compact subset of $M\cup l_2^+$, it is a Z-set in $M\cup l_2^+$. By Lemma 4, for each $i>0$, let $T_i^n=\bigcup\limits_{j>0}J_{ij}^n$ be a starset at s_i^{n-1} such that $T_i^n\subset V_i^n\setminus F$ and $T_i^n\cap(S^{n-1}\cup A)=\{s_i^{n-1}\}$. For each $i>0$, let $\{V_{ij}^{n+1}\}_{j>0}$ denote the infinite set of elements of $\mathcal{V}_{n+1}$ which are contained in V_i^n and let $t_{ij}^n\in V_{ij}^{n+1}\setminus(F\cup S^{n-1}\cup A\cup M)$. Denote the free tips of T_i^n by $\{s_{ij}^n\}_{j>0}$ and observe that, as in the first stage, S^{n-1}, $\bigcup\limits_{i>0,j>0}\{s_{ij}^n\}$, $\bigcup\limits_{i>0,j>0}\{t_{ij}^n\}$, M and A are all Z-sets in $(M\cup l_2^+)F$. Let $\mathcal{B}_n=\{B_{1/2^n}(x)\colon\ x\in(M\cup l_2^+)\setminus F\}$. Note that $t_{ij}^n\in B_{1/2^n}(s_{ij}^n)$, $J_{ij}^n\subset B_{1/2^n}(s_{ij}^n)$ and $B_{1/2^n}(s_{ij}^n)$ is arcwise connected. Thus, let $f_n^*\in\mathcal{H}\big((M\cup l_2^+)\setminus F\big)$ such that $f_n^*(s_{ij}^n)=t_{ij}^n$,

$$f_n^*|(M\cup A\cup S^{n-1})=\mathrm{id}$$

and f_n^* is limited by $\mathrm{st}^4(\mathcal{B}_n)$. Observe that $\operatorname{diam}f_n^*(J_{ij}^n)<18/2^n$. As before, let $S_i^n=\bigcup\limits_{j>0}f_n(J_{ij}^n)$ and let $S^n=S^{n-1}\cup(\bigcup\limits_{i>0}S_i^n)$. Then it is easily verified that the derived ω-stage starset system, S^*, has the properties listed in the statement of this lemma.

If S^* is an ω-stage starset system which fulfills the requirements of Lemma 5, then we say that S^* *converges* to F.

6. Proof of Theorem 6. The first lemma of this section is essentially the inductive step in the proof of Theorem 6 which follows. A *halfspace* in l_2 is a pair $[Y,K]$ of subsets of l_2 for which there exists $h\in\mathcal{H}(l_2)$ such that $h(Y)=M\cup l_2^+$ and $h(K)=M$, where as before

$$M=\{(x_i)_{i>0}\in l_2\colon\ x_1=0\}\quad\text{and}\quad l_2^+=\{(x_i)_{i>0}\in l_2\colon\ x_1>0\}.$$

LEMMA 6. *Let $[Y,K]$ be a halfspace in l_2, let A be an arc in Y with endpoints p and q where $A\cap K=\{p\}$, and let $S=\bigcup\limits_{i>}J_i$ be a starset at q where $A\cap K=\{p\}$, and let $S=\bigcup\limits_{i>0}I_i$ be a starset at q where $S\cap(A\cup K)=\{q\}$. If W is an open set in l_2 containing A, then for any open set V in l_2 containing q, there exist (1) a relative basic open set U in K with $p\in U\subset W$ and (2) a $G\in\mathcal{H}(l_2)$ such that $G|l_2\setminus W=\mathrm{id}$, $G(U)\subset V\cap W$ and for each $i>0$, $G(U)\cap J_i$ is a point different from q.*

Proof. Since $[Y,K]$ is a halfspace in l_2, there exists $f\in\mathcal{H}(l_2)$ such that $f(Y)=M\cup l_2^+$ and $f(K)=M$. We may also assume that $f(p)=\theta$. If $A'=\{te_1\colon 0\leqslant t\leqslant 1\}$ and for each $i>0$, $J_i'=\{(1+t)e_1+te_{i+1}\colon 0\leqslant t\leqslant 1\}$ and if $S'=\bigcup\limits_{i>0}J_i'$, then there is a homeomorphism h' of $M\cup f(A)\cup f(S)$ onto $M\cup A'\cup S'$ such that $h'|M=\mathrm{id}$, $h'\big(f(A)\big)=A'$ and $h'\big(f(S)\big)=S'$. As $M\cup f(A)\cup f(S)$ and $M\cup A'\cup S'$ are Z-sets in $M\cup l_2^+$, let $h\in\mathcal{H}(M\cup l_2^+)$ be an extension of h' and define $H\in\mathcal{H}(l_2)$ by $H|(M\cup l_2^+)=h$ and $H|l_2\setminus(M\cup l_2^+)=\mathrm{id}$. Then $g=H\circ f$ is in $\mathcal{H}(l_2)$ and carries $A\cup S$ onto $A'\cup S'$. Considering l_2 as $R\times M$ and $B_\varepsilon(x)=\{y\in M\colon\ d(x,y)<\varepsilon\}$, let r be such that $[-r,1+r]\times B_{2r}(\theta)\subset g(W)$ and $\{1+r/2\}\times B_r(\theta)\subset g(W\cap V)$. Note that for each $i>0$,

$$\big(\{1+r/2\}\times B_r(\theta)\big)\cap J_i'=\{(1+r/2)e_1+(r/2)e_{i+1}\}.$$

Let φ be a piecewise linear homeomorphism of $[-r,1+r]$ onto itself such that $\varphi(-r)=-r$, $\varphi(0)=1+r/2$ and $\varphi(1+r)=1+r$. For each t, $0\leqslant t\leqslant 2r$, put $\varphi_t=\varphi$ if $t\leqslant r$ and $\varphi_t=\big((2r-t)/r\big)\varphi+\big((t-r)/r\big)\mathrm{id}$ if $t\geqslant r$. Now define $F\in\mathcal{H}(R\times M)$ by $F(u,x)=\big(\varphi_{\|x\|}(u),x\big)$ where $u\in[-r,1+r]$ and $x\in B_{2r}(\theta)$ and $F|(R\times M)\setminus\big([-r,1+r]\times B_{2r}(\theta)\big)=\mathrm{id}$. Let $G\in\mathcal{H}(l_2)$ be defined by $G=g^{-1}\circ F\circ g$ and let $U=g^{-1}\big(B_r(\theta)\big)$.

Proof of Theorem 6. Let $(\mathcal{V}_n)_{n>0}$ be a sequence of covers of F as described in Lemma 3. Let D be a Z-set in M which is homeomorphic to the space of irrationals. (D can be obtained by letting $\alpha\subset N$ be infinite, letting $M'=\{(x_i)_{i>0}\in l_2\colon\ x_1=0$ and $x_i=0$ if $i\in\alpha\}$ and using Proposition 1.) By Lemma 2, let $\mathcal{G}=\{G_i\}_{i>0}$ be an infinite discrete collection of relative open sets in M such that $\operatorname{mesh}\mathcal{G}<\frac{1}{2}$ and for each $i>0$, $G_i\cap D\neq\emptyset$. Now for each $i>0$ let $p_i\in G_i\cap D$ and let $A_i=\{te_1+p_i\colon 0\leqslant t\leqslant 1\}$. Also let $q_i\in V_i^1\setminus F$, where $\mathcal{V}_1=\{V_i^1\}_{i>0}$, and define f' by $f'(e_1+p_i)=q_i$ and $f'|M=\mathrm{id}$. Since $\bigcup\limits_{i>0}\{e_1+p_i\}$, $\bigcup\limits_{i>0}\{q_i\}$ and M are all Z-sets in the l_2-manifold $(M \cup l_2^+) \setminus F$ and since f' is homotopic to the identity on $(M \cup l_2^+) \setminus F$, by Proposition 3, let $f \in \mathcal{H}\big((M \cup l_2^+) \setminus F\big)$ be an extension of f'. Observe that for each $i > 0$, $f(A_i) \cap F = \emptyset$. By Lemma 5, let S^* be an ω-stage starset system in l_2^+ which is based on $A = \bigcup_{i>0} f(A_i)$ at $\{q_i\}_{i>0}$ and which converges to F. In the remainder of this proof we use the notation developed in the proof of Lemma 5.

By Lemma 1, let $\{W_i^1\}_{i>0}$ be a discrete collection of open sets in l_2 such that for each $i > 0$, $f(A_i) \subset W_i^1$, $W_i^1 \cap F = \emptyset$ and $W_i^1 \cap M \subset G_i$. Since for each $i > 0$, $S_i^1 = \bigcup_{j>0} J_{ij}^1$ is a starset in S^* at q_i, by Lemma 6 there exist (1) a relative basic open set U_i^1 in M where $p_i \in U_i^1 \subset (W_i^1 \cap M)$ and (2) $h_i^1 \in \mathcal{H}(l_2)$ such that $h_i^1 | l_2 \setminus W_i^1 = \mathrm{id}$, $h_i^1(U_i^1) \subset (W_i^1 \cap V_i^1)$ and for each $j > 0$, $h_i^1(U_i^1) \cap J_{ij}^1$ is a point different from q_i. Define h_1 by $h_1 | W_i^1 = h_i^1 | W_i^1$ and $h_1 | (l_2 \setminus \bigcup_{i>0} W_i^1) = \mathrm{id}$. Then $h_1 \in \mathcal{H}(l_2)$ since $\{W_i^1\}_{i>0}$ is a discrete collection.

In the following we let $H_i = h_i \circ h_{i-1} \circ \ldots \circ h_1$ and assume $h_1, \ldots, h_{n-1} \in \mathcal{H}(l_2)$ have been defined so as to have the following properties:

(1) for each starset $S_i^{n-1} = \bigcup_{j>0} J_{ij}^{n-1}$ in $\mathrm{cl}(S^{n-1} \setminus S^{n-2})$, there exists a relative basic open set U_i^{n-1} in M such that $\operatorname{diam} U_i^{n-1} < 1/2^{n-1}$, $U_i^{n-1} \cap D \neq \emptyset$, $H_{n-1}(U_i^{n-1}) \subset V_i^{n-1} \setminus F$ where $\mathcal{V}_{n-1} = \{V_i^{n-1}\}_{i>0}$, and for each $j > 0$, $H_{n-1}(U_i^{n-1}) \cap J_{ij}^{n-1}$ is a point other than the base point of S_i^{n-1},

(2) $\{\overline{U}_i^{n-1}\}_{i>0}$ is a refinement of $\{U_i^{n-2}\}_{i>0}$;

(3) $h_{n-1} | H_{n-2}(M \setminus \bigcup_{i>0} U_i^{n-1}) = \mathrm{id}$;

(4) $d(h_{n-1}, \mathrm{id}) < 3 \cdot (18/2^{n-2})$; and

(5) $H_{n-1}(M) \cap F = \emptyset$.

For each $i > 0$ and $j > 0$, let $A_{ij}^n = J_{ij}^{n-1} \cap H_{n-1}(M \cup l_2^+)$ and $p_{ij} = J_{ij}^{n-1} \cap H_{n-1}(M)$. Since $\{A_{ij}^n\}_{i>0, j>0}$ and $\{p_{ij}\}_{i>0, j>0}$ are countable we may denote these collections by $\{A_i^n\}_{i>0}$ and $\{p_i\}_{i>0}$ respectively. By Lemma 1, let $\{W_i^n\}_{i>0}$ be a discrete collection of open sets in l_2 such that $A_i^n \subset W_i^n$, $W_i^n \cap F = \emptyset$, $\big(W_i^n \cap H_{n-1}(M)\big) \subset H_{n-1}(U_i^{n-1})$ and $\operatorname{diam} W_i^n < 2 \operatorname{diam} A_i^n$. Since for each $i > 0$, there is a starset, S_i^n, in S^* at the free tip of A_i^n, we apply Lemma 6 again to obtain (1) a relative basic open set U_i^n in M such that $p_i \in H_{n-1}(U_i^n) \subset W_i^n$ and $U_i^n \cap D \neq \emptyset$, and (2) $h_i^n \in \mathcal{H}(l_2)$ such that $h_i^n | l_2 \setminus W_i^n = \mathrm{id}$, $h_i^n\big(H_{n-1}(U_i^n)\big) \subset W_i^n \cap V_i^n$ and the intersection of $h_i^n\big(H_{n-1}(U_i^n)\big)$ with each basic arc of S_i^n is a point other than the base point of S_i^n. Define h_n by $h_n | W_i^n = h_i^n | W_i^n$ and $h_n | l_2 \setminus \bigcup_{i>0} W_i^n = \mathrm{id}$. Then $h_n \in \mathcal{H}(l_2)$ since $\{W_i^n\}_{i>0}$ is a discrete collection. Note that $d(h_n, \mathrm{id}) \leqslant \operatorname{mesh}(\{W_i^n\}_{i>0}) < 3 \operatorname{mesh}(\{A_i^n\}_{i>0}) < 3 \cdot (18/2^{n-1})$ (Lemma 5).

Since for each $n > 0$, $d(h_n, \mathrm{id}) < 3 \cdot (18/2^{n-1})$, we have that for each $x \in M$, $\big(H_n(x)\big)_{n>0}$ is a Cauchy sequence and thus converges to a point in l_2. If H of M into l_2 is defined by $H(x) = \lim_{n \to \infty} H_n(x)$, then, by the above and the fact that each H_n is continuous, it is easily seen that H is continuous. Since for each $n > 0$, $\mathcal{V}_n = \{V_i^n\}_{i>0}$ is a cover of F and $H_n(U_i^n) \subset V_i^n$, then $F \subset H(M)$. To show H is one-to-one, let $x, y \in M$ with $x \neq y$. If there exists $n > 0$ such that $x, y \in M \setminus \bigcup_{i>0} U_i^n$, then $H_n(x) = H(x)$ and $H_n(y) = H(y)$ so that $H(x) \neq H(y)$. If there exists n_0 such that for $n \geqslant n_0$, $x \in U_i^n$ and $y = U_j^n$ with $i \neq j$, then for each $n \geqslant n_0$, $H_n(x) \in V_i^n$ and $H_n(y) \in V_j^n$ and thus $H(x) \neq H(y)$. Finally, if for each $n > 0$, $x \in \bigcup_{i>0} U_i^n$ and there exists $m > 0$ such that $y \in M \setminus \bigcup_{i>0} U_i^m$, then $H(x) \in F$, $H(y) \in H_m(M)$ and $H_m(M) \cap F = \emptyset$ so that $H(x) \neq H(y)$ and H is one-to-one. As $h_{n+1} | H_n(M \setminus \bigcup_{i>0} U_i^n) = \mathrm{id}$ and for each $i > 0$, $H_n(U_i^n) \subset V_i^n$, we have that H^{-1} is continuous. Thus H is an embedding. Since $H^{-1}(F) \subset D$ and D is a Z-set in M, by Property 3, $H^{-1}(F)$ is a Z-set in M. Moreover, $h_{n+1} | H_n(M \setminus \bigcup_{i>0} U_i^n) = \mathrm{id}$ implies that $H(M)$ is locally bicollared at each point in $H(M \setminus F)$.

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