Radicals of semi-group rings

by

Jan Krempa (Warszawa)

Abstract. The aim of this paper is to investigate radicals of semi-group rings. In particular, radicals of rings of matrices and polynomials are considered. The results obtained in the paper generalize known theorems on radicals of rings of matrices and polynomials. Methods used here are similar to those of Amitsur and Ortiz.

Introduction. The aim of this paper is to investigate radicals of semi-group rings. In particular, radicals of rings of matrices and polynomials will be considered. All the necessary informations about radical properties can be found in [4]. The results obtained can be transferred, methods of investigation unchanged, to narrower classes of rings or algebras. Some part of the results concerning semi-group rings is known in the associative case [11]. Since in this paper rings need not be associative, then, for an arbitrary ring R, we shall denote by $N(R)$ the set of all $a \in R$ such that for any $x, y \in R$ the following equalities hold:

$$(ax)y - a(xy) = 0\,,$$

$$(xa)y - x(ay) = 0\,,$$

$$(xy)a - x(ya) = 0\,.$$

Thus $N(R)$ is the nucleus of R. Moreover, by $Z(R)$ we denote the center of the ring R, i.e., the set of all $a \in N(R)$ such that $ax = xa$ for all $x \in R$.

When saying that I is an ideal of R we shall always mean that I is a two-sided ideal of R.

For elements of an arbitrary semi-group we shall use terms applied in the case of a multiplicative semi-group of an associative ring.

Let R be a ring and G a semi-group. Consider the set of formal sums $\sum_{g \in G} r_g g$, where $r_g \in R$ and $r_g = 0$ for almost all $g \in G$. Two such sums are considered identical, $\sum_{g \in G} r_g g = \sum_{g \in G} r'_g g$ if and only if $r_g = r'_g$ for every $g \neq 0$. This set, together with operations of addition and multiplication defined as follows:

$$\sum_{g\in G} r_g g + \sum_{g\in G} r'_g g = \sum_{g\in G} (r_g + r'_g) g\,, \tag{1}$$

$$\sum_{g\in G} r_g g \cdot \sum_{g'\in G} r'_{g} g' = \sum_{g\in G}\sum_{g'\in G} r_g r'_{g} g g' \tag{2}$$

is a ring which we shall call a *semi-group ring* and denote by $R[G]$.

The set of formal sums $\sum_{g\in G} r_g g$ may be considered without any relation and such a set together with operations (1) and (2) is also a ring called sometimes a *non-contracted semi-group ring*. Let us denote this ring by $R[G]_{nc}$. If the semigroup G does not contain a zero element then $R[G]_{nc} = R[G]$. If G contains a zero element then $R[G]_{nc} = R[G']$ where G' is identical with the semi-group G but we forget that the zero is the zero element of the semi-group. Therefore all the results obtained for semi-group rings of semi-groups without zero are valid for non-contracted semi-group rings. In the sequel we shall always assume that every semi-group contains at least one non-zero element.

1. Semi-group rings. The following lemma, which is a modification of well known results [3], [6] will be usefull when investigating the radicals of semi-group rings.

LEMMA 1. *Let B be a ring, C an ideal of B, and $h\colon C\to B$ a mapping such that*

1^0 $h(c_1)+h(c_2)-h(c_1+c_2)\in C$ *for* $c_1, c_2\in C$;

2^0 $B\big(h(C)\big)\subset C+h(C)$;

3^0 $\big(h(C)\big)B\subset C+h(C)$;

4^0 $h(c_1c_2)-h(c_1)h(c_2)\in C$ *for* $c_1, c_2\in C$.

Then $C = S(B)$ for some radical property S implies $h(C)\subset C$.

Proof. Denote by D the set $C+h(C)$. From the conditions 1^0, 2^0 and 3^0 it follows that the set D is an ideal of B. Denote by φ the mapping of the ring C onto the ring D/C defined as follows: $\varphi(c) = h(c)+C$, where $c\in C$. It follows from the conditions 1^0-4^0 and from the definition of D that φ is a homomorphism onto the ring D/C. If C is an S-ring, then D/C is also an S-ring, and thus D is an S-ideal of B. Hence $D\subset S(B)$, and as $S(B) = C$, we have $h(C)\subset C$.

THEOREM 1. *If B is an ideal of A such that $A = B+N(A)$, then for any radical property S, $S(B)$ is an ideal of A.*

Proof. Denote by C the set $S(B)$ and by A' the set of all those $a\in A$, for which $aC\subset C$ and $Ca\subset C$. Of course A' is a subgroup of A and $B\subset A'$. Let $a\in N(A)$. Denote by h a function defined as follows: $h(c) = ac$ for $c\in C$. Since $C\subset B$ and B is an ideal of A, then $h(C)\subset B$. We shall verify that the function h satisfies conditions 1^0-4^0 from Lemma 1. It is evident that h satisfies the condition 1^0. Now let $b\in B$, $c\in C$ and $c_1\in C$. Then

$$bh(c) = b(ac) = (ba)c\in Bc\subset C\subset C+h(C);$$

$$h(c)b = (ac)b = a(cb)\in aC\subset h(C)\subset C+h(C);$$

$$h(cc_1)-h(c)h(c_1) = a(cc_1)-(ac)(ac_1) = (ac)c_1-\big((ac)a\big)c_1$$
$$= \big(ac-(ac)a\big)c_1\in BC\subset C\,.$$

Now, by Lemma 1, $h(C) = aC\subset C$. Analogously, one can prove that $Ca\subset C$ and thus $a\in A'$. Since $A = B+N(A)$, $A' = A$. Therefore C is an ideal of A.

THEOREM 2. *If G is any semi-group and S any radical property then, for any ring R, $S(R[G])$ is an ideal of $R^*[G]$, where R^* is obtained from R by an adjunction of the unity element by the ring of integers in the usual way.*

Proof. Since $R^* = R+Z$ then $R^*[G] = R[G]+Z[G]$. Moreover $R[G]$ is an ideal of $R^*[G]$ and $Z[G]\subset N(R^*[G])$. Therefore Theorem 2 follows from Theorem 1.

COROLLARY 1. *If G is a semi-group and contains a unity element, then* $\big(S(R[G])\cap R\big)[G]\subset S(R[G])$.

Proof. Of course

$$S(R[G])\cap R\subset S(R[G])\,,$$

whence

$$(R^*[G])\big(S(R[G])\cap R\big)\subset (R^*[G])S(R[G])\,.$$

From Theorem 2 it follows that

$$(R^*[G])S(R[G])\subset S(R[G])\,.$$

Moreover

$$(R^*[G])\big(S(R[G])\cap R\big) = \big(S(R[G])\cap R\big)[G]\,,$$

therefore

$$\big(S(R[G])\cap R\big)[G]\subset S(R[G])\,.$$

DEFINITION 1. Let S be any property and G a semi-group. A ring R is called a *GS-ring* if and only if $R[G]$ is an S-ring (cf. [12]).

In the following, we state several theorems about the property GS. Some of them can be found in ([11]). Easy proofs will be left to the reader.

THEOREM 3. *If the property S is radical, so is the property GS.*

Proof. It is evident that a homomorphic image of a GS-ring is a GS-ring. Suppose now that any non-zero homomorphic image of the ring R contains a non-zero GS-ideal.

We will show that R is then a GS-ring, that is, $R[G]$ is an S-ring. Let J be the union of all these ideals I of R for which $I[G]\subset S(R[G])$.

Then $J[G] \subset S(R[G])$. Suppose that $J \neq R$. Then there exists a non-zero GS-ideal A/J in the ring R/J. Since $A \neq J$, $A[G] \not\subset S(R[G])$ and hence $(A[G]+S(R[G]))/S(R[G])$ is a non-zero ideal of the ring $R[G]/S(R[G])$. But $(A[G]+S(R[G]))/S(R[G])$ is a homomorphic image of $(A/J)[G]$ and $(A/J)[G]$ is isomorphic to $(A[G])/(J[G])$. Thus $(A[G]+S(R[G]))/S(R[G])$ is an S-ideal. This is a contradiction with the fact that $R[G]/S(R[G])$ is an S-semi-simple ring. Then $J = R$ and so $R[G] \subset S(R[G])$, whence $R[G] = S(R[G])$.

We shall say that a property S is *radically inherited by ideals* (*one-sided ideals, subrings*) if any ideal (one-sided ideal, subring) of an S-ring is also an S-ring.

PROPOSITION 1. *If the property S is radically inherited by ideals, one-sided ideals or subrings, so is the property GS.*

A radical property S is supernilpotent if every zero-ring is an S-radical ring.

PROPOSITION 2. *The property S is supernilpotent if and only if the property GS is supernilpotent.*

Proof. If R is a ring with zero multiplication then the ring $R[G]$ is a direct sum of its ideals Rg, $g \neq 0$, which are isomorphic with R. Therefore $R[G]$ is an S-ring if and only if R is an S-ring.

PROPOSITION 3. *If h is a homomorphism of the semi-group G onto a semi-group G', then for any radical property S the inequality $GS \leqslant G'S$ is satisfied.*

PROPOSITION 4. *If G is a semi-group without zero then $GS \leqslant S$ for any radical property S.*

PROPOSITION 5. *If G and G' are semi-groups then for any property S we have $G(G'S) = G'(GS) = (G \otimes G')S$, where $G \otimes G' = G \times G'$ is a semi-group with operations defined co-ordinate-wise if $0 \notin G$ and $0 \notin G'$; if at least one of the semi-groups contains the zero element then $G \otimes G' = G \times G/\sim$, where $\sim$ is a relation defined as follows:*

$$(g, g')\sim(h, h') \Leftrightarrow [(g = h \wedge g' = h') \vee (g = h = 0) \vee (g' = h' = 0) \vee (g = h' = 0) \vee (g' = h = 0)]$$

for $g, h \in G$, $g', h' \in G'$ and where the class of a pair with at least one zero co-ordinate is considered as a zero element of the semi-group $G \otimes G'$.

PROPOSITION 6. *For any semi-group G and any properties S and S', if $S \leqslant S'$ then $GS \leqslant GS'$.*

PROPOSITION 7. *Let G be any semi-group with unity element and S — any radical property. Then $GS(R) \subset S(R[G]) \cap R$. If $(S(R[G]) \cap R)[G]$ is an S-ideal of $R[G]$, then $GS(R) = S(R[G]) \cap R$ for any ring R.*

COROLLARY 2. *If S is radically inherited by ideals and G is a semi-group with a unity element, then $GS(R) = R \cap S(R[G])$ for any ring R.*

An example of a semi-group G and a radical property S such that $GS(R) \neq S(R[G]) \cap R$ is not known to the author.

DEFINITION 2. The property S is G-*normal* if for any ring R there exists an ideal $A \subset R$ such that $S(R[G]) = A[G]$.

PROPOSITION 8. *If G is a semi-group with unity element, then for any radical property S the following conditions are equivalent:*

1^0 *S is a G-normal property;*

2^0 *for any ring R $S(R[G]) = (S(R[G]) \cap R)[G]$;*

3^0 *for any ring R, if $S(R[G]) \neq 0$ then $S(R[G]) \cap R \neq 0$.*

The proof is obtained by taking into account the Corollary 1.

PROPOSITION 9. *If the property S is G-normal, then*

1^0 *if S is a strong radical property, so is the property GS;*

2^0 *if the property S is semi-simply inherited by ideals (one-sided ideals or subrings), so is the property GS.*

DEFINITION 3. If the property S is G-normal and $GS = S$ then the property S is called G-*invariant*.

2. Matrices. Let us consider now a semi-group M_k composed of all $k \times k$ matrices e_{ij} and the element zero, where e_{ij} is the matrix with 1 in the ijth place and 0's elsewhere. For such a semi-group, the ring $R[M_k]$ is isomorphic to the ring R_k of all $k \times k$ matrices with elements in the ring R. Therefore the results of the first part of this paper may be applied to matrices.

From now on the M_k-invariant properties for any k will be called *matrically invariant*.

PROPOSITION 10. *Any radical property is M_k-normal for every $k \geqslant 1$.*

Proof. It follows from Theorem 2 that for any ring R, $S(R_k)$ is an ideal in $(R^*)_k$ and so it is of the form A_k for some ideal A of R^*, since R^* is a ring with unity element. Clearly $A \subset R$.

THEOREM 4. *For any $k \geqslant 1$ and a radical property S the following implications are true:*

(a) *if the property S is radically inherited by one-sided ideals then $M_k S \geqslant M_{k+1} S$;*

(b) *if S is a strong radical property* [5], *then $M_k S \leqslant M_{k+1} S$.*

Proof of (a). Let R be any $M_{k+1}S$-ring. Then R_{k+1} is an S-ring. Denote by I the set of all those matrices of R_{k+1} whose last row is zero, and by $\mathfrak{J}$ the set of all those matrices of R_{k+1} whose last column is zero. The set I is a right ideal of R_{k+1} and $\mathfrak{J}$ is a left ideal of R_{k+1}, hence $I \cap \mathfrak{J}$ is a left ideal in I. From the fact that the property S is radically inherited by one-sided ideals it follows that I and $I \cap \mathfrak{J}$ are S-rings.

The rings $I \cap \mathfrak{J}$ and R_k are isomorphic to each other and so R_k is an S-ring, i.e., R is an M_kS-ring which establishes the implication (a).

Proof of (b). Suppose that R is an M_kS-ring. Let I and $\mathfrak{J}$ be as defined above. Since R_k is an S-ring then $I \cap \mathfrak{J}$ is a left S-ideal of the ring I. From the fact that S is a strong radical property it follows that $I \cap \mathfrak{J}$ is contained in $S(I)$ and $(I \cap \mathfrak{J})\, I$ is contained in $S(I)$. But $(I \cap \mathfrak{J})\, I = I^2$, whence $(I \cap \mathfrak{J}) + I^2$ is an ideal of I and $(I \cap \mathfrak{J}) + I^2$ is contained in $S(I)$. Let h be a mapping of the ring R_k into I such that

$$h\begin{bmatrix} a_{11} & \dots & a_{1k} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & a_{11} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{k1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}}_{k}.$$

Clearly for any two matrices A and B in R_k, $h(A+B) = h(A)+h(B)$ and $h(A\cdot B) \in I^2 \subset S(I)$, $h(A)h(B) \in I^2 \subset S(I)$. Furthermore for any matrix C of I there exist matrices $A \in I \cap \mathfrak{J}$ and $B \subset R_k$ such that $A + h(B) = C$. Let $\bar{h}$ be the superposition of h with the natural homomorphism of I onto $I/S(I)$. It follows from the conditions above that $\bar{h}$ is a homomorphism of the S-ring R_k onto the S-semi-simple ring $I/S(I)$. Hence $I/S(I) = 0$, i.e., $S(I) = I$.

Denote by I' the set of those matrices of the ring R_{k+1} whose first row is zero. It can be proved, similarly as for I, that $S(I') = I'$. Using again the fact, that S is a strong radical property we obtain the following inclusions: $I \subset S(R_{k+1})$ and $I' \subset S(R_{k+1})$. Hence $I+I' \subset S(R_{k+1})$, but $I+I' = R_{k+1}$ and so R_{k+1} is an S-ring, that is, R is an $M_{k+1}S$-ring.

COROLLARY 3. *If a strong radical property S is radically inherited by one-sided ideals then S is a matrically invariant property.*

This can be proved by induction on k.

THEOREM 5. *If a strong radical property S is radically inherited by ideals and R is an S-ring, then $\bar{R}$ is also an S-ring, where $\bar{R}$ is the zero ring on the additive group of R.*

Proof. Similarly as in the proof of Theorem 4(b) for $k = 1$, we obtain for an S-ring R that the ring I of matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, where $a, b \in R$, is an S-ring. The set L of matrices of the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where $b \in R$ is an ideal of the S-ring I. Therefore L is an S-ring. The theorem follows now from the fact that L and $\bar{R}$ are isomorphic to each other.

COROLLARY 4. *If a strong radical property S is radically inherited by ideals in the class of algebras over an arbitrary but fixed field F and there exists a non-zero S-algebra over F, then the property S is supernilpotent in the class of all F-algebras.*

Proof. By Theorem 5 there exists a non-zero S-algebra over F with zero multiplication, hence, for algebras over a fixed field F, every algebra with zero multiplication is an S-ring.

COROLLARY 5. *If a strong radical property S is radically inherited by ideals, then the property S is supernilpotent in the class of all rings if and only if there exists an S-ring R, whose additive group contains an element of infinite rank.*

Proof. If the property S is supernilpotent, then we may take $R = \bar{Z}$ where $\bar{Z}$ is the zero ring on the group of integers.

If R is an S-ring, then by Theorem 4, $\bar{R}$ is also an S-ring. Let a be an element of R of infinite additive rank. Then the set A of multiplicities of a is an ideal in $\bar{R}$. It is clear that A and $\bar{Z}$ are isomorphic to each other. Since the property S is inherited by ideals then $\bar{Z}$ is an S-ring and thus every cyclic group considered as a ring with zero multiplication is an S-ring. Therefore every ring with zero multiplication is an S-ring.

THEOREM 6. *Let $\mathfrak{T}$ be any class of semi-prime rings such that 0° if every non-zero ideal of a ring R can be mapped homomorphically onto some non-zero ring of $\mathfrak{T}$, then the R must be in $\mathfrak{T}$; then if the following three conditions are equivalent*

1° $R \in \mathfrak{T}$,

2° $R_k \in \mathfrak{T}$ *for every* $k \geqslant 1$,

3° *there exists an integer* $k \geqslant 1$ *such that* $R_k \in \mathfrak{T}$, *the upper radical property determined by the class* $\mathfrak{T}$ *is matrically invariant.*

To prove this theorem we need the following lemma:

LEMMA 2. *If I is a semi-prime ideal in the ring R_k then there exists a semi-prime ideal $A \subset R$ such that $A_k = I$.*

Proof. Denote by $\bar{A}$ the set of those $a \in R$ for which there exists a matrix in I having a as one of its elements, and by A the set of finite sums of elements of $\bar{A}$. From the fact that I is an ideal of R_k it follows that A is an ideal of R. Moreover I contains $(R_kA_k)R_k$ and $R_k(A_kR_k)$ and A_k is an ideal of R_k. It follows from the inclusions above that $I + A_k^2$ is an ideal of R_k and $(I+A_k^2)^2 \subset I$. Since I is semi-prime, I contains $I + A_k^2$, therefore I contains A_k^2, whence I contains A_k. Since I is contained in A_k, $I = A_k$. It can be easily verified that A is a semi-prime ideal of R.

Proof of Theorem 6. Denote by S the upper radical property determined by the class $\mathfrak{T}$, and let k be a positive integer. It suffices to show that $M_kS = S$. Let R be a non-M_kS-ring, i.e., R_k is a non-S-ring. Then there exists an ideal I in R_k such that the ring R_k/I is in the class $\mathfrak{T}$. Hence I is a semi-prime ideal. By Lemma 2 there exists an ideal $A \subset R$

such that $I = A_k$. The rings $R_k/I = R_k/A_k$ and $(R/A)_k$ are isomorphic to each other, $(R/A)_k \in \mathfrak{T}$. From the latter it follows that also $(R/A) \in \mathfrak{T}$ what means that R is a non-S-ring.

Suppose now that R is a non-S-ring. Then there exists an ideal A in R such that the ring R/A is in the class $\mathfrak{T}$. Hence, by 2^0, R_k/A_k is also in the class $\mathfrak{T}$ and thus R_k is a non-S-ring. This means that R is a non-M_kS-ring.

Corollary 6. *The Brown-McCoy radical property is matrically invariant.*

Proof. It is sufficient to prove that for an arbitrary ring R the following conditions are equivalent:

(i) R is a simple ring with unity element;

(ii) for any $k \geqslant 1$, R_k is a simple ring with unity element;

(iii) there exists an integer $k \geqslant 1$ such that R_k is a simple ring with unity element, and the corollary follows from Theorem 10.

We have thus shown that the Corollary 3 is not an if and only if condition, as the Brown-McCoy radical property is not strong [5]. Moreover this property is not radically inherited by one-sided ideals (see Example 3).

There exist radical properties which are not matrically invariant.

Example 1. Let A be an arbitrary non-empty subset of positive integers and let F be a field. Denote by $\mathfrak{T}$ the set of all those rings of matrices F_k for which $k \in A$. Since the set $\mathfrak{T}$ consists of simple rings with unity element, there exists the upper radical property determined by $\mathfrak{T}$, which we denote by S. It is easy to verify that $M_iS(F) = 0$ if and only if $i \in A$. Hence, if A is not the set of all positive integers, the property S is not matrically invariant.

Most of the results of this section can be extended to infinite matrices with finite number of non-zero entries.

3. Polynomials. Denote by P_τ a free abelian semi-group with unity element, the set of generators of which has cardinal number τ. Then the ring $R[P_\tau]$ is isomorphic to the ring of polynomials $R[X_\tau]$, where X_τ is the set of commutative indeterminates with cardinal number τ. We can therefore apply here the results of the first part of this paper (cf. [12], [13]).

The P_τ-invariant properties will be called *polynomially invariant*. The P_τ-normal properties will be called *polynomially normal properties* or *Amitsur properties*.

There exist radical properties which are not polynomially normal. One can verify this fact considering the upper radical determined by an arbitrary finite field [8].

Let τ and τ' be any two cardinal numbers. Then the semi-groups $P_\tau \times P_{\tau'}$ and $P_{\tau+\tau'}$ are isomorphic to each other. Therefore, for any radical property S, we have $P_{\tau+\tau'}S = P_\tau(P_{\tau'}S) = P_{\tau'}(P_\tau S)$. Moreover, if $\tau' \leqslant \tau$, there exists a homomorphism of the semi-group P_τ onto $P_{\tau'}$, and therefore for an arbitrary radical property S we obtain the inequality $P_\tau S \leqslant P_{\tau'}S$. In particular, $P_\tau S \leqslant S$.

Theorem 7. *For any cardinal number τ there exists a field F and a radical property S in the class of algebras over the field F such that if $\tau' < \tau$ then $P_{\tau'}S > P_\tau S$.*

Proof. There are three possibilities to consider

(a) $\tau > \aleph_0$. Let F be an arbitrary field with cardinal number τ and let $F(y)$ be the field of rational functions over F in one indeterminate y. Let S denote the upper radical property determined by the field F. Consider the field F. We show that F is a $P_{\tau'}S$-ring for any $\tau' < \tau$, but is not a $P_\tau S$-ring. The dimension of $F(y)$ over F is equal τ and for any $\tau' < \tau$ the dimension of $F[X_{\tau'}]$ over F is $\tau'\aleph_0$. Since $\tau > \aleph_0$, $\tau'\aleph_0 < \tau$. Comparing the dimensions of $F(y)$ and $F[X_{\tau'}]$ we see that there does not exist an F-homomorphism from $F[X_{\tau'}]$ onto $F(y)$ and therefore $F[X_{\tau'}]$ is a $P_{\tau'}S$-ring for any $\tau' < \tau$.

Now let T be a set with cardinal number τ. The field $F(y)$ has a base with cardinal number τ over F. Denote by a_t, where $t \in T$, the elements of this base. Similarly, indeterminates in the ring $F[X_\tau]$ will be denoted by x_t, where $t \in T$. Define the mapping $h\colon F[X_\tau] \to F(y)$ by $h(a) = a$, $a \in F$ and $h(x_t) = a_t$, $t \in T$. One can easily see that h is a homomorphism onto.

From the fact that such a homomorphism exists it follows that $F[X_\tau]$ is not an S-ring and therefore it is not a $P_\tau S$-ring.

(b) $\tau = \aleph_0$. Let F be any denumerable field and $F(y)$ the field of rational functions in one indeterminate y over F. Denote by S the upper radical property determined by the field $F(y)$ in the class of algebras over F. Consider the field F. We show that F is a P_kS-ring for any positive integer k, but is not a $P_{\aleph_0}S$-ring. The Hilbert's Nullstellensatz [10] implies that, for any $k > 0$, there does not exist an F-homomorphism of the ring $F[x_1, \ldots, x_k]$ onto the field $F(y)$. Hence F is a P_kS-ring for every $k > 0$. The dimension of $F(y)$ over F is $\aleph_0$. Let the elements a_l, $l = 1, 2, \ldots$ form a base of $F(y)$ over F. Similarly as in (a), an F-homomorphism of $F[X_{\aleph_0}] = F[x_1, x_2, \ldots]$ onto $F(y)$ can be defined. This implies that F is not a $P_{\aleph_0}S$-ring.

(c) $\tau = n$, where n is a positive integer. Let F_0 be any field of characteristic $p > 0$. Denote by F the field $F_0(y_1, \ldots, y_n)$ of rational functions in n commutative indeterminates $y_1, \ldots, y_n$ over F_0 and by G its extension $F_0(\sqrt[p]{y_1}, \ldots, \sqrt[p]{y_n})$. Let S be the upper radical property determined by the field G in the class of all algebras over F. Consider the field F. We show that F is a P_kS-ring for any $k < n$ and that F is not a P_nS-ring. The field G is an algebra over F generated by the elements $\sqrt[p]{y_1}, \ldots, \sqrt[p]{y_n}$.

The dimension of G over is p^n and $a^p \in F$ for every a in G. Hence G is not generated by k elements for $k < n$. For any $k < n$ the radical $P_k S(F)$ is equal F since every F-homomorphism of the ring $F[x_1, \dots, x_k]$ sends it onto an F-algebra generated by k elements. However, there exists an F-homomorphism h of the ring $F[x_1, \dots, x_n]$ onto the field G defined by the equalities $h(x_i) = \sqrt[p]{y_i}$, where $i = 1, \dots, n$. The existence of this homomorphism implies that $P_n S(F)$ is not equal F. This completes the proof.

The following theorem is a weaker version of the preceding one for the class of all rings.

THEOREM 8. *For any cardinal number $\tau \geqslant \aleph_0$ there exists a radical property S such that $P_{\tau'} S > P_\tau S$ for $\tau' < \tau$.*

Proof. We consider two cases.

(a) $\tau > \aleph_0$. Let F be any algebraically closed field with cardinal number τ and let $F(y)$ be the field of rational functions over F in one indeterminate y. Let S denote upper radical property determined by $F(y)$ in the class of all rings. Consider the field F. We show that F is a $P_{\tau'} S$-ring for any $\tau' < \tau$ but is not a $P_\tau S$-ring. Let $\tau' < \tau$. Suppose there exists a homomorphism h of the ring $F[X_{\tau'}]$ onto the field $F(y)$. This homomorphism gives the field $F(y)$ a now structure of vector space if we put $\alpha \circ a = h(\alpha) a$ for $\alpha \in F$, $a \in F(y)$. Since F is algebraically closed with cardinal number τ and $F(y)$ is not algebraically closed, the dimension of the vector spaces $F(y)$ defined above is greater or equal τ. The homomorphism h is an F-linear mapping of the vector space $F[X_{\tau'}]$ onto the considered space $F(y)$. The comparison of dimensions of these vector spaces leads to a contradiction. Analoguously as in the case (a) of Theorem 7 we obtain that $P_\tau S(F) \neq F$.

(b) $\tau = \aleph_0$. Let F be a prime field and $F(y)$ the field of rational functions over F in one indeterminate y. Denote by S the upper radical property determined by the field $F(y)$ in the class of all rings. Since every homomorphism of the ring $F[x_1, \dots, x_k]$ into the field $F(y)$ is an F-linear mapping, then, similarly as in the preceding theorem, we have $P_k S(F) = F$ for any positive integer k and $P_{\aleph_0} S(F) = 0$.

We shall show in the sequel, that for many radical properties S the following condition is satisfied: $P_\tau S = P_{\aleph_0} S = \bigcap_{n=0}^{\infty} P_n S$ for any $\tau \geqslant \aleph_0$, where $\bigcap_{n=0}^{\infty} P_n S$ is the largest radical property smaller then $P_i S$ for $i = 0, 1, 2, \dots$ Properties satisfying this condition will be called *properties of finite type.*

LEMMA 3. *If $S = P_1 S$ and S is a property of finite type then, for any τ, $P_\tau S = S$.*

Proof. Since $P_{n+1} S = P_n(P_1 S)$, we have $P_m S = S$ for any m and therefore $S = \bigcap_{n=0}^{\infty} P_n S = P_{\aleph_0} S = P_\tau S$ for any τ, as the property S is of finite type.

We shall denote the Brown-McCoy property by M.

PROPOSITION 11. *The Brown-McCoy property is of finite type.*

Proof. From the preceding we obtain the following inequalities for any property S and any cardinal number $\tau \geqslant \aleph_0$:

$$P_0 S \geqslant P_1 S \geqslant \dots \geqslant \bigcap_{n=0}^{\infty} P_n S \geqslant P_{\aleph_0} S \geqslant P_\tau S .$$

Since every polynomial in the ring $R[X_\tau]$ depends only on a finite number of variables, then, by the definition of the property M, we have that $P_\tau M \geqslant \bigcap_{n=0}^{\infty} P_n \mathrm{X}$. This completes the proof.

4. Associative rings. In this section only associative rings will be considered.

LEMMA 4. *If R is an algebra over a commutative ring F and h is a homomorphism of the ring $R[x]$ into some ring A with unity element, then the homomorphism h may be in one and only one way extended to a homomorphism $\bar{h}$ of the ring $R^*[x]$ onto A, where R^* is obtained from R by an adjunction of unity element by the ring F.*

Proof. The ring $R[x]$ is an ideal of $R^*[x]$. Therefore the homomorphism h (as it is well known [4]) may be extended to $\bar{h}$. It can be easily verified that the homomorphism $\bar{h}$ is uniquely determined.

Consider now a ring R and a homomorphism h of the ring $R[x]$ onto a simple ring A with unity element. The homomorphism h determines in R a sequence of ideals $I_n(h)$ defined as follows: an element r of R is in $I_n(h)$ if and only if there exist elements $r_0, r_1, \dots, r_{n-1}$ in R such that the polynomial $r_0 + r_1 x + \dots + r_{n-1} x + r x^n$ is in the kernel of h. Lemma 4 implies the following properties of the ideals $I_n(h)$:

(i) $I_k(h) \subseteq I_{k+1}(h)$ for $k = 0, 1, 2, \dots;$

(ii) there exists a positive integer j such that $I_j(h) \neq I_0(h)$;

(iii) $I_0(h)$ is a prime ideal of R;

(iv) if R is an algebra over a commutative ring F then the ideals $I_n(h)$ are F-ideals of R.

LEMMA 5. *If R is an algebra over a field F such that $M(R) = R$ and h is a homomorphism of the ring $R[x]$ onto a simple ring A with unity element then $\bar{h}(x)$ is transcedental over $\bar{h}(F)$, where $\bar{h}$ is the extension of h defined in Lemma 4.*

Proof. The field F being contained in the center of $R^*[x]$, $\bar{h}(F)$ is contained in the center of A. Hence the ring A may be considered as an F-algebra if we put $f\cdot a = \bar{h}(f)\cdot a$ for $f\in F$ and $a\in A$. The homomorphism h is an F-linear homomorphism of $R^*[x]$ onto the F-algebra A. The same holds for h and $R[x]$.

Suppose there exists a polynomial φ of degree n with coefficients in F such that $\varphi(h(x)) = 0$. Let k be the smallest integer such that $I_k(h) \neq I_0(h)$. First, we show that there exists a polynomial $p(x)$ in $R[x]$ such that $h(p) = 1$ and the degree of p is smaller or equal k. Since $I_0(h)$ is a prime ideal of R, $(I_k(h))^n \not\subset I_0(h)$, $(I_k(h))^n[x]$ is an ideal of $R[x]$ and $(I_k(h))^n x \neq \ker h$. By the homomorphism theorem there exists a polynomial q in $(I_k(h))^n[x]$ such that $h(q) = 1$. If the degree of q is $\geqslant n$ we have $q = q_0\varphi + q_1$, where the degree of q_1 is smaller then the degree of φ and $h(q_1) = 1$. We may therefore assume that the degree of q is $<n$. Let $q = \sum_{i=0}^{l} b_i x^i$, where $l<n$, $b_i \in (I_k(h))^n$. If $l \geqslant k$, then, since $b_k \in (I_k(h))^n$, we can find a polynomial q' of degree l in $(I_k(h))^{n-1}[x]$, $q' = \sum_{i=0}^{l} c_i x^i$, $c_i \in (I_k(h))^{n-1}$, such that $c_l = b_l$ and $h(q') = 0$. Thus $h(q-q') = 1$, the degree of $q-q'$ is smaller than l and $q-q'$ is in $(I_k(h))^{n-1}[x]$. Denote the degree of $q-q'$ by s. If $s \geqslant k$ then we can find a polynomial q'' in $(I_k(h))^{n-2}[x]$ such that $h(q'') = 0$, the degree of $(q-q')-q''$ is smaller than s and $h(q-q'-q'') = 1$. Proceeding in this manner, one can obtain a polynomial $p \in I_k(h)[x]$ of degree smaller than k and such that $h(p) = 1$ not more than $n-k$ times. Let $p_0 = r_0 + r_1 x + \ldots + r_m x^m$ be a polynomial of minimal degree such that $h(p_0) = 1$ and $m<k$. Since $M(R) = R$, m is positive. Let r be any element of the ring R. Then $h(p_0 r - r) = 0$ and therefore $r_m\cdot r$ belongs to $I_m(h) = I_0(h)$. The ring $I_0(h)$ being a prime ideal of R, r_m belongs to $I_0(h)$. Hence

$$1 = h\Big(\sum_{i=0}^{m-1} r_i x^i\Big) + h(r_m)\,h(x^m) = h\Big(\sum_{i=0}^{m-1} r_i x^i\Big),$$

as $h(r_m) = 0$. This is a contradiction with the choice of the polynomial p_0. Therefore $h(x)$ must be transcendental over F.

THEOREM 9. *If R is an algebra over a field F with cardinal number τ, $\tau > \aleph_0$, and the base of R over F has cardinal number $\tau' < \tau$, then $P_1M(R) = M(R)$.*

Proof. Clearly, $M(R) \supseteq P_1M(R)$. Suppose there exists a homomorphism h of the ring $(M(R))[x]$ onto some simple ring A with unity element. Then, by Lemma 5, $h(x)$ is transcendental over F. Since elements of the form $1/(h(x)-a)$ for $a\in F$ are linearly independent over F, the cardinal number of the base of A over F is also greater than τ. But the cardinal number of the base of $(M(R))[x]$ over F is $\leqslant \tau'\aleph_0 = \max\{\tau', \aleph_0\} < \tau$. As h is F-linear, then A cannot be the image of $(M(R))[x]$. Then $(M(R))[x]$ is an M-ring and therefore $M(R)$ is a P_1M-ring. Therefore $M(R) \subseteq P_1M(R)$ and so $M(R) = P_1M(R)$.

COROLLARY 7. *In the class $\mathcal{A}_F$ of F-algebras satisfying the conditions of Theorem 9 the equality $M = P_\tau M$ holds for any cardinal number τ.*

Proof. It follows from Theorem 9 that in the class $\mathcal{A}_F$ the equality $P_1M = M$ is satisfied. Since the class $\mathcal{A}_F$ together with R contains the algebra $R[x]$ and the Brown-McCoy property is of finite type (Proposition 11), Corollary 7 follows from Lemma 3.

COROLLARY 8. *If R is a finitely generated algebra over an undenumerable field F, then, for any cardinal number τ, $M_\tau(R) = M(R)$.*

Proof. This corollary follows immediately from the fact that the dimension of any finitely generated algebra over a field is not greater than $\aleph_0$.

THEOREM 10. *If R is a simple ring then $M(R) = P_1M(R)$.*

Proof. Clearly $P_1M(R) \subseteq M(R)$. Now, let R be an M-ring. To show that R is a P_1M-ring we prove that $R[x]$ is an M-ring. Suppose that there exists a homomorphism h of the ring $R[x]$ into a simple ring A with unity element. Let k be the smallest integer such that $I_k(h) \neq I_0(h)$. Since R is simple, $I_0(h) = 0$ and $I_k(h) = R$. Let $p \in R[x]$, $p = r_0 + r_1 x + \ldots + r_n x^n$, $r_n \neq 0$ be a polynomial of minimal degree such that $h(p) = 1$, $1\in A$. Since R is an M-ring, $n > 0$. Were $n \geqslant k$, r_n would be in $I_k(h) = I_n(h)$ and one would find a polynomial $q = s_0 + s_1 x + \ldots + s_{n-1}x^{n-1} + r_n x^n$ such that $h(q) = 0$. Then $h(p-q) = 1$ and the degree of $p-q$ would be smaller than n. This contradicts the choice of the polynomial p. Therefore $n<k$. Let r be any element of the ring R. Then $pr-r$ is in the kernel of h and therefore $r_n\cdot r$ belongs to $I_n(h) = I_0(h) = 0$. Hence $r_nR = 0$ and $Rr_n = 0$ and thus $R^2 = 0$, R being a simple ring. Therefore $(R[x])^2 = R^2[x] = 0$. In particular $1 = h(p^2) = h(0)$. This contradiction shows that R is a P_1M-ring.

Baer, Levitzki and Jacobson radical properties are inherited by one-sided ideals and are strong. Therefore, from the results of section 2 it follows that these properties are matrically invariant. The Brown-McCoy radical property is also matrically invariant. The question whether Köethe radical property is matrically invariant is equivalent to the Köethe problem which may be formulated as follows: are the one-sided nil-ideals contained in two-sided nil-ideals ([9], [12])?

THEOREM 11 ([1] and [8]). *Baer, Levitzki, Köethe, Jacobson and Brown-McCoy radical properties are Amitsur properties.*

THEOREM 12 ([1]). *Baer and Levitzki radical properties are polynomially invariant and therefore of finite type.*

The problem whether $P_1 J = K$ is equivalent to the problem of Köethe mentioned above ([9], [12]).

Amitsur ([2]) introduced the following definition:

DEFINITION 4. An algebra R over a field F has the LBI *property* if and only if every finite-dimensional F-subspace V of R consists of bounded index nilpotent elements.

Amitsur has shown that the LBI property is radical in the class of algebras over a fixed and infinite field F and that $\mathrm{LBI} = P_{\aleph_0} K = P_{\aleph_0} J$. The following theorem is also due to Amitsur ([2]).

THEOREM 13. *In the class of algebras over an arbitrary undenumerable field F $K = P_{\aleph_0} K = \mathrm{LBI}$.*

Below we give another proof of this theorem.

Let R be an arbitrary nil-algebra over an undenumerable field F and let the polynomial $p(x)$ be in $R[x]$. For any $a \in F$ the mapping h_a defined by $h_a(q) = q(a)$, where $q \in R[x]$, is an F-homomorphism of the algebra $R[x]$ onto R. For m, a positive integer, denote by F_m the set of all those a in F for which $(h_a(p))^m = 0$. Since R is a nil-algebra, then for any a in F there exists an integer $k > 0$ such that $(h_a(p))^k = 0$, i.e. $a \in F_k$. From this is follows that $F = \bigcup_{m=1}^{\infty} F_m$. Since the field F is undenumerable, there exists an integer $i > 0$, such that F_i is an infinite set. From the fact that mappings h_a are homomorphisms it follows that $0 = (h_a(p))^i = h_a(p^i)$ for a in F_i. The set F_i being infinite, $p^i = 0$. Thus, we have shown that the algebra $R[x]$ is nil and therefore $K = P_1 K$. Our theorem now follows from Lemma 3.

Now, if we confine ourselves to the class A_F of algebras R over a fixed field F with cardinal number $\tau > \aleph_0$ such that the cardinal number of the base of R over F is $\tau' < \tau$, then, as it is well-known (2), $J = K$, i.e. $J = \mathrm{LBI}$. In the class of all rings the only sequence of radical properties we can write is the following:

$$J > K \geqslant P_1 J \geqslant P_1 K \geqslant P_2 J \geqslant P_2 K \geqslant \ldots \geqslant \bigcap_{n=0}^{\infty} P_n J = \bigcap_{n=0}^{\infty} P_n K > L .$$

In fact, since $J > K$ (see Example 2), $P_n J \geqslant P_n K$ for all n. It is well known (1) that $K \geqslant P_1 J$, so $P_n K \geqslant P_{n+1} J$ for all n. Moreover, one can verify, that nil-algebras constructed by Golod [7] are $P_{\aleph_0} K$-algebras. Those nil-algebras are not L-algebras and therefore $P_{\aleph_0} K > L$. It seems to be interesting to investigate the above sequence of properties.

In the class of all rings neither $J \leqslant P_1 M$ nor $P_1 M \leqslant J$ holds.

EXAMPLE 2. Let F be any field and let R be the ring of all rational functions over F in one indeterminate y of the form $yf(y)/g(y)$, $f(y)$ and $g(y)$ in $F[y]$ and $g(0) \neq 0$. One can easily verify that $M(R) = J(R) = R$ and $K(R) = 0$. Since the ring R is commutative then, as it is well known [1], $M(R[x]) = J(R[x]) = (K(R))[x] = 0$. Thus we have shown that $P_1 M(R) = 0$. Now, if F is an undenumerable field, $P_1 M(R) \neq M(R) = J(R) = R$. Therefore we cannot ommit the condition on the dimension of R in Theorem 9.

EXAMPLE 3. Let R be the ring of all matrices with elements in an arbitrary field F. The ring R is simple without unity element. Therefore, by Theorem 10, R is a $P_1 M$-ring. However R is not J-semi-simple. Let I be an ideal of R composed of all matrices which have non-zero elements in the first row only. It is easily seen that there exists a homomorphism of the ring I onto the field F. Therefore I is not an M-ring.

References

[1] S. A. Amitsur, *Radicals of polynomial rings*, Canad. J. Math. 8 (1956), pp. 355–361.

[2] — *Algebras over infinite fields*, Proc. Amer. Math. Soc. 7 (1956), pp. 35–48.

[3] T. Anderson, N. J. Divinsky and A. Suliński, *Hereditary radicals in associative and alternative rings*, Canad. J. Math. 17 (1965), pp. 594–603.

[4] N. J. Divinsky, *Rings and Radicals*, London 1965.

[5] — J. Krempa and A. Suliński, *Strong radical properties of alternative and associative rings*, Journal of Algebra 17 (3) (1971).

[6] — and A. Suliński, *Kurosh radicals of rings with operators*, Canad. J. Math. 17 (1965), pp. 278–288.

[7] Е. С. Голод, *О ниль-алгебрах и финитноаппроксимируемых р-группах*, ИАН СССР, сер. матем. 28 (1964), стр. 273-276.

[8] J. Krempa, *On radical properties of polynomial rings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), pp. 545–548.

[9] — *Logical connections between some open problems on non-commutative rings*, Fund. Math. 76 (1972), pp. 121–130.

[10] S. Lang, *Algebra*, Reading 1965.

[11] Ortiz, *A construction in general radical theory*, Canad. J. Math. 22 (1970), pp. 1097–1100.

[12] S. A. Amitsur, *Nil radicals. Historical notes and some new results*, Rings, modules and radicals, Amsterdam (1973), pp. 47–65.

[13] A. Sierpińska, *Radicals of polynomials in non-commutative indeterminates*, Bull. Acad. Polon. Sci. Sér Sci. Math. Astronom. Phys. 21 (1973), pp. 805–808.

Reçu par la Rédaction le 20. 2. 1973