References

[1] J. C. E. Dekker, *Infinite series of isols*, Proc. Sympos. Pure Math. 5, Amer. Math. Soc., Providence, R. I. (1962), pp. 77–96.
[2] — *The minimum of two regressive isols*, Math. Z. 83 (1964), pp. 345–366.
[3] C. G. Jockusch, Jr., *Uniformly introreducible sets*, J. Symbolic Logic 33 (4) (1968), pp. 521–536.
[4] S. C. Kleene, *Introduction to Metamathematics*, 1952.
[5] J. R. Shoenfield, *On degrees of unsolvability*, Ann. of Math. 69 (1959), pp. 644–653.

Reçu par la Rédaction le 10. 11. 1972

On h-regular graded algebras

by

A. Tyc (Toruń)

Abstract. Let R be a commutative ring with identity. In the paper the well-known notion of a regular sequence in R (or an R-sequence) is generalized as follows: A sequence $u_1, \ldots, u_n$, $u_i \in R$, is called an h-regular sequence in R if $(u_1, \ldots, u_n) \neq R$ and $(u_1, \ldots, u_{k-1}) : (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$ $(k = 1, \ldots, n)$, where $h_i = h(u_i)$ is the minimum of integers $n > 0$ such that $u_i^n = 0$ (if there is no such an integer $h(u_i) = \infty$ and $u_i^\infty = 0$). A local Noetherian ring R is said to be *h-regular* if its unique maximal ideal is generated by an h-regular sequence. It is shown that any commutative graded R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ with the ideal $I = \bigoplus_{i>0} A_i$ generated by an h-regular set is of the form $\bigoplus_i R[X]/(X^{h_i})$ for some $h_i \in N \cup \{\infty\}$ (N is the set of positive integers). Moreover, the Tate resolution of such algebras is found provided R is an h-regular local Noetherian ring.

Introduction. Let R be a commutative local Noetherian ring with the unique maximal ideal m. Recall that a sequence $u_1, \ldots, u_n$, $u_k \in m$, is called an *R-sequence* if $(u_1, \ldots, u_{k-1}) : (u_k) = (u_1, \ldots, u_{k-1})$ for $k = 1, \ldots, n$. In [1] T. Józefiak adapts this definition for commutative graded R-algebras. Namely, if $A = \bigoplus A_i$ is such an algebra, then a sequence $u_1, \ldots, u_n$ of homogeneous element from the ideal $I = m \oplus (\bigoplus_{i>0} A_i) \subset A$ is said to be *normal* (or *regular*) (1) in A provided

$$(u_1, \ldots, u_{k-1}) : (u_k) = \begin{cases} (u_1, \ldots, u_{k-1}) & \text{if } \deg u_k \text{ is even,} \\ (u_1, \ldots, u_k) & \text{if } \deg u_k \text{ is odd} \end{cases}$$

for $k = 1, \ldots, n$ (we assume $x^2 = 2$ for any homogeneous element $x \in A$ of odd degree). In this paper the notion of a regular sequence in A is

(1) The term "regular sequence" instead of "normal sequence" is used in Józefiak's next paper [3]. We prefer the term "regular sequence" also.

generalized as follows: Let $u_1, \ldots, u_n$ be as above and let $h(u_k) = h_k$ be the minimum of integers $n > 0$ such that $u_k^n = 0$ (if $u_k^n \neq 0$ for all n, then we put $h(u_k) = \infty$ and $u_k^\infty = 0$). We say that $u_1, \ldots, u_n$ is an *h-regular sequence in* A if the following condition holds:

$$(u_1, \ldots, u_{k-1}) : (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$$

for $k = 1, \ldots, n$. Clearly any regular sequence in A is an h-regular sequence in A. A set U of homogeneous elements of a commutative graded R-algebra A is called *h-regular* if every finite sequence $u_1, \ldots, u_n$, $u_k \in U$, is h-regular in A. A commutative graded R-algebra A is called *h-regular* if the ideal I is generated by an h-regular set of homogeneous elements.

In Section 1, making use of the methods of [1], we prove that a commutative graded R-algebra A is h-regular if and only if A is isomorphic with the tensor product of graded algebras $\bigoplus_i A_i$, where $A_i = R[X]/(X^{h_i})$ for some $h_i \in N \cup \{\infty\}$, $h_i > 0$. The Tate resolution (see [1]) of any h-regular finitely generated R-algebra is found in Section 2.

Throughout the paper all local rings are assumed to be Noetherian.

I wish to thank dr. T. Józefiak for helpful discussions.

1. h-regular sequences. Let R be a commutative ring with identity. By a graded R-algebra we mean in this paper a positively graded R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ satisfying the following conditions:

(i) $xy = (-1)^{pq} yx$ if $x \in A_p$, $y \in A_q$,

(ii) $z^2 = 0$ if $z \in A_p$ and p is odd,

(iii) $A_0 = R$.

We write $\partial(x) = p$ if $x \in A_p$ and say that x is a homogeneous element of degree p. The ring R is a graded R-algebra with trivial grading $R_0 = R$, $R_i = 0$ for $i > 0$. If A is a graded R-algebra, then the ideal $\bigoplus_{i>0} A_i$ will be denoted by I'. One can easily check that for any graded A-module N the equality $NI' = N$ implies $N = 0$.

1.1. Definition. Let x be an element of a graded R-algebra A. The height of x (shortly $h(x)$) is the minimum of integers $n \geq 1$ such that $x^n = 0$. If there is no such an integer, then we put $h(x) = \infty$.

Observe that $h(x) = 2$ for any non-zero homogeneous element x of odd degree.

1.2. Lemma. *If $h(x) = h$ and $(0) : (x) = (x^{h-1})$* [2], *then $(0) : (x^i) = (x^{h-i})$ for $0 \leq i \leq h$.*

[2] In this paper we use the following conventions: $\infty - i = \infty$ if $i < \infty$, $\infty - \infty = 0$, $x^\infty = 0$, $x^0 = 1$.

The proof is easy and we omit it.

1.3. Definition. Let A be a graded R-algebra. A sequence $u_1, \ldots, u_n$ of homogeneous elements of A with $h(u_i) = h_i$ is called *h-regular* in A if the following conditions hold:

1° $(u_1, \ldots, u_n) \neq A$,

2° $(u_1, \ldots, u_{k-1}) : (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$, $k = 1, \ldots, n$ (for $k = 1$ we set $(u_1, \ldots, u_{k-1}) = 0$).

Let N^∞ denote the set $N \cup \{\infty\}$, where N is the set of positive integers.

Example 1. Suppose R' is a commutative local ring and $u_1', \ldots, u_n'$ is an R'-sequence. It is not difficult to show (see [4]) that for any sequence $h_1, \ldots, h_n$, $h_i \in N^\infty$, $h_i > 1$, $i = 1, \ldots, n$, the images $u_1, \ldots, u_n$ of u''s in $R = R'/(u_1'^{h_1}, \ldots, u_n'^{h_n})$ form an h-regular sequence in R. Moreover, $h(u_i) = h_i$ for $1 \leq i \leq n$.

Example 2. Let $n > 0$ be a fixed integer and let $k_1, \ldots, k_n, h_1, \ldots, h_n$ be a sequence of elements of N^∞ such that $k_i < \infty$ for $i = 1, \ldots, n$ and $h_i = 2$ for k_i odd. Then clearly $R[X_i]/(X_i^{h_i})$ with the grading given by $\partial(X_i) = k_i$ is a graded R-algebra for all $1 \leq i \leq n$. Denote by T the graded R-algebra $\bigotimes_{i=1}^{n} R[X_i]/(X_i^{h_i})$, where "$\otimes$" is the tensor product in the category of graded R-algebras. One can easily check that the images $x_1, \ldots, x_n$ of X's in T form an h-regular sequence in T and $h(x_i) = h_i$, $1 \leq i \leq n$.

1.4. Lemma. *Let $u_1, \ldots, u_n$ be a sequence of homogeneous elements of a graded R-algebra A with $h(u_i) = h_i$. Then $u_1, \ldots, u_n$ is an h-regular sequence in A if and only if for some k $u_1, \ldots, u_{k-1}$ is an h-regular sequence in A and the images $\bar{u}_k, \ldots, \bar{u}_n$ of u's in $\bar{A} = A/(u_1, \ldots, u_{k-1})$ form an h-regular sequence in $\bar{A}$ with $h(\bar{u}_i) = h_i$, $i = 1, \ldots, n$.*

Proof. Easy.

1.5. Proposition. *If $u_1, \ldots, u_n$ is an h-regular sequence in a graded R-algebra A, $h(u_i) = h_i$ and $u_i \in I'$, then for any permutation $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$, $u_{i_1}, \ldots, u_{i_n}$ is an h-regular sequence in A.*

Proof. In view of Lemma 1.4 and the fact that each permutation is a composition of transpositions of the form $(k, k+1)$, it suffices to prove the proposition for the transposition changing 1 and 2. Consequently we have to show equalities 1 and 2 below:

1. $(0) : (u_2) = (u_2^{h_2-1})$, 2. $(u_2) : (u_1) = (u_2, u_1^{h_1-1})$.

For the proof of 1 we need the following

1.6. Lemma. *If u_1, u_2 is an h-regular sequence in A with $h(u_i) = h_i$ and $yu_2 = 0$ for some $y \in A$, then for all k, $0 \leq k < h_1$, there are $a, b \in A$ such that $y = a \cdot u_2^{h_2-1} + b \cdot u_1^{k+1}$.*

The Lemma easily follows by induction on k.

Now one can prove 1. Consider two cases:

(a) $h(u_1) = h_1 < \infty$, (b) $h_1 = \infty$.

In case (a) if $yu_2 = 0$, then, applying Lemma 1.6 to y and $k = h_1-1$, we get $y = a\cdot u_2^{h_2-1} + b\cdot u_1^{h_1} = a\cdot u_2^{h_2-1} \in (u_2^{h_2-1})$. In case (b) it is sufficient to show that $\big((0):(u_2)\big)I' + (u_2^{h_2-1}) = (0):(u_2)$ since this equality gives

$$\big((0):(u_2)/(u_2^{h_2-1})\big)I' = (0):(u_2)/(u^{h_2-1})$$

and consequently $(0):(u_2) = (u_2^{h_2-1})$. Let $y \in (0):(u_2)$. By Lemma 1.6 $y = a\cdot u_2^{h_2-1} + bu_1$ for some $a, b \in A$. Hence $0 = yu_2 = (bu_2)u_1$. It follows that $bu_2 = 0$ since $(0):(u_1) = 0$ $\big(h(u_1) = \infty\big)$. Thus $y = a\cdot u_2^{h_2-1} + bu_1 \in (u_2^{h_2-1}) + \big((0):(u_2)\big)I'$ $(u_1 \in I'!)$ and part 1 is proved.

To prove 2 assume $y \in (u_2):(u_1)$. Then $yu_1 = bu_2$, which implies $b = c\cdot u_2^{h_2-1} + au_1$. Hence $yu_1 = au_1u_2$. Hence we conclude that $y \pm au_2 \in (u_1^{h_1-1})$, i.e. $y \in (u_2, u_1^{h_1-1})$. The proposition is proved.

Proposition 1.5 permits us to speak about finite h-regular sets contained in the ideal I' and justifies the following

1.7. Definition. A set U of homogeneous elements of a graded R-algebra A contained in I' is called *h-regular in A* if every finite subset of U is an h-regular set in A in the sense of Definition 1.3.

Now we shall characterize those graded R-algebras for which the ideal I' is generated by an h-regular set.

1.8. Lemma. *If $u_1, \dots, u_n$ is an h-regular sequence of homogeneous elements of a graded R-algebra A, $u_i \in I'$, $h(u_i) = h_i$, and*

$$T = \bigotimes_{i=1}^{n} R[X_i]/(X_i^{h_i})$$

with $\partial(X_i) = \partial(u_i)$, then the natural map of graded R-algebras $\varphi\colon T \to A$ defined by $\varphi(X_i) = u_i$ is an injection.

The proof is a slight modification of the proof of Lemma 2.5 in [1] and we leave it to the reader.

1.9. Theorem. *Suppose A is a graded R-algebra and the ideal I' $(= \bigoplus_{i>0} A_i)$ is generated by a set of homogeneous elements $\{u_i, i \in \Lambda\}$ with $h(u_i) = h_i$. Then the following statements are equivalent:*

(i) *$\{u_i, i \in \Lambda\}$ is an h-regular set in A.*

(ii) *$A \simeq \bigotimes_i A_i$, where $A_i = R[X_i]/(X_i^{h_i})$ and $\partial(X_i) = \partial(u_i)$.*

Proof. The implication (ii) $\Rightarrow$ (i) is a straightforward computation. For implication (i) $\Rightarrow$ (ii) observe that the natural map of graded R-algebras

$$\varphi\colon \bigotimes_i A_i \to A$$

given by $\varphi(X_i) = u_i$ is an epimorphism. Moreover, φ is an injection by Lemma 1.8. Consequently φ is an isomorphism and the theorem is proved.

In what follows we assume that the basic ring R is a local ring with the unique maximal ideal m and that all the graded R-algebras $A = \bigoplus_{i=0}^{\infty} A_i$ and all graded A-modules $M = \bigoplus_{j=0}^{\infty} M_j$ under consideration are of finite type, i.e. A_i, M_j are finitely generated R-modules. Denote by I the ideal $m \oplus I'$. I is the unique maximal homogeneous ideal in A and hence any h-regular homogeneous set is contained in I. Furthermore, in this situation we have:

1.10. Nakayama Lemma. *If M is a graded A-module, then $MI = M$ implies $M = 0$.*

Using the Nakayama Lemma in part 2 (b) of the proof of Proposition 1.5 (instead of the fact that $MI' = M$ gives $M = 0$) one can prove that this proposition is true for any h-regular sequence (not necessarily contained in I'). Consequently, repeating Definition 1.7, we may speak about h-regular sets of arbitrary cardinality. Finally, recall that a set $\{m_i, i \in J\}$ of homogeneous elements of a graded A-module M is called a *minimal set* of generators of M if $\{m_i + MI, i \in J\}$ is a base of the vector space M/MI over the field A/I.

1.11. Proposition *If $u_1, \dots, u_n$ is an h-regular set of homogeneous generators of the ideal I, then $u_1, \dots, u_n$ is a minimal set of generators of I.*

Proof. Easy.

1.12. Definition. A graded R-algebra A is called *h-regular* if the ideal I is generated by an h-regular set of homogeneous elements. If the trivial graded R-algebra R is h-regular, then R is called an *h-regular local ring*.

1.13. Remark. Any regular local ring is obviously an h-regular local ring.

The following is a generalization of [1], Theorem 2.6.

1.14. Proposition. *A graded R-algebra A is h-regular if and only if R is the h-regular local ring and*

$$A \simeq \bigotimes_{i\in J} R[X_i]/(X_i^{h_i})$$

for some index set J and a set $\{h_i, i \in J\}$, $h_i \in N^{\infty}$.

Proof. This is a consequence of Theorem 1.9.

1.15. Remark. It is shown in [1] that if the ideal I of a graded R-algebra A is generated by a normal (= regular) set of homogeneous elements $u_1, \dots, u_n$, then any minimal set of homogeneous generators of I is regular. This does not hold in the case of h-regular sets.

EXAMPLE 3. Let $A = k[X, Y]/(X^3, Y^3)$, where k is a field and $\partial(X) = \partial(Y) = 2$. Clearly $(\bar{X}, \bar{Y})$ is an h-regular set of homogeneous generators of the ideal I. On the other hand, the elements $u = \bar{X}$, $v = u+\bar{Y}$ form a minimal but not h-regular set of homogeneous generators of I because $h(v) = 5$ whereas $(u):(v) = (u, v^2) \neq (u, v^4)$.

2. Tate resolution of h-regular, finitely generated R-algebras. As before, let R be a local ring with the maximal ideal m and the residue field $k = R/m$. In [1] an analogue of the Tate resolution of a local ring was defined for graded R-algebras. We shortly recall the basic constructions used there. For notions of a bigraded commutative A-algebra and a differential A-algebra, which we use below, see [1] also.

Let A be a fixed graded R-algebra and let $\mathbf{A}$ be a differential A-algebra. For any homogeneous cycle $u \in \mathbf{A}_{p,q}$ we define

$$\mathbf{A}\langle T, dT = u\rangle = \begin{cases} \mathbf{A} \otimes_A \Lambda(AT) & \text{if } p+q \text{ is even,} \\ \mathbf{A} \otimes_A \Gamma(AT) & \text{if } p+q \text{ is odd,} \end{cases}$$

where $\Lambda(AT)$, $\Gamma(AT)$ are bigraded commutative A-algebras given by the equalities:

$$\Lambda(AT) = A \oplus AT, \quad T^2 = 0, \quad \partial(T) = p, \quad w(T) = q+1,$$

$$\Gamma(AT) = AT^{(0)} \oplus AT^{(1)} \oplus \ldots, \quad T^{(i)}T^{(j)} = \frac{(i+j)!}{i!\,j!}\, T^{(i+j)},$$

$$\partial T^{(i)} = pi, \quad wT^{(i)} = i(q+1).$$

(Recall that for a symbol X and the integer $p \geqslant 0$ AX is a graded A-module $A \otimes_R RX$, where $(RX)_p = RX$ and $(RX)_j = 0$ if $j \neq p$.) Clearly $\mathbf{A} \subset \mathbf{A}\langle T, dT = u\rangle$. In the bigraded commutative A-algebra $\mathbf{A}\langle T, dT = u\rangle$ one can define a differential d such that $d(a) = d_{\mathbf{A}}(a)$ for $a \in \mathbf{A}$, $dT = u$ and $\mathbf{A}\langle T, dT = u\rangle$ with d is a differential A-algebra. The variable T "kills" a given cycle u. In fact, $H_n(\mathbf{A}\langle T, dT = u\rangle) = H_n(\mathbf{A})/\sigma A$, where $\sigma = u + B_n(\mathbf{A}) \in H_n(\mathbf{A})$. If $u_1, \ldots, u_n$ is a sequence of homogeneous cycles, then "killing" successively $u_1, \ldots, u_n$ we get a differential A-algebra denoted by $\mathbf{A}\langle T_1, \ldots, T_n, dT_i = u_i\rangle$. The above construction may be generalized in such a way that one can "kill" an arbitrary set of homogeneous cycles $\{u_j, j \in J\}$. In this case the corresponding differential A-algebra is denoted by $\mathbf{A}\langle T_j, dT_j = u_j\rangle$. It is proved in [1] that for every cyclic graded A-module N there exists a differential A-algebra $\mathbf{A}$ which is a free resolution of the A-module N, i.e.

1° $\mathbf{A}_{*,n}$ are free graded A-modules for all $n \geqslant 0$.

2° $H_i(\mathbf{A}) = 0$ for $i > 0$, $H_0(\mathbf{A}) = N$.

If $N = k = A/I$ $\big(I = m \oplus (\bigoplus_{i>0} A_i)\big)$, then one can find a differential A-algebra $\mathbf{A}$ which is a minimal free resolution of k, i.e. it satisfies the additional condition: $d(\mathbf{A}) \subset \mathbf{A}I$. Moreover, such an algebra $\mathbf{A}$ is unique up to an isomorphism of differential A-algebras and is called the *Tate resolution of* A. We denote it by $\mathbf{X}$. The algebra $\mathbf{X}$ is obtained as the union of an ascending chain of differential A-algebras $F_0\mathbf{X} \subset F_1\mathbf{X} \subset \ldots$ By definition: $F_0\mathbf{X} = (F_0\mathbf{X})_{*,0} = A$, $F_1\mathbf{X} = F_0\mathbf{X}\langle T_i, dT_i = u_i\rangle$, where $\{u_i\}$ is a minimal set of homogeneous generators of the ideal I and $F_{n+1}\mathbf{X} = F_n\mathbf{X}\langle T_j, dT_j = v_j\rangle$, where $\{v_j + B_n(F_n\mathbf{X})\}$ is a minimal set of homogeneous generators of the graded A-module $H_n(F_n\mathbf{X})$.

In what follows the Tate resolution of the graded R-algebra A will be found provided A is h-regular and finitely generated.

We start with the following

2.3. DEFINITION. A sequence $u_1, \ldots, u_p, \ldots, u_n$ of homogeneous elements of a graded R-algebra A is called *p-ordered* if $u_1, \ldots, u_p$ is the maximal subset of the set $u_1, \ldots, u_n$ such that $h(u_i) = \infty$ or $\partial(u_i) = 2$ for $i = 1, \ldots, p$. In particular, it follows $h(u_i) < \infty$ and $\partial(u_i)$ is even for $i = p+1, \ldots, n$.

2.4. PROPOSITION. *Let $u_1, \ldots, u_p, \ldots, u_n$, $h(u_i) = h_i$, be an h-regular, p-ordered sequence of homogeneous elements of a graded R-algebra A and let $\mathbf{Y} = A\langle T_1, \ldots, T_n, dT_i = u_i\rangle$, where $\partial(T_i) = \partial(u_i)$, $w(T_i) = 1$ $\big(u_i \in Z_0(A)!\big)$. Then $\{v_{p+1}T_{p+1} + B_1(\mathbf{Y}), \ldots, v_nT_n + B_1(\mathbf{Y})\}$, with $v_j = u_j^{h_j-1}$, is a minimal set of homogeneous generators of the graded A-module $H_1(\mathbf{Y})$ if $n-p > 0$ and $H_1(\mathbf{Y}) = 0$ if $n = p$.*

Proof. We apply induction on $s = n-p$. If $s = 0$, i.e. $n = p$, then $u_1, \ldots, u_n$ is a regular sequence and by [1], Proposition 5.1, $H_1(\mathbf{Y}) = 0$. Now assume that $s > 0$ and denote by $\mathbf{Y}'$ the algebra $A\langle T_1, \ldots, T_{n-1}, dT_i = u_i\rangle$. Then clearly $\mathbf{Y} = \mathbf{Y}'\langle T_n, dT_n = u_n\rangle$ and by the induction hypothesis $\{y'_{p+1}, \ldots, y'_{n-1}$ with $y'_i = v_iT_i + B_1(\mathbf{Y}')$ is a minimal set of homogeneous generators of the A-module $H_1(\mathbf{Y}')$. Since $\partial(u_n)$ is even $(s > 0)$, $\mathbf{Y} = \mathbf{Y}' \otimes \Lambda(AT_n)$ and we have the exact sequence

$$0 \to \mathbf{Y}' \xrightarrow{\sigma} \mathbf{Y} \xrightarrow{\tau} \mathbf{Y}' \to 0,$$

where $\sigma(a') = a' \oplus 1$ and $\tau(a' \otimes T_n) = a'$. This sequence produces the long homology sequence

$$\ldots H_1(\mathbf{Y}') \xrightarrow{\Delta} H_1(\mathbf{Y}') \xrightarrow{\sigma_*} H_1(\mathbf{Y}) \xrightarrow{\tau_*} H_0(\mathbf{Y}') \xrightarrow{\Delta} H_0(\mathbf{Y}') \ldots$$

where, as is easy to verify, Δ is a multiplication by u_n and $H_0(\mathbf{Y}') = A/(u_1, \ldots, u_{n-1})$. Hence and from the h-regularity of the sequence $u_1, \ldots, u_n$ it follows that

$$\operatorname{Im}\tau_* = \operatorname{Ker}\Delta = A\big(v_n + B_0(\mathbf{Y}')\big), \quad v_n = u_n^{h_n-1}.$$

Let $x \in H_1(Y)$ and let $\tau_*(x) = av_n + B_0(Y')$ for some $a \in A$. Then $\tau_*(x) = \tau_*(ay_n)$, where $y_n = v_n T_n + B_1(Y)$, which implies $x - ay_n \in \operatorname{Ker}\tau_* = \operatorname{Im}\sigma_* = Ay_1 + \dots + Ay_{n-1}$, $y_i = v_i T_i + B_1(Y')$, since $y_1', \dots, y_{n-1}'$ generate A-module $H_1(Y')$ and $\sigma_*(y_i') = y_i$. Consequently $x \in Ay_1 + \dots + Ay_n$, i.e. $y_1, \dots, y_n$ is a set of generators of the A-module $H_1(Y)$. We have to prove yet that it is a minimal set of generators. For this aim it is sufficient to show that the equality $\sum_{i=p+1}^{n} a_i y_i = 0$ $(a_i \in A)$ implies $a_i \in I$. Taking in view the form of y_i, we see that the above equality is equivalent to the equality $\sum_{i=p+1}^{n} a_i v_i T_i = d_2 b$ for some $b \in Y_{*,2}$. However, $Y_{*,2} = Y'_{*,2} \oplus \oplus (Y'_{*,1} \otimes AT_n)$, so $b = b_2' + b_1' \otimes T_n$, where $b_j' \in Y'_{*,j}$, $j = 1, 2$, and therefore $\sum_{i=p+1}^{n} a_i v_i T_1 = db_2' + d(b_1') T_n - b_1' u$. Since $Y_{*,1} = Y'_{*,1} \oplus AT_n$, it follows that

$$(*) \qquad \sum_{i=p+1}^{n-1} a_i v_i T_i = db_2' - b_1' u_n\,, \qquad a_n v_n T_n = d(b_1') T_n\,.$$

Hence $a_n v_n = a_n u_n^{h_n-1} = d(b_1') \in B_0(Y') = (u_1, \dots, u_{n-1})$ and by Lemma 1.2 $a_n = ru_n + s$ for some $r \in A$ and $s \in (u_1, \dots, u_{n-1}) = B_0(Y')$. Let $s = d(b')$, where $b' \in Y'_{*,1}$. As a result we obtain $d(b_1') = a_n v_n = (ru_n + d(b'))v_n = d(b')v_n = d(b' \cdot v_n)$ since d is an A-homomorphism. Thus

$$b_1' - b' v_n \in Z_1(Y')\,.$$

Using again the fact that $y_1', \dots, y_{n-1}'$ generate $H_1(Y')$, we conclude that there are $c_i \in A$ such that

$$b_1' - b' v_n - \sum_{i=p+1}^{n-1} c_i v_i T_i \in B_1(Y')\,.$$

This formula together with equality $(*)$ gives

$$\sum_{i=p+1}^{n-1} a_i v_i T_i + \sum_{i=p+1}^{n-1} u_n c_i v_i T_i \in B_1(Y')\,.$$

Consequently $\sum_{i=p+1}^{n-1} (a_1 - u_n c_i) y_i' = 0$ and from the minimality of the set $y_1', \dots, y_{n-1}'$ it follows that $a_i - u_n c_i \in I$ for $i = p+1, \dots, n-1$. Therefore $a_i \in I$, $i = p+1, \dots, n-1$ $(u_n \in I)$. This finishes the proof since $a_n = ru_n + s$ with $s \in (u_1, \dots, u_{n-1})$ belongs to I also.

2.5. Lemma. *Let A be a differential A-algebra with $H_0(A) = A/\mathfrak{A}$ and let $u \in I_p \subset A_{p,0}$, p-even, $h(u) = h$. Further, let*

$$B = A\langle T, dT = u\rangle\langle S, dS = u^{h-1}T\rangle$$

(note that $u^{h-1}T \in Z_1(A\langle T, dT = u\rangle)$. Then we have

1° if $H_1(B) = H_2(B) = 0$, then $\mathfrak{A}\colon (u) = (\mathfrak{A}, u^{h-1})$ and $H_1(A) = H_2(A) = 0$,

2° if $H_1(A) = 0$ for $i > 0$ and $\mathfrak{A}\colon (u) = (\mathfrak{A}, u^{h-1})$, then $H_i(B) = 0$ for $i > 0$ and $H_0(B) = A/(\mathfrak{A}, u)$.

Proof. Let $H_1(B) = H_2(B) = 0$. It is obvious that

$$B = A'\langle S, dS = u^{h-1}T\rangle$$

with $\partial(S) = (h-1)p$, $w(S) = 2$ and $A' = A\langle T, dT = u\rangle$ with $\partial(T) = p$ $w(T) = 1$. Since $\partial(u^{h-1}T) + w(u^{h-1}T) = (h-1)p+1$ is odd, $B = A' \otimes \Gamma(AS)$ and we have the exact sequence

$$0 \to A' \xrightarrow{\sigma} B \xrightarrow{\tau} B \to 0$$

with $\sigma(a') = a' \otimes 1$ and $\tau(a \otimes S^{(k)}) = a \otimes S^{(k-1)}$. Hence we obtain the long homology sequence

$$(*) \qquad \dots H_k(A') \xrightarrow{\sigma_*} H_k(B) \xrightarrow{\tau_*} H_{k-2}(B) \xrightarrow{\Delta} H_{k-1}(A') \to \dots$$
$$\dots H_1(B) \xrightarrow{\Delta} H_2(A') \xrightarrow{\sigma_*} H_2(B) \xrightarrow{\tau_*} H_0(B) \xrightarrow{\Delta} H_1(A') \xrightarrow{\sigma_*} H_1(B) \to 0$$

(τ is of degree -2). It follows that $H_1(A') \overset{\Delta}{\underset{\sim}{\leftarrow}} H_0(B) = A/(\mathfrak{A}, u)$ and $H_2(A') = 0$. Moreover, it is easy to check that $\Delta(\bar{a}) = au^{h-1}T + B_1(A')$ for $\bar{a} = a + B_0(B) \in H_0(B)$. Now observe that the equality $A' = A \otimes \Lambda(AT)$ furnishes us with the exact sequence

$$0 \to A \xrightarrow{\alpha} A' \xrightarrow{\beta} A \to 0$$

$(\alpha(a) = a \otimes 1,\ \beta(a' \otimes T) = a')$ which induces the long homology sequence

$$(**) \qquad \dots H_k(A) \xrightarrow{\delta} H_k(A) \xrightarrow{\alpha_*} H_k(A') \xrightarrow{\beta_*} H_{k-1}(A) \to \dots H_2(A) \xrightarrow{\delta} H_2(A)$$
$$\to H_2(A') \to H_1(A) \xrightarrow{\delta} H_1(A) \xrightarrow{\alpha_*} H_1(A') \xrightarrow{\beta_*} H_0(A) \xrightarrow{\delta} H_0(A) \to \dots$$

Since δ is a multiplication by u and $H_2(A') = 0$, then $H_2(A) = uH_2(A)$. Therefore, $H_2(A) = 0$ by the Nakayama Lemma. It remains to prove that $\mathfrak{A}\colon (u) = (\mathfrak{A}, u^{h-1})$ and $H_1(A) = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & H_0(A) = A/(\mathfrak{A}, u) \arrow[r, "u^{h-1}"] \arrow[d, "\Delta"', "\simeq"] & A/\mathfrak{A} \arrow[r, "u"] \arrow[d, equal] & A/\mathfrak{A} \arrow[d, equal] \\
0 \arrow[r] & H_1(A) \arrow[r, "u"] & H_1(A) \arrow[r, "\alpha_*"] & H_1(A') \arrow[r, "\beta_*"] & H_0(A) \arrow[r, "u"] & H_0(A)
\end{tikzcd}$$

The low sequence in this diagram is exact, as a part of the exact sequence $(**)$, and hence the upper sequence is exact. It follows that

$\mathfrak{A}$: $(u) = (\mathfrak{A}, u^{h-1})$. Moreover, if $su^{h-1} \in \mathfrak{A}$, i.e. $s \in (0)$: $(\bar{u}^{h-1})$ in $A/\mathfrak{A}$, then by Lemma 1.2 $s \equiv ru(\mathrm{mod}\,\mathfrak{A})$. Therefore $s \in (\mathfrak{A}, u)$ and the map u^{h-1}: $A/(\mathfrak{A}, u) \to A/\mathfrak{A}$ is a monomorphism. As a consequence we have: $0 = \operatorname{Ker}\beta_* = \operatorname{Im}\alpha_*$ which implies that $u\colon H_1(A) \to H_1(A)$ is an epimorphism. Hence $H_1(A) = uH_1(A)$ and again by the Nakayama Lemma $H_1(A) = 0$. Thus 1° is proved.

To prove 2° assume that $\mathfrak{A}$: $(u) = (\mathfrak{A}, u^{h-1})$ and $H_i(A) = 0$ for $i > 0$. Clearly $H_0(B) = A/(\mathfrak{A}, u)$. So we have still to show that $H_i(B) = 0$ for $i > 0$. In virtue of the exactness of the sequence $(**)$, $H_i(A') = 0$ for $i > 1$ and $H_1(A') \simeq \operatorname{Ker}\delta = A\cdot\bar{u}^{h-1}$ where $\bar{u} = u + \mathfrak{A}$. Hence, making use of the exactness of the sequence $(*)$, we get $H_k(B) = H_{k-2}(B)$ for $k > 2$, $H_1(B) \simeq \operatorname{Coker}\Delta$ and $H_2(B) \simeq \operatorname{Ker}\Delta$, where $\Delta\colon A/(\mathfrak{A}, u) = H_0(B) \to H_1(A') = A\cdot\bar{u}^{h-1} \subset A/\mathfrak{A}$ is a multiplication by u^{h-1}. However, Δ is an isomorphism since, by the assumption, $\mathfrak{A}$: $(u) = (\mathfrak{A}, u^{h-1})$. Consequently $H_1(B) = H_2(B) = 0$ and the required equality $H_i(B) = 0$ for $i > 0$ follows from the above-mentioned formula $H_k(B) = H_{k-2}(B)$, $k > 2$. This completes the proof of the lemma.

Now we are in a position to prove the main result of the section. As before, let A be a graded R-algebra and let $u_1, \ldots, u_p, \ldots, u_n$ be a p-ordered sequence of homogeneous elements of the ideal I with $h(u_i) = h_i$. Moreover, let $Y_k = A\langle T_1, \ldots, T_k, dT_i = u_i\rangle$.

2.6. Proposition. *The following conditions are equivalent:*

(1) *$u_1, \ldots, u_n$ is an h-regular sequence in A.*

(2) *$B = Y_n\langle S_{p+1}, \ldots, S_n, dS_i = u_i^{h_i-1}T_i\rangle$ is a free resolution of the graded A-module $A/(u_1, \ldots, u_n)$,*

(3) *$H_1(B) = H_2(B) = 0$.*

Proof. For the proof of implication (1)$\Rightarrow$(2) we apply induction on $s = n-p$. If $s = 0$, then the proposition follows from [1], Proposition 5.1. Let $s > 0$ and let the proposition hold for all h-regular sequences with $n-p < s$. It is obvious that $B = B'\langle T_n, dT_n = u_n\rangle\langle S_n, dS_n = u_n^{h_n-1}T_n\rangle$, $\partial(T_n) = \partial(u_n)$, $w(T_n) = 1$, $\partial(S_n) = h_n\cdot\partial(u_n)$, $w(S_n) = 2$, where $B' = Y_{n-1}\langle S_{p+1}, \ldots, S_{n-1}, dS_j = u_j^{h_j-1}\cdot T_j\rangle$. Using the induction assumption, we have

$$H_0(B') = A/(u_1, \ldots, u_{n-1}),$$
$$H_i(B') = 0, \quad i > 0.$$

Now implication (1)$\Rightarrow$(2) follows from Lemma 2.5 ($\partial(u_n)$ is even since $s > 0$).

Implication (2)$\Rightarrow$(3) is clear. To prove the last implication, (3)$\Rightarrow$(1), again induction on $s = n-p$ will be used. If $s = 0$, then $B = Y_n = A\langle T_1, \ldots, T_n, dT_i = u_i\rangle$ and by [1], Proposition 5.1, $u_1, \ldots, u_n$ is a regular (hence h-regular) sequence in A. Let $s > 0$. As above, $B = B'\langle T_n, dT_n = u_n\rangle\langle S_n, dS_n = u_n^{h_n-1}T_n\rangle$ and $\partial(T_n) = \partial(u_n)$ is even ($s > 0$). By the assumption, $H_1(B) = H_2(B) = 0$; thus, in virtue of Lemma 2.5, $H_1(B') = H_2(B') = 0$ and $(u_1, \ldots, u_{n-1})$: $(u_n) = (u_1, \ldots, u_{n-1}, u_n^{h_n-1})$. Moreover, applying the induction hypothesis to the h-regular sequence $u_1, \ldots, u_p, \ldots, u_{n-1}$ and the corresponding A-algebra B', we conclude that $u_1, \ldots, u_{n-1}$ is an h-regular sequence in A. This finishes the proof.

2.7. Remark. Proposition 2.6 is valid for graded algebras over any commutative (not necessarily local) ring R provided the set $u_1, \ldots, u_n$ under consideration is contained in the ideal $I' = \bigoplus_{i>0} A_i$.

2.8. Corollary. *If $u_1, \ldots, u_p, \ldots, u_n$ is an h-regular p-ordered sequence of homogeneous generators of the ideal I, then the Tate resolution X of A is equal to*

$$X = F_2X = A\langle T_1, \ldots, T_n, dT_i = u_i\rangle\langle S_{p+1}, \ldots, S_n, dS_j = u_j^{h_j-1}\cdot T_j\rangle.$$

Proof. This is a consequence of Propositions 2.4, 2.6 and the construction of X.

2.9. Corollary. *Suppose that R, R' are local rings and A, A' are graded algebras over R and R', respectively. Moreover, let $f\colon A \to A'$ be a homomorphism of graded rings and let $u_1, \ldots, u_p, \ldots, u_n$ be an h-regular p-ordered, homogeneous sequence in A such that $v_1, \ldots, v_n$, $v_i = f(u_i)$, is an h-regular sequence in A' with $h(v_i) = h(u_i)$. Then*

$$\operatorname{Tor}_i^A(A/(u_1, \ldots, u_k), A') = 0 \quad \textit{for } k = 0, \ldots, n \textit{ and } i > 0.$$

Proof. By Proposition 2.6,

$$B = A\langle T_1, \ldots, T_k, dT_i = u_i\rangle\langle S_{p+1}, \ldots, S_k, dS_j = u_j^{h_j-1}T_j\rangle$$

is a free resolution of the A-module $A/(u_1, \ldots, u_k)$ and

$$B' = A'\langle T_1, \ldots, T_k, dT_i = v_i\rangle\langle S_{p+1}, \ldots, S_k, dS_j = v_j^{h_j-1}T_j\rangle$$

is a free resolution of the A'-module $A'/(v_1, \ldots, v_k)$. Furthermore, it is easy to see that $B' = B \otimes_A A'$. Consequently

$$\operatorname{Tor}_i^A(A/(u_1, \ldots, u_k), A') = H_i(B \otimes_A A') = 0 \quad \text{for } i > 0.$$

2.10. Theorem. *Assume that A is a finitely generated, graded R-algebra. Then the following conditions are equivalent:*

(1) *R is the h-regular local ring and the ideal $I = m \oplus (\bigoplus_{i>0} A_i)$ is generated by an h-regular set of homogeneous elements.*

(2) *R is the h-regular local ring and $A \simeq \bigotimes_i R[X]/(X^{h_i})$ for some $h_i \in N^\infty$.*

(3) *There exists a p-ordered sequence of homogeneous generators of I $u_1, \ldots, u_p, \ldots, u_n$, $h(u_i) = h_i$, such that the Tate resolution X of A is equal to*

$$X = F_2X = A\langle T_1, \ldots, T_n, dT_i = u_i\rangle\langle S_{p+1}, \ldots, S_n, dS_j = u_j^{h_j-1}T_j\rangle$$

with $\partial(T_i) = \partial(u_i)$, $w(T_i) = 1$, $\partial(S_j) = h_j\partial(u_j)$, $w(S_j) = 2$.

(4) *There exists a p-ordered sequence $u_1, \ldots, u_p, \ldots, u_n$ of homogeneous generators of the ideal I, $h(u_i) = h_i$, such that*

$$H_1(B) = H_2(B) = 0\,,$$

where $B = A\langle T_1, \ldots, T_n, dT_i = u_i\rangle\langle S_{p+1}, \ldots, S_n, dS_j = u_j^{h_j-1}T_j\rangle$.

Proof. The equivalence of (1) and (2) is contained in Proposition 1.14. Implication (2) $\Rightarrow$ 3) follows from Corollary 2.8 and (3) $\Rightarrow$ (4) is obvious. Finally, implication (4) $\Rightarrow$ (3) holds in virtue of Proposition 2.6.

2.11. Remark. We do not know if the above theorem is true for graded R-algebras which are not finitely generated.

The following example shows that not every graded R-algebra A with $X = F_2X$ is h-regular, i.e. satisfies one of the equivalent conditions from Theorem 2.10.

2.12. Example. Let k be a field and let $A = k[X, Y]/(XY)$, where $\partial(X) = \partial(Y) = 2$. Since XY is a non-zero divisor in $k[X, Y]$, we have $X = F_2X$ by [2], IV, § 2 Theorem 1. At the same time one can easily prove that A is not an h-regular graded k-algebra.

References

[1] T. Józefiak, *Tate resolution for commutative graded algebras over a local ring*, Fund. Math. 74 (1972), pp. 209–231.

[2] — *Thesis*, Toruń 1969.

[3] — *A homological characterization of graded complete intersections*, to appear.

[4] A. Tyc, *On h-regular local Noetherian rings*, to appear.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 8. 1. 1973

Absolute retracts as factors of normed linear spaces

by

H. Toruńczyk (Warszawa)

Abstract. It is shown that if X is a (complete) $\mathrm{AR}(\mathfrak{M})$-space, then, for a suitable (complete) normed linear space E, $X\times E$ and E are homeomorphic. This implies that l_2-manifolds are characterized as separable, complete, l_2-stable $\mathrm{ANR}(\mathfrak{M})$'s.

In this paper we deal with products of absolute retracts and normed linear spaces. We shall show that any absolute retract is a factor of a normed linear space, i.e. if $X \in \mathrm{AR}(\mathfrak{M})$, then there is a normed space E such that $X\times E$ is homeomorphic to E. We also show that normed spaces E of a more special type (e.g. all infinite-dimensional Hilbert spaces) have the property that each retract of E is also a factor of E.

The paper is a sequel to [28] and we shall use some terms and notation of [28]. In particular, we shall say that a retraction r of a metric space (Y, d) is *regular*, if r is continuous and

(*) for every $\varepsilon > 0$ the set $\{y \in Y\colon d(r(y), y) \geqslant \varepsilon\}$ is of positive d-distance from $r(Y)$.

The main result of [28] was:

Theorem 0. *Let r be a regular retraction of a normed linear space $(E, \|\ \|)$ and let $X = r(E)$. Then $X\times\sum_{l_1}E \cong \sum_{l_1}E$, and if, moreover, X is complete in the norm $\|\ \|$, then also $X\times\prod_{l_1}E \cong \prod_{l_1}E$. Here, $\prod_{l_1}E = \{(t_i)\in E^\infty\colon \sum_{i\geqslant 1}\|t_i\| < \infty\}$ and $\sum_{l_1}E = \{(t_i)\in E^\infty\colon t_i = 0$ for almost all $i\}$, both spaces being equipped with the norm $|||(t_i)||| = \sum_{i\geqslant 1}\|t_i\|$.*

To apply this theorem, we examine here regular retractions more accurately. Unexpectedly enough, it appears (see Section 2) that on any absolute retract X there is an admissible metric ϱ with the property that every closed isometric embedding of (X, ϱ) into a metric space Y maps X