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Boolean-valued selectors for families of sets *

by

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Abstract. Let $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ be a family of sets. We say that $\mathfrak{X}$ has a selector if there is a set S such that $|S\cap X_\alpha = 1$ for every $\alpha<\varkappa$. $\mathfrak{X}$ has partial selectors if for every $\beta<\varkappa$ the family $\mathfrak{X}\upharpoonright\beta = \langle X_\alpha\rangle_{\alpha<\beta}$ has a selector. Let $\mathbf{E}(\varkappa,\varkappa)$ denotes the following statement: *For every family $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ of sets of powers $<\lambda$, if $\mathfrak{X}$ has partial selectors then $\mathfrak{X}$ has a selector.* In this paper we prove a theorem on the invariance of $\mathbf{E}(\varkappa,\varkappa)$ under some generic extensions, namely: *Let $|\mathfrak{B}| = \lambda$, $\mathfrak{B}$ satisfy σ-cc, and $\lambda^{\underline{\sigma}} < \varkappa$. Moreover, suppose that for each* ZF*-formula Φ with parameters from $\check{V}$ we have $\|\Phi\|\in\{0,1\}$. Then $\mathbf{E}(\varkappa,\varkappa)$ implies $\|\mathbf{E}(\check{\varkappa},\check{\varkappa})\| = 1$ in $V^{(\mathfrak{B})}$.*

This paper is a continuation of [3]. For the readers' convenience we repeat the main notions and results of [3].

If $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ is a family of sets then $\mathfrak{X}$ has a *selector* if there is a set S such that $|S\cap X_\alpha| = 1$ for every $\alpha<\varkappa$. We say that $\mathfrak{X}$ has *partial selectors* if for every $\beta<\varkappa$ the family $\mathfrak{X}\upharpoonright\beta = \langle X_\alpha\rangle_{\alpha<\beta}$ has a selector. In [3] the following statement, denoted by $\mathbf{E}(\varkappa,\lambda)$, has been studied: "For every family $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ of sets of powers $<\lambda$ if $\mathfrak{X}$ has partial selectors then $\mathfrak{X}$ has a selector".

The main results of [3] can be presented as follows:

THEOREM. (a) *$\mathbf{E}(\varkappa,\varkappa)$ implies that $\varkappa$ is regular.*
(b) *If $\varkappa$ is weakly compact then $\mathbf{E}(\varkappa,\varkappa)$ holds.*
(c) *$\mathbf{E}(\varkappa,\varkappa)$ implies that $\varkappa$ has the tree property.*
(d) [GCH]. *$\mathbf{E}(\varkappa,\varkappa)$ if and only if $\varkappa$ is weakly compact.*

In this paper we give a theorem about the invariance of the property $\mathbf{E}(\varkappa,\varkappa)$ under some generic extensions. We shall work in the Boolean version of forcing; thus for the readers' convenience we recall the main notions and notations concerning the Boolean-valued universe $V^{(\mathfrak{B})}$. For more information see e.g. [2].

Let $\mathfrak{B}$ be a complete Boolean algebra. We say that $\mathfrak{B}$ *satisfies σ-cc* (*σ-chain condition*) if every family of non-zero disjoint elements of $\mathfrak{B}$ has

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the cardinality smaller than σ. The Boolean-valued universe $V^{(\mathcal{B})}$ is defined by induction as follows:

$$V^{(\mathcal{B})} = \{u \in V\colon \mathrm{Function}(u) \wedge \mathrm{dom}(u) \subseteq V^{(\mathcal{B})} \wedge \mathrm{rng}(u) \subseteq \mathcal{B}\},$$

where V denotes the standard universe, i.e. the class of all sets. The logical values of $\|\cdot \in \cdot\|$ and $\|\cdot = \cdot\|$ in $V^{(\mathcal{B})}$ are defined by induction:

(1) $\|u \in v\| = \sum_{x \in \mathrm{dom}(v)} v(x) \cdot \|u = x\|$, and

(2) $\|u = v\| = \prod_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow \|x \in v\|\big) \cdot \prod_{x \in \mathrm{dom}(v)} \big(v(x) \Rightarrow \|x \in u\|\big)$.

We define the Boolean value of any **ZF**-formula Φ with parameters in $V^{(\mathcal{B})}$ by induction:

(I) For atomic formulas as in (1) and (2).

(II) If $\Phi = \neg\Psi$, then $\|\Phi\| = -\|\Psi\|$.

(III) If $\Phi = \Psi_1 \wedge \Psi_2$, then $\|\Phi\| = \|\Psi_1\| + \|\Psi_2\|$.

(IV) If $\Phi = \exists x \Psi$, then $\|\Phi\| = \sum_{x \in V^{(\mathcal{B})}} \|\Psi(x)\|$.

We also define the natural embedding of V into $V^{(\mathcal{B})}$ by induction:

$$\check{x} = \{\langle \check{y}, 1\rangle\colon y \in x\}.$$

Let us denote by $\check{V}$ the image of V by this embedding.

Now, our problem can be formulated as follows: Suppose $\mathbf{E}(\varkappa, \varkappa)$ holds and $\mathcal{B}$ is a complete Boolean algebra. What should we assume about the cardinality of $\mathcal{B}$ to have $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$ in $V^{(\mathcal{B})}$?

The assumptions on the algebra $\mathcal{B}$ presented in our Theorem (in § 3) are rather strong and only under the assumption of GCH seem to be completely natural. (Then this Theorem gives the invariance of $\mathbf{E}(\varkappa, \varkappa)$ under "mild" extensions). But in this case our Theorem is a consequence of the well-known result about the invariance of weak compactness under "mild" extensions, see e.g. [1]. Thus the difficulty of the proof that $\mathbf{E}(\varkappa, \varkappa)$ is invariant under some extensions suggests that $\mathbf{E}(\varkappa, \varkappa)$ is different from the weak compactness of $\varkappa$.

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§ 1. A Boolean counterpart for $\mathbf{E}(\varkappa, \varkappa)$. In this section we define a condition $\mathbf{E}(\mathcal{B}, \varkappa)$ which, under some natural assumptions on $\mathcal{B}$, implies that $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$ in $V^{(\mathcal{B})}$. For convenience we introduce the following abbreviations:

[A] $\mathcal{B}$ satisfies $\varkappa$-cc.

[B] $\|\check{\varkappa}$ is a cardinal$\| = 1$.

[C] $\|\check{\varkappa}$ is regular$\| = 1$.

[D] For each **ZF**-formula Φ with constants from $\check{V}$ we have either $\|\Phi\| = 0$ or $\|\Phi\| = 1$.

Let us remark that [A] always implies [B], and for regular $\varkappa$ also [A] implies [C]. Assumption [D] will be used to simplify calculations in $V^{(\mathcal{B})}$.

DEFINITION 1.1. A function $F\colon \varkappa \times \varkappa \to \mathcal{B}$ is called a *$\varkappa$-covering of $\mathcal{B}$* if F satisfies the following two conditions:

(.1) $(\forall_{\alpha<\varkappa})\big(\sum_{\xi<\varkappa} F(\alpha, \xi) = 1\big)$.

(.2) $(\forall_{\alpha<\varkappa})(\exists_{\beta<\varkappa})(\forall_{\gamma<\varkappa})\big(\gamma \geqslant \beta \to F(\alpha, \gamma) = 0\big)$.

LEMMA 1.2. *Let $F\colon \varkappa \times \varkappa \to \mathcal{B}$ be a $\varkappa$-covering of $\mathcal{B}$. Then there exists an element $\mathfrak{X}_F = \langle X_\alpha\rangle_{\alpha<\check{\varkappa}}$ of $V^{(\mathcal{B})}$ such that:*

(i) *$\|\mathfrak{X}_F$ is a sequence of length $\check{\varkappa}$ of elements of $P(\check{\varkappa})\| = 1$,*

(ii) *$\|(\forall \alpha < \check{\varkappa})(X_\alpha \neq 0)\| = 1$,*

(iii) *if [B] then $\|(\forall \alpha < \check{\varkappa})(|X_\alpha| < \check{\varkappa})\| = 1$.*

Proof. Let $F\colon \varkappa \times \varkappa \to \mathcal{B}$ be a $\varkappa$-covering of $\mathcal{B}$. For each $\alpha < \varkappa$, we define $f_\alpha\colon \varkappa \to \mathcal{B}$, putting $f_\alpha(\xi) = F(\alpha, \xi)$ for $\xi < \varkappa$. Let $X_{\check{\alpha}} \in V^{(\mathcal{B})}$ be defined by the conditions: $\mathrm{dom}(X_{\check{\alpha}}) = \check{\varkappa}$ and $X_{\check{\alpha}}(\check{\xi}) = f_\alpha(\xi)$, for $\xi < \varkappa$. Then $\|X_{\check{\alpha}} \subseteq \check{\varkappa}\| = 1$. Let $Y_{\check{\alpha}} \in V^{(\mathcal{B})}$ be such that $\|Y_{\check{\alpha}} = \langle \check{\alpha}, X_{\check{\alpha}}\rangle\| = 1$. Then we define $\mathfrak{X}_F \in V^{(\mathcal{B})}$ by the conditions: $\mathrm{dom}(\mathfrak{X}_F) = \{Y_{\check{\alpha}}\colon \alpha < \varkappa\}$ and $\mathfrak{X}_F(Y_{\check{\alpha}}) = 1$ for all $\alpha < \varkappa$. Then of course $\mathfrak{X}_F$ satisfies (i). (ii) follows from (1.1.1). By (1.1.2), for each $\alpha < \varkappa$ there is a $\beta < \varkappa$ such that $\|X_{\check{\alpha}} \subseteq \check{\beta}\| = 1$. Thus (iii) follows from assumption [B].

LEMMA 1.3. [A] & [C]. *Each element $\mathfrak{X} \in V^{(\mathcal{B})}$ which fulfils conditions* (i), (ii) *and* (iii) *of* 1.2, *determines some $\varkappa$-covering $F_{\mathfrak{X}}$ of $\mathcal{B}$. Moreover, if $F_{\mathfrak{X}} = F_{\mathfrak{Y}}$ then $\|\mathfrak{X} = \mathfrak{Y}\| = 1$.*

Proof. Since $\|\mathfrak{X} \in {}^{\check{\varkappa}}P(\check{\varkappa})\| = 1$, $\mathfrak{X}$ determines a function $F_{\mathfrak{X}}\colon \varkappa \times \varkappa \to \mathcal{B}$ defined by $F_{\mathfrak{X}}(\alpha, \beta) = \|\check{\beta} \in X_{\check{\alpha}}\|$. Now, since for each $\alpha < \varkappa$, $\|X_{\check{\alpha}} \neq 0\| = 1$, (1.1.1) holds. By [C] and (iii), for each $\alpha < \varkappa$, we have $\|(\exists \beta < \check{\varkappa})(X_{\check{\alpha}} \subseteq \beta)\| = 1$. Hence, by [A], there is a $\beta < \varkappa$ such that $\|X_{\check{\alpha}} \subseteq \check{\beta}\| = 1$. Consequently (1.1.2) holds.

Now, suppose that $\|\mathfrak{X}, \mathfrak{Y} \in {}^{\check{\varkappa}}P(\check{\varkappa})\| = 1$ and $F_{\mathfrak{X}} = F_{\mathfrak{Y}}$. Then

$$\|\mathfrak{X} = \mathfrak{Y}\| = \|(\forall \alpha < \check{\varkappa})(X_\alpha = Y_\alpha)\| = \prod_{\alpha<\varkappa} \|X_{\check{\alpha}} = Y_{\check{\alpha}}\|.$$

But $\|X_{\check{\alpha}}, Y_{\check{\alpha}} \subseteq \check{\varkappa}\| = 1$ for each $\alpha < \varkappa$; consequently we have

$$\|X_{\check{\alpha}} = Y_{\check{\alpha}}\| = \big\|(\forall \beta < \check{\varkappa})\big((\beta \in X_{\check{\alpha}}) \leftrightarrow (\beta \in Y_{\check{\alpha}})\big)\big\| = \prod_{\beta<\varkappa} \big(F_{\mathfrak{X}}(\alpha, \beta) \Leftrightarrow F_{\mathfrak{Y}}(\alpha, \beta)\big) = 1.$$

Thus $\|\mathfrak{X} = \mathfrak{Y}\| = 1$.

COROLLARY. [A], [B] & [C]. $F_{\mathfrak{X}_F} = F$ *and* $\|\mathfrak{X}_{F_{\mathfrak{X}}} = \mathfrak{X}\| = 1$.

Proof. By Lemmas 1.2 and 1.3.

DEFINITION 1.4. Let $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ be a family of subsets of $\varkappa$. A function $S\colon \varkappa\to\varkappa$ is a *selector of* $\mathfrak{X}$ *of length* δ if the following two conditions are satisfied:

(.1δ) $(\forall\xi<\delta)\big(S(\xi)\in X_\xi\big)$.

(.2δ) $(\forall\xi,\eta<\delta)\big(S(\xi)\in X_\eta\to S(\xi)=S(\eta)\big)$.

A selector of length $\varkappa$ will be called a *simply selector of* $\mathfrak{X}$.

LEMMA 1.5. *Let F and G be $\varkappa$-coverings of $\mathcal{B}$, and suppose that G satisfies the following condition:*

$$(*)\qquad (\forall\alpha<\varkappa)(\forall\beta_1,\beta_2<\varkappa)(\beta_1\neq\beta_2\to G(\alpha,\beta_1)G(\alpha,\beta_2)=0)\,.$$

If F and G satisfy the following two conditions:

(.1δ) $(\forall\alpha,\beta<\delta)(\forall\xi<\varkappa)\big(F(\alpha,\xi)G(\beta,\xi)=F(\beta,\xi)G(\alpha,\xi)\big)$,

(.2δ) $(\forall\alpha<\delta)(\forall\xi<\varkappa)\big(G(\alpha,\xi)\leqslant F(\alpha,\xi)\big)$,

then G determines an element $S\in V^{(\mathcal{B})}$ such that

(.3δ) $\|S$ *is a selector of* $\mathfrak{X}_F$ *of length* $\check{\delta}\|=1$.

Proof. Since G is a $\varkappa$-covering of $\mathcal{B}$, by Lemma 1.2, G determines in $V^{(\mathcal{B})}$ an element $\mathcal{Y}_G=\langle Y_\alpha\rangle_{\alpha<\check{\varkappa}}$ such that $\|\mathcal{Y}_G$ is a sequence of length $\check{\varkappa}$ of non-void subsets of $\check{\varkappa}\|=1$. Using $(*)$ we can check that $\|(\forall\alpha<\check{\varkappa})$ $(Y_\alpha$ is a one-element set$)\|=1$. Thus, we can define $S\in V^{(\mathcal{B})}$ in such a way that $\|S\colon\check{\varkappa}\to\check{\varkappa}\|=1$ and $\|(\forall\alpha<\check{\varkappa})(S_\alpha\in Y_\alpha)\|=1$. Then $G(\alpha,\xi)=\|S_{\check{\alpha}}=\check{\xi}\|$ for all $\alpha,\xi<\varkappa$. Consequently, by (1.5.2δ), we have $\|(1.4.1\delta)\|=1$.

To prove (.3δ), it suffices to show that $\|(1.4.2\delta)\|=1$. Let us remark that by (1.5.1δ), for all $\alpha,\beta<\delta$ and for each $\xi<\varkappa$, we have

$$\big(G(\alpha,\xi)\cdot F(\beta,\xi)\Leftrightarrow G(\beta,\xi)\cdot F(\alpha,\xi)\big)=1\,.$$

Since $G(\beta,\xi)\leqslant F(\beta,\xi)$, we obtain

$$\big(F(\alpha,\xi)\cdot G(\beta,\xi)\Rightarrow G(\alpha,\xi)\big)=1\,.$$

Thus in $V^{(\mathcal{B})}$: $(\|\check{\xi}\in X_{\check{\alpha}}\|\cdot\|\check{\xi}=S_{\check{\beta}}\|\Rightarrow\|\check{\xi}=S_{\check{\alpha}}\|)=1$. Hence $\|\check{\xi}\in X_{\check{\alpha}}\wedge\check{\xi}=S_{\check{\beta}}\to\check{\xi}=S_{\check{\alpha}}\|=1$ and consequently $\|S_{\check{\beta}}\in X_{\check{\alpha}}\to S_{\check{\beta}}=S_{\check{\alpha}}\|=1$. But this gives $\|(1.4.2\delta)\|=1$ and finishes the proof of (.3δ).

LEMMA 1.6. [A]. *Let F be a $\varkappa$-covering of $\mathcal{B}$. Let $S\in V^{(\mathcal{B})}$ satisfy* (1.5.3δ). *Then the function $G_S\colon\varkappa\times\varkappa\to\mathcal{B}$ defined by $G_S(\alpha,\beta)=\|S_{\check{\alpha}}=\check{\beta}\|$ is a $\varkappa$-covering of $\mathcal{B}$ and conditions* $(*)$, (1.5.1δ) *and* (1.5.2δ) *are satisfied. Moreover, if $G_{S_1}=G_{S_2}$ then $\|S_1=S_2\|=1$.*

Proof. Let $\mathfrak{X}_F\in V^{(\mathcal{B})}$ be a sequence in $V^{(\mathcal{B})}$ determined by F and let $G_S(\alpha,\beta)=\|S_{\check{\alpha}}=\check{\beta}\|$. Then of course $G_S\colon\varkappa\times\varkappa\to\mathcal{B}$ satisfies (1.1.1). Since $\|S\colon\check{\varkappa}\to\check{\varkappa}\|=1$, for all $\xi,\eta<\varkappa$, we have $\|\check{\xi}=S_{\check{\alpha}}\|\cdot\|\check{\eta}=S_{\check{\alpha}}\|\leqslant\|\check{\xi}=\check{\eta}\|$ and $(*)$ is satisfied. Now, using $(*)$ and [A], we see that (1.1.2) also holds. Consequently G_S is a $\varkappa$-covering of $\mathcal{B}$ satisfying condition $(*)$.

Proof of (1.5.2δ). By (1.5.3δ) we have $\|(\forall\alpha<\check{\delta})(S_\alpha\in X_\alpha)\|=1$; thus $\prod_{\alpha<\delta}\|S_{\check{\alpha}}\in X_{\check{\alpha}}\|=1$ and therefore, for each $\alpha<\delta$, we have $\|S_{\check{\alpha}}\in X_{\check{\alpha}}\|=1$. Consequently $\sum_{\xi<\varkappa}F(\alpha,\xi)\cdot G_S(\alpha,\xi)=1$. But, for $\xi\neq\xi'$, $G_S(\alpha,\xi)\cdot G_S(\alpha,\xi')=0$, since G_S satisfies $(*)$. Thus $F(\alpha,\xi)\cdot G_S(\alpha,\xi)=G_S(\alpha,\xi)$ and consequently $G_S(\alpha,\xi)\leqslant F(\alpha,\xi)$ for each $\alpha<\delta$, which proves (1.5.2δ).

Proof of (1.5.1δ). Since $\|S\colon\check{\varkappa}\to\check{\varkappa}\|=1$, for each $\alpha<\varkappa$ we have $\|(\exists\eta<\check{\varkappa})(S_{\check{\alpha}}=\eta)\|=1$. By (1.5.3$\delta$) we have

$$\|S_{\check{\beta}}\in X_{\check{\alpha}}\to S_{\check{\alpha}}=S_{\check{\beta}}\|=1\,.$$

Thus for each $\beta<\delta$: $\|S_{\check{\beta}}\in X_{\check{\alpha}}\|\leqslant\|S_{\check{\alpha}}=S_{\check{\beta}}\|$. Hence

$$\|(\exists\eta<\check{\varkappa})(S_{\check{\beta}}=\eta\wedge\eta\in X_{\check{\alpha}})\|\leqslant\|(\exists\eta<\check{\varkappa})(S_{\check{\beta}}=\eta\wedge S_{\check{\alpha}}=\eta)\|$$

and consequently

$$\sum_{\eta<\varkappa}G_S(\beta,\eta)\cdot F(\alpha,\eta)\leqslant\sum_{\eta<\varkappa}G_S(\beta,\eta)\cdot G_S(\alpha,\eta)\,.$$

Thus

$$G_S(\beta,\eta)\cdot F(\alpha,\eta)\leqslant\sum_{\eta<\varkappa}G_S(\beta,\eta)\cdot G_S(\alpha,\eta)$$

for each $\eta<\varkappa$. But using $(*)$, we obtain

$$G_S(\beta,\eta)\cdot F(\alpha,\eta)\leqslant G_S(\beta,\eta)\cdot G_S(\alpha,\eta)\,.$$

Finally, $G_S(\beta,\eta)\leqslant F(\beta,\eta)$ by (1.5.2δ) and consequently

$$G_S(\beta,\eta)\cdot F(\alpha,\eta)\leqslant F(\beta,\eta)\cdot G_S(\alpha,\eta)\,.$$

By symmetry we get (1.5.1δ).

Finally suppose that S_1, S_2 satisfy all the assumptions of Lemma 1.6 and let $G_{S_1}=G_{S_2}$. Then

$$\|S_1=S_2\|=\|(\forall\alpha<\check{\varkappa})(S_{1\alpha}=S_{2\alpha})\|=\prod_{\alpha<\varkappa}\|S_{1\alpha}=S_{2\alpha}\|\,.$$

Because of $\|S_1,S_2\colon\check{\varkappa}\to\check{\varkappa}\|=1$, we have

$$\|S_{1\alpha}=S_{2\alpha}\|=\|(\forall\eta<\check{\varkappa})(S_{1\alpha}=\eta\leftrightarrow S_{2\alpha}=\eta)\|$$
$$=\prod_{\eta<\varkappa}\big(G_{S_1}(\alpha,\eta)\Leftrightarrow G_{S_2}(\alpha,\eta)\big)=1\,.$$

Thus $\|S_1=S_2\|=1$.

DEFINITION 1.7. A $\varkappa$-covering G of $\mathcal{B}$ fulfilling $(*)$ is said to be a δ-*refinement* of a given $\varkappa$-covering F of $\mathcal{B}$ if F and G satisfy conditions (1.5.1δ) and (1.5.2δ).

DEFINITION 1.8. $\mathbf{E}(\mathcal{B}, \varkappa)$ denotes the following statement: "Each $\varkappa$-covering of $\mathcal{B}$ which has a δ-refinement for all $\delta < \varkappa$, has a $\varkappa$-refinement".

THEOREM 1.9. (.1) [A]. *If* $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$, *then* $\mathbf{E}(\mathcal{B}, \varkappa)$ *holds.*

(.2) [A] & [D]. *If* $\mathbf{E}(\mathcal{B}, \varkappa)$ *holds, then* $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$.

Proof of (.1). Since $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$, by the Theorem (clause (a)) in the introduction assumption [C] holds and consequently [B] also holds.

Suppose that F is a $\varkappa$-covering of $\mathcal{B}$ having for each $\delta < \varkappa$ some δ-refinement. Then, by Lemma 1.2, F determines $\mathfrak{X}_F \in V^{(\mathcal{B})}$ satisfying conditions (i), (ii), and (iii) from Lemma 1.2. Moreover, by Lemma 1.5, for each $\delta < \varkappa$, $\|\mathfrak{X}_F$ has a selector of length $\check{\delta}\| = 1$. Thus, since $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$, we conclude that $\|\mathfrak{X}_F$ has a selector of length $\check{\varkappa}\| = 1$. Thus, by Lemma 1.6, we see that $\mathbf{E}(\mathcal{B}, \varkappa)$ holds.

Proof of (.2). Since $\mathbf{E}(\varkappa, \varkappa)$ is a **ZF**-formula with constants from V, by [D], $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 0$ or 1. We shall exclude the case $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 0$.

Suppose, on the contrary, that $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 0$. Then there exists an $\mathfrak{X} \in V^{(\mathcal{B})}$ such that:

(a) $\|\mathfrak{X} \in {}^{\check{\varkappa}}P(\check{\varkappa})\| = 1$,

(b) $\|(\forall a < \check{\varkappa})(|X_a| < \check{\varkappa})\| = 1$,

(c) $\|(\forall \delta < \check{\varkappa})$ ($\mathfrak{X}$ has a selector of length δ)$\| = 1$,

(d) $\|\mathfrak{X}$ has a selector of length $\check{\varkappa}\| = 0$.

Thus, by Lemmas 1.3, and 1.6, we get a $\varkappa$-covering of $\mathcal{B}$ which has for each $\delta < \varkappa$ a δ-refinement. By $\mathbf{E}(\mathcal{B}, \varkappa)$ this $\varkappa$-covering has a $\varkappa$-refinement. Thus, by Lemma 1.5, we get

(e) $\|X$ has a selector of length $\check{\varkappa}\| = 1$, which contradicts (d). Consequently $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$. Q.E.D.

§ 2. Main Lemma. In this section we shall show that under some assumptions concerning the cardinality of $\mathcal{B}$, $\mathbf{E}(\varkappa, \varkappa)$ implies $\mathbf{E}(\mathcal{B}, \varkappa)$.

For this purpose let us introduce some notations: Let F be a given $\varkappa$-covering of $\mathcal{B}$. For $\alpha, \beta, \gamma < \varkappa$ let

$$\varphi(\alpha, \beta) = \min\{\zeta < \varkappa\colon (\forall \xi < \varkappa)\big(\xi \geqslant \zeta \rightarrow F(\alpha, \xi) + F(\beta, \xi) = 0\big)\},$$

and

$$\varphi(\alpha, \beta, \gamma) = \max\big(\varphi(\alpha, \beta), \varphi(\alpha, \gamma)\big).$$

Let us remark that, by (1.1.2), $\varphi(\alpha, \beta) < \varkappa$ is well defined for all $\alpha, \beta < \varkappa$.

Let $\mathcal{R}$ be the set of all partitions of the unity in $\mathcal{B}$, i.e., $r \in \mathcal{R}$ if and only if (i) $r \subseteq \mathcal{B}$, (ii) $0 \notin r$, (iii) $x, y \in r$ and $x \neq y$ implies $x \cdot y = 0$, and (iv) $\Sigma r = 1$

Let $r \in \mathcal{R}$. We call r *acceptable* for α if for each $x \in r$ there is a $\xi < \varkappa$ such that $x \leqslant F(\alpha, \xi)$. Let $\mathcal{R}_\alpha$ be the set of all partitions acceptable for α, let $\mathcal{R}^\alpha = \{\alpha\} \times \mathcal{R}_\alpha$ and $\mathcal{R}^{\alpha\beta} = \mathcal{R}^\alpha \times \mathcal{R}^\beta$.

For $\alpha_0, \alpha_1, \xi < \varkappa$, $r_0, r_1 \in \mathcal{R}$, $x_0, x_1 \in \mathcal{B}$, the symbol $(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1, \xi)$ denotes the following 5-tuple:

$$\langle\{\alpha_0, \alpha_1\}, \{\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\}, \{\langle\alpha_0, x_0\rangle, \langle\alpha_1, x_1\rangle\}, x_0 \cdot F(\alpha_1, \xi), \xi\rangle.$$

For $\alpha_0 \neq \alpha_1$, $r_i \in \mathcal{R}_{\alpha_i}$, $i = 0, 1$ we define the following sets:

$$R^{\alpha_0\alpha_1r_0r_1} = \mathcal{R}^{\alpha_0\alpha_1} - \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\},$$

$$A_\xi^{0\alpha_0\alpha_1r_0r_1} = \{(\alpha_0, r_0, y_0, \alpha_1, r_1, y_1, \xi)\colon y_i \in r_i \text{ and } y_i \leqslant F(\alpha_i, \xi) \text{ for } i = 0, 1\},$$

$$A_\xi^{\alpha_0\alpha_1r_0r_1} = R^{\alpha_0\alpha_1r_0r_1} \cup A_\xi^{0\alpha_0\alpha_1r_0r_1}.$$

PROPOSITION 2.1. *For* $\alpha_0 \neq \alpha_1$ *and* $\gamma < \varkappa$, *let* $\mathcal{A}_\gamma^{0\alpha_0\alpha_1} = \{\mathcal{R}^{\alpha_0\alpha_1}\} \cup \{A_\xi^{\alpha_0\alpha_1r_0r_1}\colon r_i \in \mathcal{R}_{\alpha_i}$ *for* $i = 0, 1$ *and* $\xi < \gamma\}$. *Then* $S \subseteq \bigcup \mathcal{A}_\gamma^{0\alpha_0\alpha_1}$ *is a selector of* $\mathcal{A}_\gamma^{0\alpha_0\alpha_1}$ *if and only if there are* $r_i \in \mathcal{R}_{\alpha_i}$ *and functions* g_i, $i = 0, 1$, *with the following properties:*

(2.1.1) $\operatorname{dom}(g_i) = \gamma$,

(2.1.2) $\operatorname{rng}(g_i) \subseteq r_i \in \mathcal{R}_{\alpha_i}$,

(2.1.3) *for each* $\xi < \gamma$, $g_i(\xi) \leqslant F(\alpha_i, \xi)$,

such that $S = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\} \cup \{(\alpha_0, r_0, g_0(\xi), \alpha_1, r_1, g_1(\xi), \xi)\colon \xi < \gamma\}$. (*we shall denote such a set* S *by* $S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$).

Proof. It is easy to see that each set of the form $S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$, where α_i, r_i, g_i, for $i = 0, 1$, satisfy (2.1.1), (2.1.2) and (2.1.3), is a selector of $\mathcal{A}_\gamma^{0\alpha_0\alpha_1}$.

Conversely, suppose that $S \subseteq \bigcup \mathcal{A}_\gamma^{0\alpha_0\alpha_1}$ is a selector of $\mathcal{A}_\gamma^{0\alpha_0\alpha_1}$. Then $|S \cap \mathcal{R}^{\alpha_0\alpha_1}| = 1$; thus S consists exactly of one pair $\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle$, where $r_i \in \mathcal{R}_{\alpha_i}$, for $i = 0, 1$. If $r_0 \neq s_0$ or $r_1 \neq s_1$, where $s_i \in R_{\alpha_i}$ for $i = 0, 1$, then $\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle \in R^{\alpha_0\alpha_1s_0s_1}$ and consequently $\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle \in A_\xi^{\alpha_0\alpha_1s_0s_1}$ for each $\xi < \gamma$. Since S is a selector, for $r_0 \neq s_0$ or $r_1 \neq s_1$ we have $S \cap A_\xi^{\alpha_0\alpha_1s_0s_1} = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$. Moreover $\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle \notin R^{\alpha_0\alpha_1r_0r_1}$, and thus $S \cap A_\xi^{\alpha_0\alpha_1r_0r_1} = S \cap A_\xi^{0\alpha_0\alpha_1r_0r_1}$. Since for each $\xi < \gamma$ we have $|S \cap A_\xi^{0\alpha_0\alpha_1r_0r_1}| = 1$, for each $\xi < \gamma$ the set S consists of exactly one 5-tuple of the form $(\alpha_0, r_0, y_0^\xi, \alpha_1, r_1, y_1^\xi, \xi)$, where $y_i^\xi \in r_i$ and $y_i^\xi \leqslant F(\alpha_i, \xi)$, for $i = 0, 1$.

Consequently, we can define functions g_0, g_1, by $g_i(\xi) = y_i^\xi$, for $\xi < \gamma$, $i = 0, 1$, in such a way that (2.1.1), (2.1.2) and (2.1.3) are satisfied and $S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$. This finishes the proof of Proposition 2.1.

Now, we shall define a new family of sets to ensure that g_0 from Proposition 2.1 is one-to-one. For this purpose let $Z_\alpha = \{\xi < \varkappa\colon F(\alpha, \xi) \neq 0\}$. Let us remark that, by (1.1.2), we have $|Z_\alpha| < \varkappa$ for each $\alpha < \varkappa$.

Let us define the following sets:

$$V_{x_0}^{\alpha_0\alpha_1r_0r_1} = R^{\alpha_0\alpha_1r_0r_1} \cup \{(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1, \xi)\colon \xi \in Z_{\alpha_0} \text{ and } x_1 \in r_1\}$$

and the family

$$\mathcal{A}^{1\alpha_0\alpha_1} = \{V_{x_0}^{\alpha_0\alpha_1 r_0 r_1}\colon\ r_1 \in \mathcal{R}_{\alpha_1},\ 0 < x_0 \in r_0 \in \mathcal{R}_{\alpha_0}\}\,.$$

PROPOSITION 2.2. *A set $S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$ is a selector $\mathcal{A}_\gamma^{0\alpha_0\alpha_1} \cup \mathcal{A}^{1\alpha_0\alpha_1}$ for $\gamma \geqslant \varphi(\alpha_0, \alpha_1)$ if and only if*

(2.2.1) $\mathrm{rng}(g_0) \supseteq r_0 - \{0\}$,

(2.2.2) *g_0 is a one-to-one mapping from the set $\{\xi < \gamma\colon g_0(\xi) \neq 0\}$ onto $r_0 - \{0\}$.*

Proof. Suppose that $S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$, where g_0 satisfies (2.2.1) and (2.2.2). Obviously, by Proposition 2.1, S is a selector of $\mathcal{A}_\gamma^{0\alpha_0\alpha_1}$. Let $X = S \cap V_{x_0}^{\alpha_0\alpha_1 s_0 s_1}$, and consider the following two cases:

Case I. $r_0 \neq s_0$ or $r_1 \neq s_1$. Then $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$.

Case II. $r_0 = s_0$ and $r_1 = s_1$. Then, for each $x_0 \in r_0$, if $x_0 \neq 0$, then there is exactly one $\xi \in Z_{\alpha_0}$ such that $g_0(\xi) = x_0$. But then $X = \{(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1, \xi)\}$. Thus S is a selector of $\mathcal{A}_\gamma^{0\alpha_0\alpha_1} \cup \mathcal{A}^{1\alpha_0\alpha_1}$.

Conversely, let us suppose that $S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$ is a selector of $\mathcal{A}^{1\alpha_0\alpha_1}$. We shall prove that (2.2.1) and (2.2.2) hold. For this purpose, let $X = S \cap V_{x_0}^{\alpha_0\alpha_1 s_0 s_1}$ and consider the following two cases:

Case I. $r_0 \neq s_0$ or $r_1 \neq s_1$. Then, as before, $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$.

Case II. $r_0 = s_0$ and $r_1 = s_1$. Then if $x_0 \in r_0$ and $x_0 \neq 0$, then there exists a $\xi \in Z_{\alpha_0}$ and some $x_1 \in r_1$ such that

$$X = \{(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1, \xi)\}\,.$$

Consequently $g_0(\xi) = x_0$, which proves (2.2.1). To prove (2.2.2) let us assume that for some $\xi_1, \xi_2 < \gamma$ we have $g_0(\xi_1) = g_0(\xi_2) = x_0 \neq 0$. Then

$$X = \{(\alpha_0, r_0, g_0(\xi_1), \alpha_1, r_1, g_1(\xi_1), \xi_1), (\alpha_0, r_0, g_0(\xi_2), \alpha_1, r_1, g_1(\xi_2), \xi_2)\}$$

and S is not a selector of $\mathcal{A}^{1\alpha_0\alpha_1}$, contrary to our assumption.

In a similar way, we shall define a family of sets which will play the same role with respect to the function g_1 as the family $\mathcal{A}^{1\alpha_0\alpha_1}$ does for g_0.

Let us define the following sets:

$$U_{x_1}^{\alpha_0\alpha_1 r_0 r_1} = R^{\alpha_0\alpha_1 r_0 r_1}\{(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1, \xi)\colon\ \xi \in Z_{\alpha_1} \text{ and } x_0 \in r_0\}\,,$$

and the family

$$\mathcal{A}^{2\alpha_0\alpha_1} = \{U_{x_1}^{\alpha_0\alpha_1 r_0 r_1}\colon\ r_0 \in \mathcal{R}_{\alpha_0} \text{ and } 0 < x_1 \in r_1 \in \mathcal{R}_{\alpha_1}\}\,.$$

COROLLARY 2.3. *A set $S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$ is a selector of the family $\mathcal{A}_\gamma^{0\alpha_0\alpha_1} \cup \mathcal{A}^{1\alpha_0\alpha_1} \cup \mathcal{A}^{2\alpha_0\alpha_1}$ for $\gamma \geqslant \varphi(\alpha_0, \alpha_1)$, if and only if, for $i = 0, 1$,*

(2.3.1) $\mathrm{rng}(g_i) \supseteq r_i - \{0\}$,

(2.3.2) *g_i is a one-to-one mapping from the set $\{\xi < \gamma\colon g_i(\xi) \neq 0\}$ onto $r_i - \{0\}$.*

Now, take the family $\mathcal{A}_\gamma^{0\alpha_0\alpha_1} \cup \mathcal{A}_\gamma^{0\alpha_1\alpha_0}$. Then, by Proposition 2.1, S is of the form:

$$(*)\qquad S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1) \cup S(\alpha_1, \alpha_0, r_1', r_0', g_1', g_0')\,.$$

The converse is not necessarily true, but it is true if, e.g., $r_i = r_i'$ and $g_i = g_i'$ for $i = 0, 1$, and for each $\xi < \gamma$ we have $g_0(\xi)F(\alpha_1, \xi) = g_1(\xi)F(\alpha_0, \xi)$. To obtain this case we shall construct some additional families.

Let us define the sets

$$F_{r_0 r_1}^{\alpha_0\alpha_1} = R^{\alpha_1\alpha_0 r_1 r_0} \cup \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}\,,$$

and the family

$$\mathfrak{C}^{0\alpha_0\alpha_1} = \{F_{r_0 r_1}^{\alpha_0\alpha_1}\colon\ r_i \in \mathcal{R}_{\alpha_i},\ i = 0, 1\}\,.$$

PROPOSITION 2.4. *A set of the form $(*)$ is a selector of $\mathfrak{C}^{0\alpha_0\alpha_1}$ if and only if $r_0 = r_0'$ and $r_1 = r_1'$.*

Proof. Indeed, if $r_0 \neq r_0'$ or $r_1 \neq r_1'$ then

$$S \cap F_{r_0 r_1}^{\alpha_0\alpha_1} = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle, \langle\langle\alpha_1, r_1\rangle, \langle\alpha_0, r_0\rangle\rangle\}\,.$$

Thus S is not a selector of $\mathfrak{C}^{0\alpha_0\alpha_1}$.

Conversely, suppose that S is of the form $(*)$ with $r_0 = r_0'$ and $r_1 = r_1'$. We wish to prove that S is a selector of the family $\mathfrak{C}^{0\alpha_0\alpha_1}$. Let $X = S \cap F_{s_0 s_1}^{\alpha_0\alpha_1}$ for some $s_i \in \mathcal{R}_{\alpha_i}$, $i = 0, 1$. We consider the following two cases.

Case I. $s_0 = r_0$ and $s_1 = r_1$. Then $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$.

Case II. $s_0 \neq r_0$ or $s_1 \neq r_1$. Then $X = \{\langle\langle\alpha_1, r_1\rangle, \langle\alpha_0, r_0\rangle\rangle\}$.

Thus in both cases $|X| = 1$, i.e., S is a selector of $\mathfrak{C}^{0\alpha_0\alpha_1}$.

Now, we shall define a family of sets such that every selector of it which is of the form $(*)$ with $r_i = r_i'$ for $i = 0, 1$ will have the additional property that $g_0 = g_0'$.

Let us define the following sets:

$$W_{0\xi}^{x_0\alpha_0\alpha_1 r_0 r_1} = R^{\alpha_0\alpha_1 r_0 r_1} \cup \{(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1, \xi)\colon\ x_1 \in r_1\} \cup$$
$$\cup \{(\alpha_1, r_1, x_1, \alpha_0, r_0, x_0', \xi)\colon\ x_1 \in r_1 \text{ and } x_0 \neq x_0' \in r_0\}\,,$$

and the family

$$\mathfrak{C}_\gamma^{1\alpha_0\alpha_1} = \{W_{0\xi}^{x_0\alpha_0\alpha_1 r_0 r_1}\colon\ \xi < \gamma,\ x_0 \in r_0 \in \mathcal{R}_{\alpha_0} \text{ and } r_1 \in \mathcal{R}_{\alpha_1}\}\,.$$

PROPOSITION 2.5. *Suppose that S is of the form $(*)$ with $r_i = r_i'$ for $i = 0, 1$. Then S is a selector of $\mathfrak{C}_\gamma^{1\alpha_0\alpha_1}$ if and only if $g_0 = g_0'$.*

Proof. Suppose that S is of the form $(*)$ with $r_i = r_i'$ for $i = 0, 1$, and $g_0 = g_0'$. Let us put $X = S \cap W_{0\xi}^{x_0\alpha_0\alpha_1 s_0 s_1}$; and consider the following three cases:

Case I. $s_0 \neq r_0$ or $s_1 \neq r_1$. Then $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$.

Case II. $s_0 = r_0$, $s_1 = r_1$ and $g_0(\xi) = x_0$. Then

$$X = \{(\alpha_0, r_0, g_0(\xi), \alpha_1, r_1, g_1(\xi), \xi)\}.$$

Case III. $s_0 = r_0$, $s_1 = r_1$ and $g_0(\xi) \neq x_0$. Then

$$X = \{(\alpha_1, r_1, g_1(\xi), \alpha_0, r_0, g_0(\xi), \xi)\}.$$

Thus S is a selector of $\mathfrak{C}_\gamma^{1\alpha_0\alpha_1}$.

Conversely, suppose that S is of the form $(*)$, $r_i = r_i'$ for $i = 0, 1$, but $g_0 = g_0'$. Then for some $\xi < \gamma$, we have $g_0(\xi) \neq g_0'(\xi)$. Put $x_0 = g_0(\xi)$ and consider $X = S \cap W_{0\xi}^{x_0\alpha_0\alpha_1r_0r_1}$. Then we have:

$$X = \{(\alpha_0, r_0, x_0, \alpha_1, r_1, g_1(\xi), \xi), (\alpha_1, r_1, g_1'(\xi), \alpha_0, r_0, g_0'(\xi), \xi)\}.$$

Thus S is not a selector of $\mathfrak{C}_\gamma^{1\alpha_0\alpha_1}$.

Similarly, we shall construct a family of sets which will play the same role for g_1 and g_1' as $\mathfrak{C}_\gamma^{1\alpha_0\alpha_1}$ does for g_0 and g_0'.

Let us define the sets

$$W_{1\xi}^{x_1\alpha_0\alpha_1r_0r_1} = R^{\alpha_0\alpha_1r_0r_1} \cup \{(\alpha_1, r_1, x_1, \alpha_0, r_0, x_0, \xi)\colon\ x_0 \in r_0\} \cup$$
$$\cup \{(\alpha_0, r_0, x_0, \alpha_1, r_1, x_1', \xi)\colon\ x_0 \in r_0 \text{ and } x_1 \neq x_1' \in r_1\}$$

and the family

$$\mathfrak{C}_\gamma^{2\alpha_0\alpha_1} = \{W_{1\xi}^{x_1\alpha_0\alpha_1r_0r_1}\colon\ \xi < \gamma,\ x_1 \in r_1 \in \mathcal{R}_{\alpha_1} \text{ and } r_0 \in \mathcal{R}_{\alpha_0}\}.$$

Then, as before, we get:

PROPOSITION 2.6. *A set S of the form $(*)$ is a selector of $\mathfrak{C}^{0\alpha_0\alpha_1} \cup \mathfrak{C}_\gamma^{1\alpha_0\alpha_1} \cup \mathfrak{C}_\gamma^{2\alpha_0\alpha_1}$ if and only if $r_i = r_i'$ and $g_i = g_i'$ for $i = 0, 1$.*

Proof. By Propositions 2.4, 2.5 and the definition of $\mathfrak{C}_\gamma^{2\alpha_0\alpha_1}$.

Thus let us consider the following set:

$$(**)\qquad S = S(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1) \cup S(\alpha_1, \alpha_0, r_1, r_0, g_1, g_0).$$

Let us define the sets

$$F_\xi^{\alpha_0\alpha_1r_0r_1} = R^{\alpha_0\alpha_1r_0r_1} \cup$$
$$\cup \{\langle\{\alpha_0, \alpha_1\}, \{\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\}, \{\langle\alpha_0, x_0\rangle, \langle\alpha_1, x_1\rangle\}, b, \xi\rangle\colon$$
$$x_i \in r_i \in \mathcal{R}_{\alpha_i},\ i = 0, 1,\ b \in \mathcal{B}\}$$

and the family

$$\mathfrak{D}_\gamma^{\alpha_0\alpha_1} = \{F_\xi^{\alpha_0\alpha_1r_0r_1}\colon\ r_i \in \mathcal{R}_{\alpha_i} \text{ for } i = 0, 1, \text{ and } \xi < \gamma\}.$$

PROPOSITION 2.7. *A set S of the form $(**)$ is a selector of $\mathfrak{D}_\gamma^{\alpha_0\alpha_1}$ if and only if the following condition is satisfied:*

(2.7.1) *For each $\xi < \gamma$: $g_0(\xi)\cdot F(\alpha_1, \xi) = g_1(\xi)\cdot F(\alpha_0, \xi)$.*

Proof. Indeed, let S be of the form $(**)$ and suppose that (2.7.1) holds. Let $X = S \cap F_\xi^{\alpha_0\alpha_1s_0s_1}$. Then we have the following two cases:

Case I. $r_0 \neq s_0$ or $r_1 \neq s_1$. Then $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$.

Case II. $r_0 = s_0$ and $r_1 = s_1$. Let $b_\xi = g_0(\xi)\cdot F(\alpha_1, \xi) = g_1(\xi)\cdot F(\alpha_0, \xi)$. Then

$$X = \{\langle\{\alpha_0, \alpha_1\}, \{\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\}, \{\langle\alpha_0, g_0(\xi)\rangle, \langle\alpha_1, g_1(\xi)\rangle\}, b_\xi, \xi\rangle\}.$$

Thus S is a selector of $\mathfrak{D}_\gamma^{\alpha_0\alpha_1}$.

Conversely, suppose that (2.7.1) does not hold. Then for some $\xi < \gamma$ we have $g_0(\xi)\cdot F(\alpha_1, \xi) \neq g_1(\xi)\cdot F(\alpha_0, \xi)$. Let us take $X = S \cap F_\xi^{\alpha_0\alpha_1r_0r_1}$. Then X has two elements, namely

$$X = \{(\alpha_0, r_0, g_0(\xi), \alpha_1, r_1, g_1(\xi), \xi), (\alpha_1, r_1, g_1(\xi), \alpha_0, r_0, g_0(\xi), \xi)\}.$$

Thus S is not a selector of $\mathfrak{D}_\gamma^{\alpha_0\alpha_1}$.

COROLLARY 2.8. *Let S be a set of the form $(*)$. Then S is a selector of $\mathcal{A}_\gamma^{0\alpha_0\alpha_1} \cup \mathcal{A}_\gamma^{0\alpha_1\alpha_0} \cup \mathfrak{C}^{0\alpha_0\alpha_1} \cup \mathfrak{C}_\gamma^{1\alpha_0\alpha_1} \cup \mathfrak{C}_\gamma^{2\alpha_0\alpha_1} \cup \mathfrak{D}_\gamma^{\alpha_0\alpha_1}$ if and only if $r_i = r_i'$, $g_i = g_i'$ for $i = 0, 1$, and (2.7.1) holds.*

Now for $\gamma \geqslant \varphi(\alpha_0, \alpha_1)$ let us define the family

$$\mathcal{E}_\gamma^{\alpha_0\alpha_1} = \mathcal{A}_\gamma^{0\alpha_0\alpha_1} \cup \mathcal{A}_\gamma^{0\alpha_1\alpha_0} \cup \mathcal{A}^{1\alpha_0\alpha_1} \cup \mathcal{A}^{2\alpha_0\alpha_1} \cup \mathfrak{C}^{0\alpha_0\alpha_1} \cup \mathfrak{C}_\gamma^{1\alpha_0\alpha_1} \cup \mathfrak{C}_\gamma^{2\alpha_0\alpha_1} \cup \mathfrak{D}_\gamma^{\alpha_0\alpha_1}.$$

It is easy to see that if γ is a limit ordinal then we have $\mathcal{E}_\gamma^{\alpha_0\alpha_1} = \bigcup_{\xi<\gamma} \mathcal{E}_\xi^{\alpha_0\alpha_1}$. Let $\mathcal{E}^{\alpha_0\alpha_1} = \bigcup_{\xi<\varkappa} \mathcal{E}_\xi^{\alpha_0\alpha_1}$.

We can summarize the preceding considerations as

COROLLARY 2.9. (.1) *A set $S \subseteq \bigcup \mathcal{E}_\gamma^{\alpha_0\alpha_1}$ (where $\gamma \geqslant \varphi(\alpha_0, \alpha_1)$) is a selector of $\mathcal{E}_\gamma^{\alpha_0\alpha_1}$ if and only if S has the following form:*

$$(+)\qquad S = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle, \langle\langle\alpha_1, r_1\rangle, \langle\alpha_0, r_0\rangle\rangle\} \cup$$
$$\cup \{\langle\{\alpha_0, \alpha_1\}, \{\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\}, \{\langle\alpha_0, g_0(\xi)\rangle, \langle\alpha_1, g_1(\xi)\rangle\}, h(\xi), \xi\rangle\colon\ \xi < \gamma\}$$

where, for $i = 0, 1$:

(a) $r_i \in \mathcal{R}_{\alpha_i}$,

(b) $g_i\colon \gamma \to r_i$ *and* $\mathrm{rng}(g_i) \supseteq r_i - \{0\}$,

(c) *for each* $\xi < \gamma$, $g_i(\xi) \leqslant F(\alpha_i, \xi)$,

(d) g_i *is a one-to-one mapping from* $\{\xi < \gamma\colon g_i(\xi) \neq 0\}$ *onto* $r_i - \{0\}$,

(e) *for each* $\xi < \gamma$, $h(\xi) = g_0(\xi)\cdot F(\alpha_1, \xi) = g_1(\xi)\cdot F(\alpha_0, \xi)$

(we shall denote such a selector by $S^*(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$).

(.2) *Each selector S of $\mathcal{E}_\gamma^{\alpha_0\alpha_1}$ has a unique extension S' to a selector of $\mathcal{E}_\eta^{\alpha_0\alpha_1}$, for $\eta > \gamma$.*

(.3) *Each selector S of $\mathcal{E}^{\alpha_0\alpha_1}$ is completely determined by a selector of $\mathcal{E}_{\varphi(\alpha_0,\alpha_1)}^{\alpha_0\alpha_1}$.*

Proof. (.1) has already been justified by 2.1-2.8.

(.2) Let $S = S^*(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$ be a selector of $\mathcal{E}_\gamma^{\alpha_0\alpha_1}$. Let us define $g_i' \supseteq g_i$, by $g_i'(\xi) = 0$, for $\gamma \leqslant \xi < \eta$. Then $S' = S^*(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1)$ is, by (.1), a selector of $\mathcal{E}_\eta^{\alpha_0\alpha_1}$. The uniqueness of this extension follows from the fact that $\eta > \gamma \geqslant \varphi(\alpha_0, \alpha_1)$ and by (c).

(.3) This follows immediately from (.2).

Let $\alpha_1 \neq \alpha_2$, $\gamma \geqslant \varphi(\alpha_0, \alpha_1, \alpha_2)$. Then

$$\left(\bigcup \mathcal{E}_\gamma^{\alpha_0\alpha_1}\right) \cap \left(\bigcup \mathcal{E}_\gamma^{\alpha_0\alpha_2}\right) = 0 .$$

Thus, by 2.9.1, each selector S of the family $\mathcal{E}_\gamma^{\alpha_0\alpha_1} \cup \mathcal{E}_\gamma^{\alpha_0\alpha_2}$ is of the form

$$\text{(S.1)} \qquad S = S^*(\alpha_0, \alpha_1, r_0^1, r_1, g_0^1, g_1) \cup S^*(\alpha_0, \alpha_2, r_0^2, r_2, g_0^2, g_2) ,$$

where r_i, r_0^i, g_i, g_0^i, $i = 1, 2$, satisfy the corresponding relations from 2.9.1.

We shall construct a family $\mathfrak{B}^{0\alpha_0\alpha_1\alpha_2}$ with the property that each selector of $\mathcal{E}_\gamma^{\alpha_0\alpha_1} \cup \mathcal{E}_\gamma^{\alpha_0\alpha_2} \cup \mathfrak{B}^{0\alpha_0\alpha_1\alpha_2}$ has the form (S.1) with the additional property that $r_0^1 = r_0^2$.

To do this, let us define for $\alpha_0 \neq \alpha_1 \neq \alpha_2 \neq \alpha_0$ and $s_0 \in \mathcal{R}_{\alpha_0}$ the set

$$\mathfrak{B}_{s_0}^{0\alpha_0\alpha_1\alpha_2} = \{\langle\langle\alpha_0, s_0\rangle, \langle\alpha_1, s_1\rangle\rangle \colon s_1 \in \mathcal{R}_{\alpha_1}\} \cup$$
$$\cup \{\langle\langle\alpha_0, s_0'\rangle, \langle\alpha_2, s_2\rangle\rangle \colon s_2 \in \mathcal{R}_{\alpha_2},\ s_0 \neq s_0' \in \mathcal{R}_{\alpha_0}\} ,$$

and the family

$$\mathfrak{B}^{0\alpha_0\alpha_1\alpha_2} = \{\mathfrak{B}_{s_0}^{0\alpha_0\alpha_1\alpha_2} \colon s_0 \in \mathcal{R}_{\alpha_0}\} .$$

PROPOSITION 2.10. *A set S of the form* (S.1) *is a selector of the family $\mathfrak{B}^{0\alpha_0\alpha_1\alpha_2}$ if and only if $r_0^1 = r_0^2$.*

Proof. Suppose that S is a set of the form (S.1), where $r_0 = r_0^1 = r_0^2$. Let us consider the set $X = S \cap \mathfrak{B}_{s_0}^{0\alpha_0\alpha_1\alpha_2}$. Then either $s_0 \neq r_0$ and then $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_2, r_2\rangle\rangle\}$, or else $s_0 = r_0$ and then $X = \{\langle\langle\alpha_0, r_0\rangle, \langle\alpha_1, r_1\rangle\rangle\}$. Thus S is a selector of $\mathfrak{B}^{0\alpha_0\alpha_1\alpha_2}$.

Conversely, suppose that S is of the form (S.1) and $r_0^1 \neq r_0^2$. Then

$$S \cap \mathfrak{B}_{r_0^1}^{0\alpha_0\alpha_1\alpha_2} = \{\langle\langle\alpha_0, r_0^1\rangle, \langle\alpha_1, r_1\rangle\rangle, \langle\langle\alpha_0, r_0^2\rangle, \langle\alpha_2, r_2\rangle\rangle\} .$$

Thus S is not a selector of $\mathfrak{B}^{0\alpha_0\alpha_1\alpha_2}$.

COROLLARY 2.11. *A set S is a selector of the family $\mathcal{E}_\gamma^{\alpha_0\alpha_1} \cup \mathcal{E}_\gamma^{\alpha_0\alpha_2} \cup \mathfrak{B}^{0\alpha_0\alpha_1\alpha_2}$ if and only if*

$$\text{(S.2)} \qquad S = S^*(\alpha_0, \alpha_1, r_0, r_1, g_0^1, g_1) \cup S^*(\alpha_0, \alpha_2, r_0, r_2, g_0^2, g_2) ,$$

where r_0, r_1, g_1 and g_0^i, $i = 1, 2$, satisfy the corresponding relations from 2.9.1.

Now we shall construct a family $\mathfrak{B}_\gamma^{1\alpha_0\alpha_1\alpha_2}$ with the property that each selector of $\mathcal{E}_\gamma^{\alpha_0\alpha_1} \cup \mathcal{E}_\gamma^{\alpha_0\alpha_2} \cup \mathfrak{B}^{0\alpha_0\alpha_1\alpha_2} \cup \mathfrak{B}_\gamma^{1\alpha_0\alpha_1\alpha_2}$ is of the form (S.2) with the additional property that $g_0^1 = g_0^2$.

For this purpose let us define the following sets:

$$B_{s_0t_0\xi}^{\alpha_0\alpha_1} = \{(\alpha_0, s_0, t_0, \alpha_1, s_1, x_1, \xi) \colon x_1 \in s_1 \in \mathcal{R}_{\alpha_1}\} ,$$
$$C_{s_0t_0\xi}^{\alpha_0\alpha_2} = \{(\alpha_0, s_0, t_0', \alpha_2, s_2, x_2, \xi) \colon x_2 \in s_2 \in \mathcal{R}_{\alpha_2} \text{ and } t_0 \neq t_0' \in s_0\} ,$$
$$D_{s_0t_0\xi}^{\alpha_0\alpha_2} = \{(\alpha_0, s_0', t_0, \alpha_2, s_2, x_2, \xi) \colon x_2 \in s_2 \in \mathcal{R}_{\alpha_0} \text{ and } s_0 \neq s_0' \in \mathcal{R}_{\alpha_0}\} ,$$
$$E_{s_0t_0\xi}^{\alpha_0\alpha_2} = \{(\alpha_0, s_0, t_0', \alpha_2, s_2, x_2, \xi) \colon x_2 \in s_2 \in \mathcal{R}_{\alpha_2},\ s_0 \neq s_0' \in \mathcal{R}_{\alpha_0} \text{ and } t_0 \neq t_0' \in s_0'\} ,$$
$$B_{s_0t_0\xi}^{1\alpha_0\alpha_1\alpha_2} = B_{s_0t_0\xi}^{\alpha_0\alpha_1} \cup C_{s_0t_0\xi}^{\alpha_0\alpha_2} \cup D_{s_0t_0\xi}^{\alpha_0\alpha_2} \cup E_{s_0t_0\xi}^{\alpha_0\alpha_2} ;$$

and the family:

$$\mathfrak{B}_\gamma^{1\alpha_0\alpha_1\alpha_2} = \{B_{s_0t_0\xi}^{1\alpha_0\alpha_1\alpha_2} \colon t_0 \in s_0 \in \mathcal{R}_{\alpha_0} \text{ and } \xi < \gamma\} .$$

PROPOSITION 2.12. *A set of the form* (S.2) *is a selector of the family $\mathfrak{B}_\gamma^{1\alpha_0\alpha_2}$ if and only if $g_0^1 = g_0^2$.*

Proof. Let $S = S^*(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1) \cup S^*(\alpha_0, \alpha_2, r_0, r_2, g_0, g_2)$. We shall show that for any $s_0 \in \mathcal{R}_{\alpha_0}$, $t_0 \in s_0$ and $\xi < \gamma$, we have $|S \cap B_\gamma^{1\alpha_0\alpha_1\alpha_2}| = 1$. For this purpose let us consider the following four cases:

Case I. $s_0 = r_0$ and $g_0(\xi) = t_0$. Then $S \cap C_{s_0t_0\xi}^{\alpha_0\alpha_2} = S \cap D_{s_0t_0\xi}^{\alpha_0\alpha_2} = S \cap E_{s_0t_0\xi}^{\alpha_0\alpha_2} = 0$. Thus

$$S \cap B_{s_0t_0\xi}^{1\alpha_0\alpha_1\alpha_2} = S \cap B_{s_0t_0\xi}^{\alpha_0\alpha_1} = \{(\alpha_0, r_0, g_0(\xi), \alpha_1, r_1, g_1(\xi), \xi)\} .$$

Case II. $s_0 = r_0$ and $g_0(\xi) \neq t_0$. Then $S \cap B_{s_0t_0\xi}^{\alpha_0\alpha_1} = S \cap D_{s_0t_0\xi}^{\alpha_0\alpha_2} = S \cap E_{s_0t_0\xi}^{\alpha_0\alpha_2} = 0$. Thus

$$S \cap B_{s_0t_0\xi}^{1\alpha_0\alpha_1\alpha_2} = S \cap C_{s_0t_0\xi}^{\alpha_0\alpha_2} = \{(\alpha_0, r_0, g_0(\xi), \alpha_2, r_2, g_2(\xi), \xi)\} .$$

Case III. $s_0 \neq r_0$ and $g_0(\xi) = t_0$. Then $S \cap B_{s_0t_0\xi}^{\alpha_0\alpha_1} = S \cap C_{s_0t_0\xi}^{\alpha_0\alpha_2} = S \cap E_{s_0t_0\xi}^{\alpha_0\alpha_2} = 0$. Thus

$$S \cap B_{s_0t_0\xi}^{1\alpha_0\alpha_1\alpha_2} = S \cap D_{s_0t_0\xi}^{\alpha_0\alpha_2} = \{(\alpha_0, r_0, g_0(\xi), \alpha_2, r_2, g_2(\xi), \xi)\} .$$

Case IV. $s_0 \neq r_0$ and $g_0(\xi) \neq t_0$. Then $S \cap B_{s_0t_0\xi}^{\alpha_0\alpha_1} = S \cap C_{s_0t_0\xi}^{\alpha_0\alpha_2} = S \cap D_{s_0t_0\xi}^{\alpha_0\alpha_2} = 0$. Thus

$$S \cap B_{s_0t_0\xi}^{1\alpha_0\alpha_1\alpha_2} = S \cap E_{s_0t_0\xi}^{\alpha_0\alpha_2} = \{(\alpha_0, r_0, g_0(\xi), \alpha_2, r_2, g_2(\xi), \xi)\} .$$

Consequently S is a selector of $\mathfrak{B}_\gamma^{1\alpha_0\alpha_1\alpha_2}$.

Conversely, suppose that S is of the form (S.2) and for some $\xi < \gamma$ we have $g_0^1(\xi) \neq g_0^2(\xi)$. Take $s_0 = r_0$ and $t_0 = g_0^1(\xi)$. Then

$$S \cap B_{s_0t_0\xi}^{\alpha_0\alpha_1} = \{(\alpha_0, r_0, g_0^1(\xi), \alpha_1, r_1, g_1(\xi)\xi)\}$$

and

$$S \cap C^{\alpha_0\alpha_2}_{s_0 t_0 \xi} = \{(\alpha_0, r_0, g_0^2(\xi), \alpha_2, r_2, g_2(\xi)\xi)\}\,.$$

Consequently S is not a selector of $\mathcal{B}^{1\alpha_0\alpha_1\alpha_2}_\gamma$.

Let $\mathcal{F}^{\alpha_0\alpha_1\alpha_2}_\gamma = \mathcal{B}^{0\alpha_0\alpha_1\alpha_2} \cup \mathcal{B}^{1\alpha_0\alpha_1\alpha_2}_\gamma$ for $\gamma \geqslant \varphi(\alpha_0, \alpha_1, \alpha_2)$. Then for the limit γ we also have $\mathcal{F}^{\alpha_0\alpha_1\alpha_2}_\gamma = \bigcup_{\xi<\gamma} \mathcal{F}^{\alpha_0\alpha_1\alpha_2}_\xi$. Let $\mathcal{F}^{\alpha_0\alpha_1\alpha_2} = \bigcup_{\xi<\varkappa} \mathcal{F}^{\alpha_0\alpha_1\alpha_2}_\xi$.

Corollary 2.13. *Let $\gamma \geqslant \varphi(\alpha_0, \alpha_1, \alpha_2)$. Then a set $S \subseteq \bigcup \mathcal{E}^{\alpha_0\alpha_1}_\gamma \cup \bigcup \mathcal{E}^{\alpha_0\alpha_2}_\gamma$ is a selector of $\mathcal{E}^{\alpha_0\alpha_1}_\gamma \cup \mathcal{E}^{\alpha_0\alpha_2}_\gamma \cup \mathcal{F}^{\alpha_0\alpha_1\alpha_2}_\gamma$ if and only if S is of the form*

$$S = S^*(\alpha_0, \alpha_1, r_0, r_1, g_0, g_1) \cup S^*(\alpha_0, \alpha_2, r_0, r_2, g_0, g_2)$$

where r_i, g_i, $i = 0, 1, 2$, satisfy the relations from 2.9.1.

Let us define $\mathcal{E}_\delta = \bigcup\{\mathcal{E}^{\alpha_0\alpha_1}\colon \alpha_0, \alpha_1 < \delta \text{ and } \alpha_0 \neq \alpha_1\}$ and

$$\mathcal{F}_\delta = \bigcup\{\mathcal{F}^{\alpha_0\alpha_1\alpha_2}\colon \alpha_0, \alpha_1, \alpha_2 < \delta \text{ and } \alpha_0 \neq \alpha_1 \neq \alpha_2 \neq \alpha_0\}\,.$$

Lemma A. *If F has a δ-refinement then $\mathcal{E}_\delta \cup \mathcal{F}_\delta$ has a selector.*

Proof. Let G be a δ-refinement of F. Let us define for each $\beta < \delta$, $r_\beta = \{G(\beta, \xi)\colon \xi < \varkappa\} \cup \{0\}$ and the function $g_\beta(\xi) = G(\beta, \xi)$ for $\xi < \varkappa$. Then, by 1.7, and 1.5, we have $r_\beta \in \mathcal{R}_\beta$ and g_β satisfies conditions (b)-(e) of 2.9.1.

For $\alpha_0, \alpha_1 < \delta$, $\alpha_0 \neq \alpha_1$, let $S_{\alpha_0\alpha_1} = S^*(\alpha_0, \alpha_1, r_{\alpha_0}, r_{\alpha_1}, g_{\alpha_0}, g_{\alpha_1})$. Then, by 2.9.1 (e), we have $S_{\alpha_0\alpha_1} = S_{\alpha_1\alpha_0}$. Let $S = \bigcup\{S_{\alpha_0\alpha_1}\colon \alpha_0, \alpha_1 < \delta \text{ and } \alpha_0 \neq \alpha_1\}$.

First we shall show that S is a selector of $\mathcal{E}_\delta$. Indeed, by 2.9.1, $S_{\alpha_0\alpha_1}$ is a selector of $\mathcal{E}^{\alpha_0\alpha_1}$. Moreover, it is easy to see that if $\{\alpha_0, \alpha_1\} \neq \{\alpha_0', \alpha_1'\}$ then $S_{\alpha_0\alpha_1} \cap S_{\alpha_0'\alpha_1'} = 0$, and also for each $X \in \mathcal{E}^{\alpha_0'\alpha_1'}$ we have $S_{\alpha_0\alpha_1} \cap X = 0$. Thus take any $X \in \mathcal{E}_\delta$. Then for some $\alpha_0, \alpha_1 < \delta$, $\alpha_0 \neq \alpha_1$ we have $X \in \mathcal{E}^{\alpha_0\alpha_1}$. Consequently $X \cap S = X \cap S_{\alpha_0\alpha_1}$ and therefore $|X \cap S| = 1$. Thus S is a selector of $\mathcal{E}_\delta$.

Now we claim that S is a selector of $\mathcal{F}_\delta$. Let us put $S^{\alpha_0}_{\alpha_1\alpha_2} = S_{\alpha_0\alpha_1} \cup S_{\alpha_0\alpha_2}$. Then, by 2.13, $S^{\alpha_0}_{\alpha_1\alpha_2}$ is a selector of $\mathcal{F}^{\alpha_0\alpha_1\alpha_2}$. We shall consider all other possibilities:

Case I. $\{\alpha_0, \alpha_1, \alpha_2\} \cap \{\alpha_0', \alpha_1', \alpha_2'\} = 0$. Then $S^{\alpha_0}_{\alpha_1\alpha_2} \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'} = 0$ and for any $X \in \mathcal{F}^{\alpha_0'\alpha_1'\alpha_2'}$ we have $S^{\alpha_0}_{\alpha_1\alpha_2} \cap X = 0$.

Case II. $|\{\alpha_0, \alpha_1, \alpha_2\} \cap \{\alpha_0', \alpha_1', \alpha_2'\}| = 1$. Then, as before, we have $S^{\alpha_0}_{\alpha_1\alpha_2} \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'} = 0$ and for any $X \in \mathcal{F}^{\alpha_0'\alpha_1'\alpha_2'}$ we have $X \cap S^{\alpha_0}_{\alpha_1\alpha_2} = 0$.

Case III. $\{\alpha_0, \alpha_1\} = \{\alpha_0, \alpha_1, \alpha_2\} \cap \{\alpha_0', \alpha_1', \alpha_2'\}$ and $\{\alpha_0, \alpha_1\} = \{\alpha_0', \alpha_1'\}$. Then $S^{\alpha_0}_{\alpha_1\alpha_2} \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'} = S_{\alpha_0\alpha_1}$ and for any $X \in \mathcal{F}^{\alpha_0'\alpha_1'\alpha_2'}$ we have $X \cap S^{\alpha_0}_{\alpha_1\alpha_2} = X \cap S_{\alpha_0\alpha_1} \subseteq X \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'}$.

Case IV. $\{\alpha_0, \alpha_1\} = \{\alpha_0, \alpha_1, \alpha_2\} \cap \{\alpha_0', \alpha_1', \alpha_2'\}$ and $\{\alpha_0, \alpha_1\} \neq \{\alpha_1', \alpha_2'\}$. Then $S^{\alpha_0}_{\alpha_1\alpha_2} \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'} = 0$ and for any $X \in \mathcal{F}^{\alpha_0'\alpha_1'\alpha_2'}$ we have $X \cap S^{\alpha_0}_{\alpha_1\alpha_2} = 0$.

Case V. $\{\alpha_0, \alpha_1, \alpha_2\} = \{\alpha_0', \alpha_1', \alpha_2'\}$ and $\alpha_0 = \alpha_0'$. Then $S^{\alpha_0}_{\alpha_1\alpha_2} = S^{\alpha_0'}_{\alpha_1'\alpha_2'}$.

Case VI. $\{\alpha_0, \alpha_1, \alpha_2\} = \{\alpha_0', \alpha_1', \alpha_2'\}$ and $\alpha_0 \neq \alpha_0'$. Then $S^{\alpha_0}_{\alpha_1\alpha_2} \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'} = S_{\alpha_0\alpha_0'}$ and for any $X \in \mathcal{F}^{\alpha_0'\alpha_1'\alpha_2'}$ we have $S^{\alpha_0}_{\alpha_1\alpha_2} \cap X = S_{\alpha_0\alpha_0'} \cap X \subseteq X \cap S^{\alpha_0'}_{\alpha_1'\alpha_2'}$.

From this consideration it easily follows that if $X \in \mathcal{F}^{\alpha_0\alpha_1\alpha_2}$ then $X \cap S = X \cap S^{\alpha_0}_{\alpha_1\alpha_2}$. Thus, since $S^{\alpha_0}_{\alpha_1\alpha_2}$ is a selector of $\mathcal{F}^{\alpha_0\alpha_1\alpha_2}$, we have $|X \cap S| = 1$. Consequently, S is a selector of $\mathcal{F}_\delta$, which finishes the proof of Lemma A.

Lemma B. *If $\mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa$ has a selector then F has a $\varkappa$-refinement.*

Proof. Let $S \subseteq \bigcup(\mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa)$ be a selector of $\mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa$. Then, for every $\alpha_0, \alpha_1 < \varkappa$ with $\alpha_0 \neq \alpha_1$, S is a selector of $\mathcal{E}^{\alpha_0\alpha_1}$. Let $S_{\alpha_0\alpha_1} = S \cap \bigcup \mathcal{E}^{\alpha_0\alpha_1}$. Then $S_{\alpha_0\alpha_1}$ is a selector of $\mathcal{E}^{\alpha_0\alpha_1}$ and, by 2.9.1, S is of the form:

$$S_{\alpha_0\alpha_1} = S^*(\alpha_0, \alpha_1, r^{(\alpha_1)}_{\alpha_0}, r^{(\alpha_0)}_{\alpha_1}, g^{(\alpha_1)}_{\alpha_0}, g^{(\alpha_0)}_{\alpha_1})\,,$$

where $r^{(\alpha_1)}_{\alpha_0}, r^{(\alpha_0)}_{\alpha_1}, g^{(\alpha_1)}_{\alpha_0}, g^{(\alpha_0)}_{\alpha_1}$ satisfy the relations (a)-(e) of Corollary 2.9.1. Moreover, S is a selector of $\mathcal{F}^{\alpha_0\alpha_1\alpha_2}$ for each $\alpha_0 \neq \alpha_1 \neq \alpha_2 \neq \alpha_0$. Thus, by 2.13, $r^{(\alpha_1)}_{\alpha_0}$ does not depend on α_1 and similarly $r^{(\alpha_0)}_{\alpha_1}$ does not depend on α_0. Thus for some $r_{\alpha_i} \in \mathcal{R}_{\alpha_i}$, $i = 0, 1$, we have $r^{(\alpha_1)}_{\alpha_0} = r_{\alpha_0}$ for each $\alpha_1 \neq \alpha_0$ and $r^{(\alpha_0)}_{\alpha_1} = r_{\alpha_1}$ for each $\alpha_0 \neq \alpha_1$. For the same reason $g^{(\alpha_1)}_{\alpha_0}$ does not depend on α_1 and $g^{(\alpha_0)}_{\alpha_1}$ does not depend on α_0. Thus we can define a function $G\colon \varkappa \times \varkappa \to \mathcal{B}$, by putting $G(\alpha_0, \xi) = g_{\alpha_0}(\xi)$ for all $\alpha_0, \xi < \varkappa$. But then, by 2.9.1, G is a $\varkappa$-refinement of F.

Main lemma. *Let $|\mathcal{B}| = \lambda$ and $\mathcal{B}$ satisfy σ-cc, and suppose that $\lambda^{\aleph_0} < \varkappa$. Then $\mathbf{E}(\varkappa, \varkappa)$ implies $\mathbf{E}(\mathcal{B}, \varkappa)$.*

Proof. To prove that $\mathbf{E}(\mathcal{B}, \varkappa)$ holds, take any $\varkappa$-convering F of $\mathcal{B}$ which for each $\delta < \varkappa$ has some δ-refinement. Let us construct the families $\mathcal{E}_\delta$ and $\mathcal{F}_\delta$. By Lemma A the family $\mathcal{E}_\delta \cup \mathcal{F}_\delta$ has a selector.

To apply $\mathbf{E}(\varkappa, \varkappa)$ to the family $\mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa$ we must estimate the cardinality of this family and also the cardinalities of its members.

First let us remark that $|\mathcal{R}| \leqslant \lambda^{\aleph_0} < \varkappa$ and consequently $|\mathcal{R}_{\alpha_0}| < \varkappa$ and $|R^{\alpha_0\alpha_1 r_0 r_1}| < \varkappa$ for each $\alpha_0, \alpha_1 < \varkappa$ and $r_i \in \mathcal{R}_{\alpha_i}$, $i = 0, 1$. Thus for each $X \in \mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa$ we have $|X| < \varkappa$. Moreover, it is easy to see that $|\mathcal{E}^{\alpha_0\alpha_1}_\gamma| \leqslant \max(\lambda^{\aleph_0}, |\gamma|)$ and similarly $|\mathcal{F}^{\alpha_0\alpha_1\alpha_2}_\gamma| \leqslant \max(\lambda^{\aleph_0}, |\gamma|)$. Therefore $|\mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa| = \varkappa$.

Finally, from $\mathbf{E}(\varkappa, \varkappa)$ it follows that $\varkappa$ is regular. Let $\mathfrak{X} = \mathcal{E}_\varkappa \cup \mathcal{F}_\varkappa$. Since $\varkappa$ is regular, we can ennumerate $\mathfrak{X}$ in such a way that for each $\beta < \varkappa$ there is an $\alpha < \varkappa$ such that $\mathfrak{X} \restriction \beta \subseteq \mathcal{E}_\alpha \cup \mathcal{F}_\alpha$. Consequently, since for each $\delta < \varkappa$ the family $\mathcal{E}_\delta \cup \mathcal{F}_\delta$ has a selector, $\mathfrak{X}$ has partial selectors. Thus, by $\mathbf{E}(\varkappa, \varkappa)$, $\mathfrak{X}$ has a selector. Now, by using Lemma B, this means that F has a refinement, which proves $\mathbf{E}(\mathcal{B}, \varkappa)$.

§ 3. Preservation Theorem.

In this part we prove that $\mathbf{E}(\varkappa, \varkappa)$ is preserved under some generic extensions.

THEOREM. *Let $|\mathfrak{B}| = \lambda$, $\mathfrak{B}$ satisfy σ-cc, and $\lambda^{\underline{\sigma}} < \varkappa$. Moreover, suppose that for each* **ZF**-*formula Φ with parameters from $\check{V}$ we have $\|\Phi\| \in \{0, 1\}$ (i.e. [D] from § 1). Then $\mathbf{E}(\varkappa, \varkappa)$ implies $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$ in $V^{(\mathfrak{B})}$.*

Proof. Since $|\mathfrak{B}| = \lambda$, $\mathfrak{B}$ satisfies σ-cc, $\lambda^{\underline{\sigma}} < \varkappa$ and $\mathbf{E}(\varkappa, \varkappa)$ are assumed, by the Main Lemma we have $\mathbf{E}(\mathfrak{B}, \varkappa)$. Next, obviously $\sigma \leqslant \varkappa$; thus $\mathfrak{B}$ satisfies also $\varkappa$-cc. Consequently all the assumptions of Theorem 1.9.2, are fulfilled. Thus, by 1.9.2, we have $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$. Q.E.D.

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Examples of disks in E^3/G which cannot be approximated by P-liftable disks*

by

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Abstract. In *Conditions under which disks are P-liftable* the author defined a set $X \subset E^3/G$ to be P-liftable if there exists a set $X' \subset E^3$ such that X and X' are homeomorphic and X is the image of X' under the natural projection mapping P. It was proved that in certain decomposition spaces, each disk $D \subset E^3/G$ can be approximated by P-liftable disks, i.e., for any $\varepsilon > 0$ there exists a P-liftable disk D_ε that is ε-homeomorphic to D. In this paper we give examples of decomposition spaces each containing a disk D that cannot be approximated by P-liftable disks.

Analogous to the problem of the existence of an approximating P-liftable disk is a question posed by Armentrout for 2-spheres when G is a pointlike decomposition. This question is answered in the negative.

An example is given of a pair of decomposition spaces that are "equivalent" in the terminology of Armentrout, Lininger, and Meyer, but differ in the property of containing P-liftable approximating disks.

A construction called a *knit Cantor set of nondegenerate elements* is defined. A newly defined property entitled *equi-locally connected* is not possessed by every point of a knit Cantor set of nondegenerate elements. Hypothesizing this property for the points in the nondegenerate elements, questions are formulated concerning the existence of P-liftable approximating disks.

Key words and phrases. Lift of a space, P-lift, topology of E^3, decomposition space, monotone decomposition, Cantor set of nondegenerate elements, equi-LC^n, equi-locally connected.

1. Introduction. In *Conditions under which disks are P-liftable* [16] the author defined a set $X \subset E^3/G$ to be *P-liftable* if there exists a set $X' \subset E^3$ such that X and X' are homeomorphic and X is the image of X' under the natural projection mapping P. The set X' is said to be the *P-lift* of X. Note that this generalizes the lifting concept (McAuley [9]) in which the projection mapping is a homeomorphism on the set that is called the lift. For spaces which (1) are definable by 3-cells, or (2) in which G has a countable number of nondegenerate elements and E^3/G is homeo-

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