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Rational approximation on arcs in C^n

by

DAVID M. WELLS (Pittsburgh, Pa.)

Abstract. Conditions on an arc in $\boldsymbol{C}^n$ are exhibited insuring that every continuous function on the arc can be uniformly approximated by rational functions with poles off the arc. Examples are given to provide negative answers to two conjectures concerning algebras of rational functions in $\boldsymbol{C}$ and $\boldsymbol{C}^2$.

Throughout this paper X will be a compact subset of $\boldsymbol{C}^n$. Whenever A is a function algebra on X we will denote the maximal ideal space of A by M_A. We are concerned with finding conditions on X which guarantee that every continuous function on X can be uniformly approximated by functions from a certain class. To this end let $C(X)$ be the uniform algebra of all continuous functions on X. Let $P(X)$, $R(X)$ and $H(X)$ respectively denote the uniform closures in $C(X)$ of the polynomials, the rational functions with poles off X, and the functions holomorphic about X. Let $\hat{X}$ and $\hat{X}_R$ respectively denote the polynomial and rational hulls of X. It is known that $\hat{X}$ and $\hat{X}_R$ are the respective maximal ideal spaces of $P(X)$ and $R(X)$, that $X \subseteq \hat{X}_R \subseteq \hat{X}$, that $P(X) = R(X)$ whenever $\hat{X}_R = \hat{X}$ and that $R(X) = H(X)$ whenever $X = \hat{X}_R$. All of this information may be found in [3] and [5].

In terms of the above notation, we are looking for conditions under which $P(X)$ or $R(X)$ coincides with $C(X)$. Since more is known about approximation in the plane than in higher dimensions, we will have occasion to consider the plane sets $X_1, \ldots, X_n$, where X_i is the image of X under the coordinate projection z_i. In case X is an arc several results, notably those due to Alexander [1] and Stolzenberg [8], give conditions on the sets X_i which guarantee that $P(X) = C(X)$. We will study $R(X)$ via the algebras $R(X_i)$, and in the case of an arc give conditions on $R(X_i)$ insuring that $R(X) = C(X)$. Our techniques will also enable us to answer some questions concerning rational convexity of sets other than arcs. The following definition will provide our starting point.

DEFINITION. A function algebra A on X is said to be *normal* on X if for any two disjoint closed sets K and L of X there is a function $f \in A$ for which $f|K = 1$ and $f|L = 0$.

Our results all hinge on the fact that normality in the algebras $R(X_i)$ can be lifted to $R(X)$.

LEMMA. *If $R(X_i)$ is normal on X_i for $i = 1, \ldots, n$, then $R(X)$ is normal on X.*

Proof. Let x and y be distinct points in X. We may assume that $p = z_1(x)$ and $q = z_1(y)$ are distinct points in C. Choose neighborhoods of p and q in X_1, say U and V, such that $\bar{U} \cap \bar{V} = \emptyset$. Choose $f \in R(X_1)$ such that $f = 1$ on U and $f = 0$ on V. Then $f \circ z_1 \in R(X)$, $f \circ z_1 = 1$ on a neighborhood of x in X, and $f \circ z_1 = 0$ on a neighborhood of y in X.

Now let K by any closed subset of X with $x \notin K$. For every $y \in K$ there is a function $f \in R(X)$ such that $f = 1$ on a neighborhood of x and $f = 0$ on a neighborhood of y. Choose finitely many functions $f_1, \ldots, f_m \in R(X)$ such that for each $j = 1, \ldots, m$, $f_j = 1$ on a neighborhood of x, $f_j = 0$ on a neighborhood U_j of some point in K, and $\{U_1, \ldots, U_m\}$ covers K. Then $g = f_1 \ldots f_m \in R(X)$, $g = 1$ on a neighborhood of x and vanishes on K.

Finally, if K and L are disjoint closed sets in X, we can use a similar argument to obtain a function in $R(X)$ equal to 1 on K and vanishing on L. Hence $R(X)$ is normal on X.

It is known (see [6] for example) that $M_A = X$ whenever A is a normal function algebra on X. If $X \subseteq C$ and $R(X)$ is normal on X, then X has no interior in the plane. We can therefore make use of the following theorem of Čirka [4]. The version stated here is given in [5].

THEOREM (ČIRKA). *If J is an arc in C^n and J_i has no interior in C for $i = 1, \ldots, n$, then $H(J) = C(J)$.*

If J is an arc in C^n and $R(J_i)$ is normal on J_i for $i = 1, \ldots, n$, then the hypothesis of Čirka's theorem is satisfied, so $H(J) = C(J)$. Furthermore by the above lemma, $R(J)$ must be normal on J. Hence $J = M_{R(J)} = \hat{J}_R$, so $R(J) = H(J)$ and we have proved the following:

THEOREM. *Let J be an arc in C^n. If $R(J_i)$ is normal on J_i for $i = 1, \ldots, n$, then $R(J) = C(J)$.*

Of course this theorem becomes much more useful if we have some clues as to when $R(J_i)$ is normal. As a tool, we examine a class of sets in C^2 considered by Wermer [9]. Let S be a subset of the unit disc in C, containing the unit circle T and having no interior in C. Let $X_S = \{(z, w) \in C^2 | \, |z| = 1, \, w \in S\} \cup \{(z, w) \in C^2 | \, z \in S, \, |w| = 1\} = (S \times T) \cup (T \times S)$.

LEMMA. *With the above notation, if $R(S)$ is normal on S, then $(\hat{X}_S)_R = X_S$.*

Proof. We need only notice that $(X_S)_1 = S = (X_S)_2$, and apply the previous lemma.

Wermer provides a characterization of the points in $(\hat{X}_S)_R$. Hence it would be nice to know whether the converse of the lemma is also true.

It should be mentioned here that following the appearance of McKissick's normal algebra [7], it was conjectured that $R(S)$ is normal on S whenever S is a plane set without interior. Unfortunately this fails to be true, and we can provide a specific counterexample.

EXAMPLE 1. *There is a compact set $S \subseteq C$ without interior such that $R(S)$ is not normal on S.*

Proof. Wermer [9] provides an example of a set S for which $(\hat{X}_S)_R \neq X_S$. By the above lemma, $R(S)$ is not normal on S.

In investigating the sets $(\hat{X}_S)_R$, Wermer searched for conditions on S which would yield information about $(\hat{X}_S)_R$. In particular, he asked whether $R(S) = C(S)$ whenever X_S is rationally convex. We can now show that this question has a negative answer. This result was also obtained by Basener [2] via a different approach.

EXAMPLE 2. *There is a compact set S contained in the unit disc of C, containing the unit circle T and having no interior, such that $R(S) \neq C(S)$ but X_S is rationally convex.*

Proof. The set S constructed by McKissick has all the properties listed above. McKissick showed $R(S) \neq C(S)$, and the normality of $R(S)$ guarantees that $(\hat{X}_S)_R = X_S$.

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UNIVERSITY OF PITTSBURGH
PITTSBURGH, PA.

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