and note that $W^1(G,\Omega)$ will be a Banach space under the norm (54). We also define $W_0^1(G,\Omega)$ to be the closure of $C_0^1(\Omega)$ in $W^1(G,\Omega)$. Then Theorem 1 and Corollary 1 will hold for $W_0^1(G,\Omega)$, instead of $H^\circ(G,\Omega)$, provided in inequality (18) we replace the function $G_+^*(f)$ by $G_+^*(0,f)$. The imbedding theorem also extends to the spaces $W^1(G,\Omega)$ for certain domains Ω but this situation will be the subject of a further investigation. The imbedding of Theorem 1 will also be compact if the Young function A increases strictly less rapidly than a function which satisfies the hypotheses of the Theorem (see [1]).

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Recognition and limit theorems for L_p-multipliers

by

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Abstract. The main theorem gives a necessary and sufficient condition for a function in $L^\infty(\Gamma)$ to be the Fourier transform of an $L_p(G)$-multiplier. Three applications of this theorem are given: to an extension of Hahn's theorem, to a limit theorem of Lévy type for L_p-multipliers, and to the study of the maximal ideal space of an algebra of L_p-multipliers.

1. Introduction. Let G denote a locally compact abelian (LCA) group with dual group Γ. Let $M_p(G)$ denote the algebra of bounded, translation invariant linear operators on $L_p(G)$, $1\leqslant p<\infty$. It is well known that $M_p(G)=M_{p'}(G)$ when p' is the conjugate index to p and that the inclusion $M_p\subset M_2$ is continuous if $p\leqslant 2$. $M_2(G)$ is isometrically isomorphic with $L^\infty(\Gamma)$ via the Fourier transform and $M_1(G)$ is isometrically isomorphic with $M(G)$, the bounded Borel measures on G by $T(f)=\mu*f$; [10]. An element T of $M_p(G)$ has a Fourier transform $\hat T(\xi)$ which is assigned by letting $\hat T(\xi)$ be the Fourier transform of T regarded as an operator on $L_2(G)$.

In this paper we shall consider the following pair of questions:

(1) *When is $\varphi\in L^\infty(\Gamma)$ the Fourier transform of an operator T in $M_p(G)$?*

(2) *If $\{T_a\}$ is a net of operators in $M_p(G)$, which converges in the weak operator topology over $L_2(G)$, when does $\{T_a\}$ converge in the weak operator topology over $L_p(G)$?*

To answer the first question we shall give a criterion on φ in $L^\infty(\Gamma)$ which is similar to the criterion given in Schoenberg's theorem [3], [11] which characterizes the Fourier transforms of measures. The theorem of Schoenberg says that φ in $L^\infty(\Gamma)$ is the Fourier transform of a bounded Borel measure μ on G if and only if there is a real number $M\geqslant 0$ such that for every H in $L_1(\Gamma)$,

$$\Big|\int_\Gamma \varphi(\gamma)H(\gamma)\,d\gamma\Big|\leqslant M\,\|\hat H\|_\infty;$$

$\hat H$ denotes the Fourier transform of H and $\|\ \|_\infty$ is the sup-norm. If $\varphi=\hat\mu$,

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a simple interchange of integrals verifies the desired inequality with $M = \|\mu\|$. The Riesz representation theorem and the density of $\{\hat{H} \mid H \in L_1(\Gamma)\}$ in $C_0(G)$ readily yield a proof of the converse. Our extension of Schoenberg's theorem will be used to prove a generalization of Hahn's Theorem 5 [7].

To answer the second question posed above, we shall prove an extension of the Lévy continuity theorem for $M_p(G)$, $1 \leqslant p < \infty$; see Section 4. Lévy's continuity theorem ([4], p. 481) says that if a sequence of Borel probability measures $\{\mu_n\}$ on the reals R, have Fourier transforms $\{\hat{\mu}\}$ which converge pointwise to a continuous function φ on R with $\varphi(0) = 1$, then φ is Fourier transform of a Borel probability measure μ and the $\{\mu_n\}$ converge in the weak*-topology of $M(R)$ to μ. If we take Schoenberg's theorem into account, we see that we could state the Lévy theorem as follows: *Let $\{T_n\}$ be a sequence of operators in $M_1(R)$ for which each T_n is given by convolution with a Borel probability measure. Suppose that the $\{T_n\}$ converge in the weak operator topology over $L_2(R)$ to an operator T which is also given by convolution with a Borel probability measure. Then $\{T_n\}$ converges in the weak operator topology over $L_1(R)$ to T.* It is this version of the Lévy theorem which we shall extend.

The classical proofs of the Lévy continuity theorem over R rely heavily on the fact that the unit ball in $M(R)$ is a compact metric space in the weak*-topology. This fact will not enter into our proof of a general theorem of Lévy type. Instead, we shall use the methods of multiplier theory.

In the fifth section of the paper we will use the recognition theorem to study an algebra of L_p-multipliers whose maximal ideal space is closed under Arens multiplication.

2. Preliminaries. We shall denote the algebra of translation invariant, bounded, operators on $L_p(G)$ by $M_p(G)$ for $1 \leqslant p < \infty$; $M_p(G)$ is referred to as the algebra of L_p-multipliers. From [5], $M_p(G)$ is isometrically isomorphic with the dual space of $A_p(G)$; $A_p(G)$ is the algebra of functions in $C_0(G)$ of the form

$$h(y) = \sum_{k=1}^{\infty} f_k * g_k(y), \qquad f_k * g_k(y) = \int_G f_k(x+y) g_k(x)\,dx,$$

where all $f_k \in L_p(G)$, all $g_k \in L_{p'}(G)$, $1/p + 1/p' = 1$, and where

$$\sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_{p'} < \infty.$$

A function h in A_p has the norm

$$|h|_p = \inf\Big\{\sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_{p'} \;\Big|\; h(y) = \sum_{k=1}^{\infty} f_k * g_k(y)\Big\}.$$

An operator T in $M_p(G)$ corresponds to $\tilde{T}$ in $A_p(G)^*$ by

$$\tilde{T}(h) = \sum_{k=0}^{\infty} (Tf_k) * g_k(0).$$

See [6] for a proof that $A_1(G) = C_0(G)$. Furthermore, it is shown in [5] that the weak operator topology on $M_p(G)$, and the weak*-topology on $M_p(G) = A_p(G)^*$ coincide on bounded sets. Regarding the algebras M_p, one should recall that M_p is isometrically isomorphic with $M_{p'}$ when $1/p + 1/p' = 1$ and $1 \leqslant p < \infty$, that the inclusions $M_1 \subset M_p \subset M_r \subset M_2$ are continuous when $1 \leqslant p \leqslant r \leqslant 2$, and that $M_2(G)$ is isometrically isomorphic with $L^\infty(\Gamma)$ via the Fourier transform (Γ denotes the character group of G). For T in $M_p(G)$ we let $\hat{T}(\gamma)$ denote the function in $L^\infty(\Gamma)$ which corresponds to T as an operator in $M_2(G)$. $A_2(G)$ is isometrically isomorphic with the algebra $A(G)$ of Fourier transforms of functions in $L_1(\Gamma)$; $\hat{f}$ in $A(G)$ has the norm $|\hat{f}|_A = \int_\Gamma |f(\gamma)|\,d\gamma$ when $f \in L_1(\Gamma)$. It follows that $A_p = A_{p'}$ and that $A_2(G)$ is dense $A_p(G)$ for $1 \leqslant p \leqslant 2$. See [5]. Herz [8] has shown that $A_p(G)$ is a Banach algebra under pointwise multiplication.

3. A characterization of Fourier transforms. In this section we fix p in $1 \leqslant p \leqslant 2$ and let Γ denote the dual group of the LCA group G. $L_1(\Gamma)$ denotes the convolution algebra of integrable, complex valued functions on Γ. By the duality theorem, G is the character group of Γ; see [10]. Let $\hat{H}(x)$ denote the Fourier transform of H in $L_1(\Gamma)$.

THEOREM 1. *φ in $L^\infty(\Gamma)$ is the Fourier transform of an L_p-multiplier T if and only if there is a non-negative real number M for which*

$$\Big|\int_\Gamma \varphi(\gamma) H(\gamma)\,d\gamma\Big| \leqslant M |\hat{H}|_p$$

for every H in $L_1(\Gamma)$. $|\hat{H}|_p$ denotes the norm of the Fourier transform of H in $A_p(G)$. If $\varphi = \hat{T}$, $\|T\|_p$ is the least constant M for which the inequality holds.

Proof. Let T be an L_p-multiplier and let $H \in L_1(\Gamma)$. Then $\hat{H}$ is in $A_2(G)$ so that there are sequences $\{f_n\}$ and $\{g_n\}$ in $L_2(G)$ such that

$$\hat{H}(x) = \sum_{n=1}^{\infty} f_n * g_n(x) \quad \text{while} \quad \sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 < \infty.$$

Then

$$\tilde{T}(\hat{H}) = \sum_{n=1}^{\infty} (Tf_n) * g_n(0) = \sum_{n=1}^{\infty} \int_\Gamma \hat{T}(\gamma) \hat{f}_n(\gamma) \hat{g}_n(\gamma)\,d\gamma,$$

by Plancherel's theorem. Since $T \in M_p$, $|\tilde{T}(\hat{H})| \leqslant \|T\|_p |\hat{H}|_p$; set $M = \|T\|_p$. Since $H(\gamma) = \sum_{n=1}^{\infty} \hat{f}_n(\gamma)\hat{g}_n(\gamma)$ (a. e.),

$$\tilde{T}(\hat{H}) = \int_\Gamma \hat{T}(\gamma) H(\gamma)\, d\gamma$$

by the dominated convergence theorem.

On the other hand, suppose that φ in $L_\infty(\Gamma)$ satisfies

$$\Big|\int_\Gamma \varphi(\gamma) H(\gamma)\, d\gamma\Big| \leqslant M |\hat{H}|_p$$

for every H in $L_1(\Gamma)$. Define

$$\Phi(\hat{H}) = \int_\Gamma \varphi(\gamma) H(\gamma) d\gamma;$$

then Φ admits a unique continuous, linear extension to all of $A_p(G)$ since $A_2(G)$ is dense in $A_p(G)$. From [5], there is a unique T in $M_p(G)$ such that $\Phi = \tilde{T}$ on $A_p(G)$. By the first part of the proof, $\hat{T}(\gamma) = \varphi(\gamma)$ almost everywhere. This completes the proof.

COROLLARY 1.1. *If $\{\hat{T}_a\}$ is a net of Fourier transforms of $L_p(G)$-multipliers $\{T_a\}$ for which $\|T_a\|_p \leqslant M < \infty$ for all a and for which $\{\hat{T}_a\}$ converges in the weak*-topology of $L^\infty(\Gamma)$ to a function φ, then φ is the Fourier transform of an L_p-multiplier T with $\|T\|_p \leqslant M$.*

Proof. By Theorem 1, for every H in $L_1(\Gamma)$,

$$\Big|\int_\Gamma \hat{T}_a(\gamma) H(\gamma)\, d\gamma\Big| \leqslant M |\hat{H}|_p$$

for every a. The convergence assumption on the $\{\hat{T}_a\}$ says that $\int_\Gamma \hat{T}_a(\gamma) H(\gamma)\, d\gamma$ converges to $\int_\Gamma \varphi(\gamma) H(\gamma)\, d\gamma$ for every H in $L_1(\Gamma)$. Therefore $\big|\int_\Gamma \varphi(\gamma) H(\gamma)\, d\gamma\big| \leqslant M |\hat{H}|_p$, and Theorem 1 implies that there is a $T \in M_p(G)$ with $\|T\|_p \leqslant M$ such that $\hat{T}(\xi) = \varphi(\xi)$ almost everywhere.

This corollary will be useful; we shall use it first to extend Hahn's Theorem 5; see [7]. Let $B_p(\Gamma)$ denote the Banach algebra of continuous functions f on Γ for which $fh \in A_p(\Gamma)$ when $h \in A_p(\Gamma)$. $B_p(\Gamma)$ is the algebra of multipliers of $A_p(\Gamma)$. $B_2(\Gamma)$ consists of the Fourier transforms of measures in $M(G)$ (see [10]) and $B_1(\Gamma)$ consists of the bounded continuous functions on G. We shall equip $B_p(\Gamma)$ with the operator norm:

$$|||f|||_p = \sup\{|fh|_p \mid |h|_p \leqslant 1\}.$$

Note that if $1 \leqslant p \leqslant r \leqslant 2$, a simple approximate identity argument shows that the inclusions $B_2 \subset B_r \subset B_p \subset B_1$ are continuous.

THEOREM 2. *If $1 \leqslant r < \infty$ and if $\left|\frac{1}{r} - \frac{1}{2}\right| \leqslant \left|\frac{1}{p} - 1\right|$, then $B_p(\Gamma)$ consists of Fourier transforms of operators in $M_r(G)$. If T_f in $M_r(G)$ denotes the operator which corresponds to $f \in B_p(\Gamma)$, then $\|T_f\|_r \leqslant |||f|||_p$.*

Proof. In Theorem 5 of [7], Hahn showed that if $1 \leqslant r < \infty$ and if $1 \leqslant p \leqslant 2$ then $h = f*g$ is the Fourier transform of an operator in $M_r(G)$ if $\left|\frac{1}{r} - \frac{1}{2}\right| \leqslant \left|\frac{1}{p} - 1\right|$ for every pair of functions f in $L_p(\Gamma)$ and g in $L_{p'}(\Gamma)$ when $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, he showed that if T_h denotes the operator in $M_r(G)$ which has Fourier transform h, then $\|T_h\|_r \leqslant \|f\|_p \|g\|_{p'}$. By Theorem 1, if $\left|\frac{1}{r} - \frac{1}{2}\right| \leqslant \left|\frac{1}{p} - 1\right|$,

$$\Big|\int_\Gamma h(\gamma) H(\gamma)\, d\gamma\Big| \leqslant \|f\|_p \|g\|_{p'} |\hat{H}|_r$$

for every $H \in L_1(\Gamma)$. If $h(\gamma) = \sum_{k=1}^{\infty} f_k * g_k(\gamma)$ is any element of $A_p(\Gamma)$, it follows from Theorem 1 that

$$\Big|\int_\Gamma h(\gamma) H(\gamma)\, d\gamma\Big| \leqslant |h|_p |\hat{H}|_r$$

for every H in $L_1(\Gamma)$. Fix r in $\left|\frac{1}{r} - \frac{1}{2}\right| \leqslant \left|\frac{1}{p} - 1\right|$. $A_2(\Gamma)$ is dense in $A_p(\Gamma)$ and the inclusion is continuous. This implies that if $\{e_a\}$ is a bounded approximate identity on $L_1(G)$, then $\{\hat{e}_a\}$ is a bounded approximate identity in $A_p(\Gamma)$. By Theorem 31E of [9], $A_2(\Gamma)$ contains an approximate identity of norm 1, $\{\hat{e}_a\}$, which consists of positive definite functions. Thus if f is in $B_p(\Gamma)$, the net $\{\hat{e}_a f\}$ converges to f uniformly over each compact subset of Γ. Fix H in $L_1(\Gamma)$ and let $\varepsilon > 0$. Choose a compact set K in Γ for which $\int_{\Gamma - K} |H(\gamma)|\, d\gamma < \varepsilon$. Then

$$\begin{aligned}\Big|\int_\Gamma f(\gamma) H(\gamma)\, d\gamma - \int_\Gamma f(\gamma)\hat{e}_a(\gamma) H(\gamma)\, d\gamma\Big| \\ \leqslant \Big|\int_K \big(f(\gamma) - f(\gamma)\hat{e}_a(\gamma)\big) H(\gamma)\, d\gamma\Big| + 2\varepsilon \|f\|_\infty.\end{aligned}$$

There is an a_0 such that $a \geqslant a_0$ implies that first integral on the right is dominated by ε. Therefore $\{fe_a\}$ converges in the weak*-topology of $L^\infty(\Gamma)$ to f. By Corollary 1.1, f is the Fourier transform of an operator in $M_r(G)$. Since $\|T_{f\hat{e}_a}\|_r \leqslant |fe_a|_p \leqslant |||f|||_p$ for every $\|T_f\|_r \leqslant |||f|||_p$. This completes the proof.

Remark. The argument used to prove the previous theorem indicates a general approach that can be used to identify functions in $L^\infty(\Gamma)$ which are Fourier transforms of operators in $M_r(G)$. Let A be a Banach algebra of bounded functions on Γ which contains a bounded approximate identity, whose elements can be identified as Fourier transforms of operators in $M_r(G)$ with $\|T_f\|_r \leqslant |f|_A$, and for which $\|h\|_\infty \leqslant |h|_A$ for $h \in A$. Then if B denotes the subalgebra of $L^\infty(\Gamma)$ which acts as multipliers on A, each function f in B is the Fourier transform of an operator T_f in $M_r(G)$ with $\|T_f\|_r \leqslant |f|_B$.

Remark. Theorem 1 and Corollary 1.1 extend immediately to the range $1 \leqslant p < \infty$. But since $M_p = M_{p'}$ when $\frac{1}{p}+\frac{1}{p'} = 1$, this is obvious.

4. Limit theorems. In this section we shall study nets of L_p-multipliers which converge in the weak operator topology over $L_2(G)$. Assume that $1 \leqslant p < \infty$.

THEOREM 3. *Let $\{T_a\}$ be a bounded net of $L_p(G)$-multipliers, $\|T_a\|_p \leqslant M < \infty$ for every a, which converges in the weak operator topology over $L_2(G)$. If T denotes the limit in $M_2(G)$, then $T \in M_p(G)$ and $\{T_a\}$ converges in the weak operator topology over $L_p(G)$ to T.*

Proof. Assume at first that $1 \leqslant p \leqslant 2$. Regard $M_p(G)$ as $A_p(G)^*$ as we may by [5] and recall that $A_2(G)$ is densely and continuously included in $A_p(G)$. If $f, g \in L_2(G)$, $\int_G (T_a f)(x) g(x)\,dx$ converges to $\int_G (Tf)(x) g(x)\,dx$ since $\{T_a\}$ converges to T in the weak operator topology over $L_2(G)$. Thus $\int_\Gamma \hat{T}_a(\gamma)\hat{f}(\gamma)\hat{g}(\gamma)\,d\gamma$ converges to $\int_\Gamma \hat{T}(\gamma)\hat{f}(\gamma)\hat{g}(\gamma)\,d\gamma$. Every H in $L_1(\Gamma)$ can be expressed as the product of a pair of functions in $L_2(\Gamma)$. Thus $\{\hat{T}_a\}$ converges to $\hat{T}$ in the weak*-topology of $L^\infty(\Gamma)$. By Corollary 1.1, $\hat{T}$ is the Fourier transform of an operator in $M_p(G)$. For every h in $A_2(G)$, $\tilde{T}_a(h)$ converges to $\tilde{T}(h)$. Since $A_2(G)$ is dense in $A_p(G)$ and since $\{\tilde{T}_a\} \cup \{\tilde{T}\}$ is a bounded set in $A_p(G)^*$, an $\varepsilon/3$-argument shows that $\{T_a\}$ converges to T in the weak operator topology over $L_p(G)$.

If $2 < p < \infty$, the net of adjoints $\{T_a^*\}$ in $M_{p'}(G)$, $1 < p' < 2$, converges in the weak operator topology over $L_{p'}(G)$ to T^*. Thus $\{T_a\}$ converges to T in the weak operator topology over $L_p(G)$. But $\{T_a\} \cup \{T\}$ is a bounded set and the weak* and weak operator topologies agree on bounded subsets of $M_p(G) = A_p(G)^*$. This completes the proof.

It is proved in [6] that $C_0(G) = A_1(G)$, and it is not hard to prove that the weak*-topology on $M(G) = M_1(G)$ agrees with the weak operator topology on bounded subsets of $M_1(G)$.

COROLLARY 3.1. *If $\{\mu_n\}$ is a bounded sequence of Borel measures on G whose Fourier transforms converge almost everywhere to a function φ, then φ is equal almost everywhere to the Fourier transform of a measure μ with $\|\mu\| \leqslant \sup_n \|\mu_n\|$. Furthermore, $\{\mu_n\}$ converges in the weak*-topology of $M(G)$ to μ.*

Proof. Since $M(G)$ is isometrically isomorphic to $M_1(G)$ and since the weak*-topology and weak operator topology agree on bounded subsets of $M_1(G)$, the conclusion follows from Theorem 3. For since $\{\hat{\mu}_n\}$ converges pointwise to φ while $\|\mu_n\| \leqslant M < \infty$, the dominated convergence theorem and Plancherel's theorem imply that the operations on $L_2(G)$ of convolution by the μ_n converge in the weak operator topology to the operator T with $\hat{T} = \varphi$.

COROLLARY 3.2. *Let $\{\mu_n\}$ be a sequence of Borel probability measures on G whose Fourier transforms converge almost everywhere on Γ to a function φ which, after correction on a set of Haar measure zero, is continuous on a neighborhood of 0 in Γ and satisfies $\varphi(0) = 1$. Then φ is equal almost everywhere to the Fourier transform of a Borel probability measure μ on G. The sequence $\{\mu_n\}$ converges in the weak*-topology of $M(G)$ to μ.*

Proof. Except for the fact that μ is a probability measure, the conclusion of the corollary follows from Corollary 3.1. $\hat{\mu}$ is a positive definite function since it is the pointwise limit of positive definite functions. Since $\hat{\mu}(0) = 1$, Bochner's representation theorem ([10], p. 19) together with the uniqueness theorem for Fourier transforms of measures implies that μ is a Borel probability measure on G.

Corollary 3.2 is the classical Lévy continuity theorem.

COROLLARY 3.3. *Let $\{T_a\}$ be a bounded net of $L_p(G)$-multipliers and suppose that $\{T_a\}$ converges in the weak operator topology over $L_q(G)$ for some q in $1 \leqslant q < \infty$. If T denotes the limit of the T_a in $M_q(G)$, then T is in $M_p(G)$ and $\{T_a\}$ converges to T in the weak operator topology over $L_p(G)$.*

Proof. Since $A_2(G)$ is dense in $A_q(G)$, the functionals $\{\tilde{T}_a\}$ satisfy $\lim_a \tilde{T}_a(h) = \tilde{T}(h)$ for every h in $A_2(G)$. Thus $\{T_a\}$ converges to T in the weak operator topology over $L_2(G)$. Theorem 3 now implies that $\{T_a\}$ converges to T in the weak operator topology over $L_p(G)$.

5. An algebra of L_p-multipliers. For any operator T in $M_p(G)$ and for any $\gamma \in \Gamma$, let T_γ denote the operator in $M_p(G)$ defined by $T_\gamma(f)(x) = \gamma(x)^{-1} T(\gamma f)(x)$. Then $\|T_\gamma\|_p = \|T\|_p$ and T_γ has Fourier transform $\hat{T}_\gamma(\xi) = \hat{T}(\gamma\xi)$. Thus Γ defines a group of isometries on $M_p(G)$. Let $CM_p(G)$ denote the subalgebra of $M_p(G)$ which consists of operators T for which $\gamma \to T_\gamma$ is strongly continuous; i. e. $\|T_\gamma - T\|_p \to 0$ as $\gamma \to 1$. Γ is regarded as a multiplicative group in this section. $CM_p(G)$ is a Banach algebra in the operator norm. Since $\|\hat{T}\|_\infty = \|T\|_2 \leqslant \|T\|_p$, it follows that every element of $CM_p(G)$ has a uniformly continuous Fourier transform. It is worth noting that $CM_2(G)$ consists of precisely those operators in

$M_2(G)$ whose Fourier transforms are uniformly continuous. It is almost as immediate that $CM_1(G) = M_1(G)$.

We need some technical information in order to prove the theorem of this section which will show that the maximal ideal space Λ_p of $CM_p(G)$ is closed under the Arens product on $CM_p(G)^*$. Let $BC_p(\Gamma)$ denote the algebra of bounded linear operators on $CM_p(G)$ which commute with the action of Γ on $CM_p(G)$; i.e. $\Phi \in BC_p(\Gamma)$ if and only if Φ is a bounded linear operator on $CM_p(G)$ which satisfies $\Phi(T)_\gamma = \Phi(T_\gamma)$ for every γ in Γ.

LEMMA 5.1. *$BC_p(\Gamma)$ is isometrically isomorphic with $CM_p(G)^*$ by the map $\Phi \to \varphi$ where $\varphi(T_\gamma) = \Phi(T)\hat{}\,(\gamma)$.*

Proof. If $\Phi \in BC_p(\Gamma)$, then the map $T \to \Phi(T)\hat{}\,(1)$ is a bounded linear functional φ on $CM_p(G)$ since

$$|\varphi(T)| = |\Phi(T)\hat{}\,(1)| \leqslant \|\Phi(T)\hat{}\,\|_\infty \leqslant \|\Phi(T)\|_p \leqslant \|\Phi\|\,\|T\|_p.$$

Thus $\|\varphi\| \leqslant \|\Phi\|$.

On the other hand let $\varphi \in CM_p(G)^*$. Let $H \in L_1(\Gamma)$ and consider the integral

$$\int_\Gamma \varphi(T_\gamma)H(\gamma)\,d\gamma = \varphi\Big(\int_\Gamma T_\gamma H(\gamma)\,d\gamma\Big).$$

If $k \in A_2(G)$, $\int_\Gamma \tilde{T}_\gamma(k)H(\gamma)\,d\gamma = \tilde{T}(k\hat{H})$ since $\tilde{T}_\gamma(k) = \tilde{T}(\gamma k)$ where $(\gamma k)(x) = \gamma(x)k(x)$. Thus $\int_\Gamma T_\gamma H(\gamma)\,d\gamma = \tilde{u} \in A_p(\Gamma)^*$ where $\tilde{u}(k) = \tilde{T}(k\hat{H})$ for every $k \in A_p(\Gamma)$. Since $\|\tilde{u}\|_p \leqslant \|T\|_p|\hat{H}|_p$, $|\varphi(\tilde{u})| \leqslant \|\varphi\|\,\|T\|_p|\hat{H}|_p$. So $\Phi(T)$, $\Phi(T)\hat{}\,(\gamma) = \varphi(T_\gamma)$, is in $M_p(G)$ by Theorem 1. Furthermore, it follows that $\|\Phi\| \leqslant \|\varphi\|$. For $\gamma_1 \in \Gamma$, $\Phi(T)_{\gamma_1}$ has Fourier transform $\Phi(T)\hat{}\,(\gamma\gamma_1) = \varphi(T_{\gamma\gamma_1}) = \Phi(T_{\gamma_1})\hat{}\,(\gamma)$; thus $\Phi \in BC_p(\Gamma)$ and $\gamma \to \Phi(T)_\gamma = \Phi(T_\gamma)$ is continuous when $T \in CM_p(G)$. This completes the proof.

LEMMA 5.2. *The algebraic homomorphisms of $CM_p(G)$ in $BC_p(\Gamma)$ correspond to the complex homomorphisms of $CM_p(G)$ in $CM_p(G)^*$ under the map $\Phi \to \varphi$, $\varphi(T_\gamma) = \Phi(T)\hat{}\,(\gamma)$.*

Proof. Let Λ_p denote the space of complex homomorphisms of $CM_p(G)$. Then if $\delta \in \Lambda_p$, let $\Delta(T)\hat{}\,(\gamma) = \delta(T_\gamma)$. Then $\Delta(TS)\hat{}\,(\gamma) = \delta([TS]_\gamma) = \delta(T_\gamma S_\gamma) = \delta(T_\gamma)\delta(S_\gamma) = \Delta(T)\hat{}\,(\gamma)\Delta(S)\hat{}\,(\gamma)$, so that Δ is an algebraic homomorphism of $CM_p(G)$. If $\Delta \in BC_p(\Gamma)$ is multiplicative, then $\delta(T) = \Delta(T)\hat{}\,(1)$ is multiplicative since

$$\delta(TS) = \Delta(TS)\hat{}\,(1) = \Delta(T)\hat{}\,(1)\Delta(S)\hat{}\,(1) = \delta(T)\delta(S).$$

DEFINITION. If $\delta_1, \delta_2 \in \Lambda_p$ and if $\delta_i(T_\gamma) = \Delta_i(T)\hat{}\,(\gamma)$ for $i = 1, 2$, define the product $\delta_1\delta_2$ of δ_1 and δ_2 to be that complex homomorphism of $CM_p(G)$ which corresponds to the composition of Δ_1 and Δ_2 in $BC_p(\Gamma)$.

Now we need more succinct notation for the correspondence between elements of $BC_p(\Gamma)$ and elements of $CM_p(G)^*$. $CM_p(G)$ is a closed subspace of $A_p(G)^*$ so that $CM_p(G)^*$ is isometrically isomorphic with $A_p(G)^{**}/CM_p(G)^\perp$ where $CM_p(G)^\perp$ denotes the annihilator of $CM_p(G)$ in $M_p(G)^* = A_p(G)^{**}$. Thus we may regard $CM_p(G)^*$ as a space of equivalence classes of functionals in $A_p(G)^{**}$ where φ and ψ in $A_p(G)^{**}$ define the same member of $CM_p(G)^*$ if and only if $(\varphi - \psi)$ vanishes on $CM_p(G)$.

Define the Arens product on $M_p(G)^*$ in the following way: (1) if $h, k \in A_p(G)$, $(hk)(x) = h(x)k(x)$; if $T \in A_p(G)^* = M_p(G)$, set $T \circ h(k) = T(hk)$; if $\varphi \in M_p(G)^*$ and $T \in M_p(G)$, set $\varphi \circ T(h) = \varphi(T \circ h)$; if $\varphi, \psi \in M_p(G)^*$, set $\varphi \circ \psi(T) = \varphi(\psi \circ T)$. See [1] where the Arens product is studied and applied to the analysis of $L^\infty(G)^* = L_1(G)^{**}$. To define the product $\varphi \circ \psi$ in $CM_p(G)^*$, we have to show that when $T \in CM_p(G)$ and $\varphi \in M_p(G)^*$, then $\varphi \circ T$ is in $CM_p(G)$. Note that $(\varphi \circ T)_\gamma(h) = \varphi \circ T(\gamma h) = \varphi\big(T \circ (\gamma h)\big)$ and that $T \circ (\gamma h)(k) = T(\gamma hk) = T_\gamma(hk)$, thus $(\varphi \circ T)_\gamma = \varphi \circ T_\gamma$. Since $\gamma \to T_\gamma$ is continuous for T in $CM_p(G)$ and since $\|\varphi \circ T\|_p \leqslant \|\varphi\|\,\|T\|_p$, $(\varphi \circ T)_\gamma$ is continuous in γ. Thus if $T \in CM_p(G)$ and if $\varphi \in M_p(G)^*$, then $\varphi \circ T \in CM_p(G)$. Therefore, we may define the product of φ and ψ in $CM_p(G)^*$ by regarding them as elements of $M_p(G)^*$, forming the functional $\varphi \circ \psi$ and then restricting this functional to $CM_p(G)$. Thus $CM_p(G)^*$ is a Banach algebra with multiplication $\circ$ which contains a right identity since $A_p(G)$ has a bounded approximate identity. $CM_p(G)^*$ need not be commutative. See [1] for the basic properties of the Arens product.

LEMMA 5.3. *Suppose that $\varphi \in CM_p(G)^*$, that $\Phi \in BC_p(\Gamma)$, and that these mappings satisfy $\Phi(T)\hat{}\,(\gamma) = \varphi(T_\gamma)$. Then $\Phi(T) = \varphi \circ T$.*

Proof. For $h \in A_2(G)$, let $H \in L_1(\Gamma)$ satisfy $\hat{H}(x) = h(x)$. Then $U = \int_\Gamma T_\gamma H(\gamma)\,d\gamma$ exists and for any function k in $A_p(G)$,

$$U(k) = T(kh) = \int_\Gamma T(\gamma k)H(\gamma)\,d\gamma.$$

Thus $\Phi(T)(h) = (\varphi \circ T)(h)$ for a dense subset of $A_p(G)$. Thus $\Phi(T) = \varphi \circ T$

THEOREM 4. *If δ_1 and δ_2 are complex homomorphisms of $CM_p(G)$, then their product $\delta_1\delta_2$ satisfies $\delta_1\delta_2 = \delta_1 \circ \delta_2$. Λ_p is closed under the Arens product.*

Proof. By definition, $\delta_1\delta_2$ is that complex homomorphism of $CM_p(G)$ which corresponds to the composition $\Delta_1\Delta_2$ of the homomorphisms Δ_1 and Δ_2 of $CM_p(G)$. Here $\delta_i(T_\gamma) = \Delta_i(T)\hat{}\,(\gamma)$ for $i = 1, 2$. By Lemma 5.3, $\Delta_1\big(\Delta_2(T)\big) = \Delta_1(\delta_2 \circ T) = \delta_1 \circ (\delta_2 \circ T)$ for every $T \in CM_p(G)$. But $\delta_1 \circ (\delta_2 \circ T) = (\delta_1 \circ \delta_2) \circ T$. Since $\Delta_1\big(\Delta_2(T)\big) = \delta_1\delta_2 \circ T$, we have the identity, $\delta_1\delta_2 \circ T = (\delta_1 \circ \delta_2) \circ T$ for every $T \in CM_p(G)$. Since $\Delta_1\Delta_2(T)\hat{}\,(1) = \delta_1\delta_2(T) = \delta_1 \circ \delta_2(T)$, it follows that $\delta_1\delta_2 = \delta_1 \circ \delta_2$. This completes the proof.

We hope to study the maximal ideal spaces of $CM_p(G)$ and of $M_p(G)$ in detail later.

6. A systems approach to $M_p(G)$. As the material in the preceding sections indicates, $M_p(G)$ can be studied from the viewpoint that $A_p(G)$, $M_p(G) = A_p(G)^*$, $M_p(G)^* = A_p(G)^{**}$ is an interrelated system of algebras. It can be shown that the algebra $B_p(G)$ of multipliers of $A_p(G)$ can be regarded as a distinguished subalgebra of $A_p(G)^{**}$. $A_p(G)$ is rich in structure since it contains the same approximate identities and local units as the Fourier algebra $A_2(G)$. For example, this implies that the Wiener–Lévy theorem, Ditkin's condition, and general forms of Wiener's (Tauberian) theorem all hold in $A_p(G)$; see [8]. We hope to pursue the study of this systems approach to $M_p(G)$ in detail later. We shall close this paper with one result which originates with this viewpoint.

Recall that if μ is a finite Borel measure on Γ, then for every T in $M_p(G)$, $\int_\Gamma \hat{T}(x+y)\,d\mu(y)$ is the Fourier transform of an operator $T^{\hat{\mu}}$ in $M_p(G)$ which can be defined as follows: If g is a continuous function with compact support, then

$$T^{\hat{\mu}}(g)(x) = \tilde{T}_y\big(g(x+y)\hat{\mu}(y)\big)$$

when $\tilde{T}$ is regarded as a functional on $A_2(G)$ which satisfies $|\tilde{T}(h)| \leqslant M|h|_p$ for some non-negative real number M. Direct computation shows that $\|T^{\hat{\mu}}\|_p \leqslant \|T\|_p\|\mu\|$; $\|\mu\|$ is the variation of μ. We shall prove

THEOREM 5. *Let $T \in M_p(G)$ and let $f \in B_p(G)$, the multipliers on $A_p(G)$. If g is a continuous function with compact support in G and if T^f is defined by $T^f(g)(x) = \tilde{T}_y\big(g(x+y)f(y)\big)$, then $T^f \in M_p(G)$ and $\|T^f\|_p \leqslant \|T\|_p|||f|||_p$.*

Proof. Notice that the functional $\tilde{T}^f$ on $A_p(G)$ satisfies $\tilde{T}^f(h) = \tilde{T}(fh)$ for every h in $A_p(G)$. Thus $T \to T^f$ is just the adjoint of the map $h \to fh$. This implies that $\|T^f\|_p \leqslant \|T\|_p|||f|||_p$ where $|||f|||_p$ is the norm of f as a multiplier on $A_p(G)$.

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