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On finite dimensional subspaces of Banach spaces with local unconditional structure

by

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Abstract. If X is a Banach space which has local unconditional structure, then X is super-reflexive or X contains subspaces uniformly isomorphic to l_∞^n for all n or X contains uniformly complemented subspaces uniformly isomorphic to l_1^n for all n. Thus if either X or Y has local unconditional structure, there is a compact, non-nuclear operator from X to Y.

Introduction and notation. The first result stated in the abstract is a local (i.e., finite dimensional) version of the theorem of James' [13] that if X is a non-reflexive Banach space with unconditional basis, then X contains a complemented subspace isomorphic to either c_0 or l_1. The result gives new information on the well-known problem whether every infinite dimensional non-reflexive Banach space contains a uniformly complemented sequence (E_n) of subspaces with

$$d(E_n, l_\infty^n) \to 1 \quad \text{or} \quad d(E_n, l_1^n) \to 1.$$

(Here $d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\|: T: X \overset{\text{onto}}{\to} Y \text{ is an isomorphism}\}$.)

In [11] Grothendieck raised the problem whether there exist infinite dimensional Banach spaces X and Y for which every (bounded, linear) operator $T: X \to Y$ is nuclear. The results herein combine with the main result of [3] to yield that if X and Y are infinite dimensional Banach spaces and either X or Y has local unconditional structure (l.u.st., in short) then there is a compact, non-nuclear operator from X to Y.

Before explaining the terminology used, we mention the organization of the paper. Section II contains a listing of known results needed later. Some proofs are given for completeness. The specialist in Banach space theory can skip all but the first two paragraphs of this section (where the definition of admissible choice of signs is given).

Section III contains preliminary results. The main fact proved here is that if X has an unconditional basis but does not contain l_∞^n uniformly

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for large n, then there is $p < \infty$ and an equivalent norm $\|\cdot\|$ on X so that $\|\sum y_n\| \geqslant (\sum \|y_n\|^p)^{1/p}$ for any $y_n \in X$ with (y_n) having pairwise disjoint supports relative to the unconditional basis. Furthermore, the basisis unconditionally monotone relative to $\|\cdot\|$.

After the work on this paper had been completed, T. Figiel pointed out to me that a recent generalization by Maurey [21] of a result of Rosenthal's [23] easily yields that if the space X does not contain l_∞^n uniformly for all n, then the identity operator on X is $(p, 1)$ absolutely summing (cf. [19]) for some $p < \infty$. This result can be used instead of Corollary III. 4 to prove the renorming lemma Corollary III.5 for those spaces X which do not contain l_∞^n uniformly for all n. This slightly weaker version of Corollary III.5 can be used in the proofs in Section IV, and thus the reader familiar with the Rosenthal–Maurey chain of ideas can skip the material in Section III preceding Corollary III.5.

The main results are in Section IV. Here we show that if X has l.u.st., then X is super-reflexive or X contains a sequence (E_n) of subspaces satisfying either $d(E_n, l_\infty^n) \to 1$, or $d(E_n, l_1^n) \to 1$ and E_n is $1+1/n$ complemented in X. Actually, the techniques prove a little more; namely, that if X is a non-super-reflexive subspace of a space Y which has l.u.st. and $d(G_n, l_\infty^n) \to \infty$ for any sequence (G_n) of subspaces of Y, then $X \supset E_n$ with $d(E_n, l_1^n) \to 1$ and E_n $1+1/n$ complemented in X. This last result and the two problems mentioned already lead naturally to the (almost certainly difficult) embedding problem mentioned at the end of the paper.

After this paper was submitted for publication, T. Figiel and I proved that if X has l.u.st. then X^{**} is isomorphic to a complemented subspace of a Banach lattice, Y, and Y is finitely representable in X. From this result and results of Tzafriri [26] it follows easily that if X has l.u.st., X is not reflexive (resp., X is not super-reflexive), and X does not contain l_∞^n for large n, then X contains a complemented copy of l_1 (resp., X contains subspaces uniformly isomorphic to l_1^n for large n). However, the argument for this result (which is rather easier than the arguments used in the present paper) does not seem to yield the existence of uniformly complemented l_1^n's in those non-super-reflexive spaces with l.u.st. which do not contain l_∞^n's for large n.

Most of the terminology is standard for Banach space theory. "Space" means real Banach space, but similar results for the case of complex scalars can be deduced from the real case. Operators are bounded and linear; subspaces are closed; $[A]$ is the closed linear span of the set A. A subspace X of Y is complemented if there is a projection (= idempotent operator) P from Y onto X. If $\|P\| \leqslant K$, X is K-complemented.

We use standard facts concerning basic sequences. A sequence $(e_n) \subset X$ is called basic if there are functionals (e_n^*) in X^* biorthogonal to (e_n) such that $x = \sum e_n^*(x) e_n$ for every $x \in [e_n]$. If the series expansion converges unconditionally for each $x \in [e_n]$, the basic sequence is called unconditional. It is known that (e_n) is unconditional iff there is a constant K such that $\|\sum \alpha_n e_n\| \leqslant K \|\sum \beta_n e_n\|$ whenever $|\alpha_n| \leqslant |\beta_n|$. The smallest such constant is denoted by $\mathscr{U}(e_n)$ and is called the unconditional constant of the basic sequence (e_n). If $\mathscr{U}(e_n) = 1$, (e_n) is said to be unconditionally monotone.

Given $x, y \in [e_n]$, we say $x \leqslant y$ if $e_n^*(x) \leqslant e_n^*(y)$ for all n (where (e_n^*) are the functionals biorthogonal to the basis (e_n)). Also, for E a set of integers, $P_E\colon [e_n] \to [e_n]_{n \in E}$ is the projection defined by $P_E x = \sum_{n \in E} e_n^*(x) e_n$. It is clear that $\|P_E\| \leqslant \mathscr{U}(e_n)$. If $x \in [e_n]$, the support $\operatorname{supp} x$ of x is the set $\{n\colon e_n^*(x) \neq 0\}$. A sequence x_n in $[e_n]$ is a block basic sequence of (e_n) provided the supports of (x_n) are pairwise disjoint and finite. Obviously $\mathscr{U}(x_n) \leqslant \mathscr{U}(e_n)$ for any block basic sequence (x_n) of (e_n). Note that our definition of block basic sequence is non-standard. However, (x_n) is a block basic sequence of (e_n) in our sense iff (x_n) is a block basic sequence in the usual sense of $(e_{p(n)})$ for some permutation $(e_{p(n)})$ of (e_n).

Basic sequences (e_n) and (x_n) are said to be K-equivalent provided the map $e_n \to x_n$ extends to an operator $T\colon [e_n] \to [x_n]$ satisfying

$$\max(1, \|T\|) \max(1, \|T^{-1}\|) \leqslant K.$$

We come now to the concept (introduced by Dubinsky, Pełczyński and Rosenthal in [4]) of local unconditional structure (l.u.st.): X has l.u.st. provided there is a constant K and a family (E_a) of finite dimensional subspaces of X, directed by inclusion, with $X = \overline{\bigcup E_a}$ and such that each E_a has a basis $(e_n^a)_{n=1}^{d(a)}$ with $\mathscr{U}(e_n^a) \leqslant K$. The infimum over all such constants K is denoted by $\mathrm{LU}(X)$ and is called the local unconditional constant of X.

The common spaces have l.u.st. For example, $\mathrm{LU}(X) = 1$ if X is a Banach lattice. However, Gordon and Lewis [10] have recently given examples of spaces which are not complemented in any space which has l.u.st.

Finally, we mention James' concept of super-reflexivity [15]: X is super-reflexive provided that every Banach space which is finitely representable in X is reflexive. (Y is finitely representable in X provided that for each $\varepsilon > 0$ and each finite dimensional subspace E of Y, there is a subspace F of X for which $d(E, F) \leqslant 1+\varepsilon$.) James [15] proved that super-reflexivity is invariant under isomorphism and Enflo [5] showed that X is super-reflexive if and only if X is isomorphic to a uniformly convex space. Of course, there are reflexive spaces with l.u.st. which are not super-reflexive, so the main theorem we prove gives new information even for spaces which have unconditional bases.

II. Background material. To construct l_1^n's in a non-super-reflexive space with l.u.st. we use an extension by James [15] (cf. [24] and [2] for alternate proofs) of one of his earlier theorems [14]. Before stating this result, we introduce notation that is used often: A choice $(\varepsilon_i)_{i=1}^n$ of signs is called *admissible* provided all + signs precede all minus signs. More formally,

Definition II.1. Suppose $\varepsilon_i = \pm 1$ for $1 \leqslant i \leqslant n$. $(\varepsilon_i)_{i=1}^n$ is called *admissible* provided that there exists k, $1 \leqslant k \leqslant n$, so that $\varepsilon_i = 1$ for $1 \leqslant i \leqslant k$ and $\varepsilon_i = -1$ for $k < i \leqslant n$.

James' theorem is:

Proposition II.2. *Suppose X is not super-reflexive, n is an integer, and $\varepsilon > 0$. Then there exist unit vectors $(x_i)_{i=1}^n$ in X so that $\|\sum_{i=1}^n \varepsilon_i x_i\| \geqslant n(1-\varepsilon)$ for every admissible choice (ε_i) of signs and there is a norm one functional x^* in X^* with $x^*(x_i) \geqslant (1-\varepsilon)$ for each $1 \leqslant i \leqslant n$.*

Remark II.3. Proposition II.2 is usually stated for X not reflexive. The extension to the non-super-reflexive case is immediate. Also, x^* does not appear in the usual formulation. However, the inequality $\|\sum_{i=1}^n x_i\| \geqslant n(1-\varepsilon)$ is derived by using a functional which satisfies the condition on x^*.

We shall need later the following elementary lemma regarding admissible choices of signs.

Lemma II.4. *If $(a_i)_{i=1}^k$ is a real sequence, then there is an admissible choice (ε_i) of signs so that $|\sum_{i=1}^k \varepsilon_i a_i| \leqslant \max |a_i|$.*

Proof. Assume, without loss of generality, that $\sum_{i=1}^k a_i \geqslant 0$. Let m be the first positive integer satisfying $\sum_{i=1}^m a_i \geqslant \frac{1}{2}\sum_{i=1}^k a_i$. For notational ease, set $A = \sum_{i=1}^k a_i$, $B = \sum_{i=1}^m a_i$, $C = \sum_{i=m+1}^k a_i$.

Now
$$B \geqslant \tfrac{1}{2}A = \tfrac{1}{2}(B+C) > B - a_m,$$
so
$$\tfrac{1}{2}B \geqslant \tfrac{1}{2}C > \tfrac{1}{2}B - a_m \quad \text{and} \quad B - C \geqslant 0 > (B-C) - 2a_m.$$
Thus
$$0 \leqslant B - C < 2a_m.$$
If $B - C \leqslant a_m$, we are finished, otherwise
$$a_m < B - C < 2a_m,$$
so
$$-a_m < (B-C) - 2a_m < a_m.$$
But $(B-C) - 2a_m = \sum_{i=1}^{m-1} a_i - \sum_{i=m}^k a_i$, and thus the proof is complete.

Next we mention some known facts concerning l_1^n and l_∞^n that are needed in the sequel. The qualitative part of Proposition II.5 goes back to James [14] (see also Giesy's paper [8] for the l_1 part). A nice (but unpublished) proof of the theorem as stated was given by Rosenthal and Lindenstrauss. Figiel discovered a similar proof of (b) independently; his proof is sketched in [7].

Proposition II.5. (a) *If $(x_i)_{i=1}^{n^2}$ is a normalized basic sequence which is K equivalent to the unit vector basis for $l_1^{n^2}$, then there is a normalized block basic sequence of (x_i) which is $K^{1/2}$ equivalent to the unit vector basis for l_1^n.*

(b) *Given $\varepsilon > 0$, $K < \infty$, and an integer n, there is an integer $q = q(\varepsilon, K, n)$ so that if $(x_i)_{i=1}^q$ is a normalized basic sequence which is K equivalent to the unit vector basis for l_∞^q, then (x_i) has a normalized block basic sequence which is $1+\varepsilon$ equivalent to the unit vector basis for l_∞^n. (In fact, given any $\varepsilon > 0$ and $K < \infty$, $q(\varepsilon, K, n)n^{-c} \to 0$ as $n \to \infty$ for some constant $c = c(\varepsilon, K)$.)*

We also need the fact (cf. [13]) that a block basic sequence of an unconditional basis spans a complemented subspace, if the block basic sequence is equivalent to the unit vector basis of l_1. Since the proof is short, we include it.

Proposition II.6. *Suppose (e_n) is an unconditionally monotone basis for X and $(y_j)_{j=1}^n$ is a normalized block basic sequence which is λ equivalent to the unit vector basis for l_1^n. Then $[y_j]_{j=1}^n$ is λ complemented in X.*

Proof. By the properties of the unit vector basis for l_1^n and the Hahn–Banach theorem, there is $x^* \in X^*$ with $\|x^*\| \leqslant \lambda$ and $x^*(y_j) = 1$ for $1 \leqslant j \leqslant n$. Let $E(j) = \operatorname{supp} y_j$. Define $P\colon X \to [y_j]$ by $Px = \sum_{j=1}^n x^*(P_{E(j)}x)y_j$. Since the $E(j)$'s are pairwise disjoint, P is a projection onto $[y_j]$. Further, for $x \in X$,
$$\begin{aligned}\|Px\| &\leqslant \sum_{j=1}^n |x^*(P_{E(j)}x)| = x^*\Big(\sum_{j=1}^n \operatorname{sign}[x^*(P_{E(j)}x)]P_{E(j)}x\Big)\\ &\leqslant \|x^*\|\Big\|\sum_{j=1}^n \operatorname{sign}[x^*(P_{E(j)}x)]P_{E(j)}x\Big\|\\ &= \|x^*\|\Big\|\sum_{j=1}^n P_{E(j)}x\Big\| \quad \text{by unconditional monotonicity}\\ &\leqslant \lambda\|x\|.\end{aligned}$$
Thus $\|P\| \leqslant \lambda$.

The final background result about l_1^n is a Bessaga–Pełczyński [1]-type perturbation lemma.

LEMMA II.7. *Suppose $(x_i)_1^n \subset X$ is λ-equivalent to the unit vector basis for l_1^n, $\|x_i\| \leqslant 1$, $\|x_i - y_i\| < \delta < 1$ for each i, $\|y_i\| \leqslant 1$, and there is a projection $P\colon X \overset{\text{onto}}{\to} [x_i]$ with $\|P\| \leqslant \lambda$. Assume $\delta\lambda < 1$. Then (y_i) is $\lambda(1-\delta\lambda)^{-1}$ equivalent to the unit vector basis for l_1^n, and there is a projection $Q\colon X \overset{\text{onto}}{\to} [y_i]$ with $\|Q\| \leqslant (1+\delta\lambda)(1-\delta\lambda)^{-1}\lambda$.*

Proof. We have, for any scalars $(a_i)_{i=1}^n$,

$$\sum |a_i| \geqslant \Big\| \sum a_i y_i \Big\| \geqslant \Big\| \sum a_i x_i \Big\| - \Big\| \sum a_i (y_i - x_i) \Big\|$$
$$\geqslant \frac{1}{\lambda} \sum |a_i| - \sum |a_i| \delta = \frac{1-\lambda\delta}{\lambda} \sum |a_i|,$$

so (y_i) is $\lambda(1-\lambda\delta)^{-1}$ equivalent to the unit vector basis for l_1^n.

Let $T\colon [x_i] \to [y_i]$ be the linear extension of the map $x_i \to y_i$. Define $S\colon X \to X$ by $S = (I-P) + TP$. Then $\|I - S\| = \|P - TP\| \leqslant \delta\lambda < 1$, so S is invertible and $\|S^{-1}\| \leqslant (1-\delta\lambda)^{-1}$. Also $Sx_i = y_i$ for each i. Then $Q = SPS^{-1}$ is a projection of X onto $[y_i]$ and $\|Q\| \leqslant (1+\delta\lambda)\lambda(1-\delta\lambda)^{-1}$.

III. Preliminary results. Suppose X is Banach space which has an unconditionally monotone basis (e_n) with biorthogonal functionals (e_n^*). The first major lemma we need is that X contains l_∞^n for all n or there exists $p < \infty$ and an equivalent monotonely unconditional norm $|\cdot|$ on X satisfying $|\sum x_n| \geqslant (\sum |x_n|^p)^{1/p}$ whenever (x_n) is a sequence in X with pairwise disjoint supports. We begin with a dual version to the required lemma.

LEMMA III.1. *Given an integer k there exists $p = p(k) > 1$ so that if (e_n) is an unconditionally monotone basis for a space X, which has the property that $(x_i)_{i=1}^k$ is not 10-equivalent to the unit vector basis of l_1^k for any normalized block basic sequence $(x_n)_{n=1}^k$ of (e_n), then $\|\sum y_n\| < 3(\sum \|y_n\|^p)^{1/p}$ for any disjointly supported sequence (y_n).*

Proof. Choose $p > 1$ so that $2k^{1/q} < 3$ $(1/p + 1/q = 1)$.

Suppose there did exist a disjointly supported sequence (y_n) in X satisfying $\|\sum y_n\| \geqslant 3(\sum \|y_n\|^p)^{1/p}$. Choose $f \in X^*$, $\|f\| = 1$, $f(\sum y_n) = \|\sum y_n\|$. Since the y_n's have disjoint supports and the basis is unconditionally monotone, we may pick such an f so that also $f(y_n) \geqslant 0$ for each n.

For $i \geqslant 1$, set $E(i) = \{n\colon [\frac12 k^{-1/q}]^{2^i} \|y_n\| < f(y_n) \leqslant [\frac12 k^{-1/q}]^{2^{i-1}} \|y_n\|\}$ and let

$$E(0) = \{n\colon [\tfrac12 k^{-1/q}] \|y_n\| < f(y_n) \leqslant \|y_n\|\}.$$

We claim that

$$(*) \qquad |E(0)| \geqslant k,$$

or there exists $i \geqslant 1$ so that $|E(i)| \geqslant k^{2^{i-1}}$. (Here $|A|$ denotes the cardinality of A.)

Indeed, if $(*)$ were false, then

$$\Big\| \sum y_n \Big\| = \sum f(y_n) = \sum_{i=0}^\infty \sum_{n \in E(i)} f(y_n) \leqslant \sum_{n \in E(0)} \|y_n\| + \sum_{i=1}^\infty [\tfrac12 k^{-1/q}]^{2^{i-1}} \sum_{n \in E(i)} \|y_n\|$$
$$\leqslant |E(0)|^{1/q} \Big(\sum_{n \in E(0)} \|y_n\|^p \Big)^{1/p} + \sum_{i=1}^\infty [\tfrac12 k^{-1/q}]^{2^{i-1}} |E(i)|^{1/q} \Big(\sum_{n \in E(i)} \|y_n\|^p \Big)^{1/p}$$

(by Hölder's inequality)

$$< k^{1/q} \Big(\sum_{n \in E(0)} \|y_n\|^p \Big)^{1/p} + \sum_{i=1}^\infty \Big(\frac12\Big)^{2^{i-1}} \Big(\sum_{n \in E(i)} \|y_n\|^p \Big)^{1/p}$$
$$< 3 \Big(\sum \|y_n\|^p \Big)^{1/p},$$

which contradicts the choice of (y_n).

Thus $(*)$ is true. Hence by normalizing $\{y_n\colon n \in E(i)\}$ for suitable i, we have that there exist a positive integer i and disjointly supported unit vectors $(z_j)_{j=1}^m$ $(m = k^{2^{i-1}})$ so that $f(z_j) \geqslant [\frac12 k^{-1/q}]^{2^i} \geqslant 3^{-2^i}$ for each j. Thus (z_j) is 3^{2^i} equivalent to the usual basis for l_1^m $(m = k^{2^{i-1}})$ and hence by Proposition II.5 there is a normalized block basic sequence $(w_j)_{j=1}^k$ of (z_j) which is 9 equivalent to the usual basis for l_1^k. Of course $(w_j)_{j=1}^k$ is still disjointly supported, so we may pick disjoint finite sets $(F(j))_{j=1}^k$ of integers such that $\|w_j - P_{F(j)} w_j\| < 90^{-1}$ for $1 \leqslant j \leqslant k$. Setting $x_j = \|P_{F(j)} w_j\|^{-1} P_{F(j)} w_j$, we have that $(x_j)_{j=1}^k$ is a normalized block basic sequence of (e_n) which is 10-equivalent to the unit vector basis of l_1^k, which contradicts the hypothesis on X.

Remark III.2. Of course, one can deduce in a purely formal way from Lemma III.1 as stated, that, given $\varepsilon > 0$, the p of Lemma III.1 may be chosen so that the inequality of the conclusion of Lemma III.1 reads $\|\sum y_n\| \leqslant (1+\varepsilon)(\sum \|y_n\|^p)^{1/p}$.

Remark III.3. The idea of obtaining and using upper or lower l_p estimates has been exploited by several authors. One striking result along these lines is the Gurarii–James theorem (cf. [12] and [16]) that every basic sequence in a super-reflexive space admits upper and lower l_p estimates.

COROLLARY III.4. *Suppose (e_n) is an unconditionally monotone basis for X and k is an integer such that $(x_i)_{i=1}^k$ is not 10-equivalent to the unit vector basis for l_∞^k for any block basic sequence $(x_i)_{i=1}^k$. Let $p = p(k)$ from Lemma III.1 and set $\dfrac1p + \dfrac1q = 1$. Then $\|\sum y_n\| \geqslant 1/3 (\sum \|y_n\|^q)^{1/q}$ for any disjointly supported sequence (y_n) in X.*

Proof. Suppose $(x_i^*)_{i=1}^k$ is a normalized block basic sequence of the functionals (e_n^*) in X^* which are biorthogonal to (e_n). Suppose that $(x_i^*)_{i=1}^k$

were 10-equivalent to the unit vector basis for l_1^k. Then there would exist $x \in X$ with $\operatorname{supp} x = \bigcup_{i=1}^k \operatorname{supp} x_i^*$, $\|x\| = 1$, and $x_i^*(x) \geqslant 10^{-1}$ for $1 \leqslant i \leqslant k$.

Let $x_i = P_{E(i)}x$, where $E(i) = \operatorname{supp} x_i$. Then $x_i^*(x) = x_i^*(x_i)$, so $10^{-1} \leqslant \|x_i\| \leqslant 1$. Also, $\|\sum_{i=1}^k x_i\| = \|x\| = 1$, and (x_i) is disjointly supported, so by the unconditional monotoneness of (e_n) $\|\sum_{i=1}^k a_i x_i\| \leqslant \max|a_i|$ for any choice (a_i) of scalars. Since $10^{-1}|a_i| \leqslant x_i^* \sum_{j=1}^k a_i x \leqslant \|\sum_{j=1}^k a_j x_j\|$ for each i, we have that $(x_i)_{i=1}^k$ is 10-equivalent to the unit vector basis of l_∞^k which contradicts the hypothesis.

Thus by Lemma III.1, $\|\sum y_n^*\| \leqslant 3\left(\sum \|y_n^*\|^{p'}\right)^{1/p'}$ for any disjointly supported (y_n^*) in $[e_n^*]$, from which the desired conclusion follows by duality.

Now we come to the renorming lemma:

COROLLARY III.5. *Suppose k is an integer and set $p = p(k)$ from Lemma* III.1. *If $(X, \|\cdot\|)$ has unconditionally monotone basis (e_n) and no block basic sequence $(x_i)_{i=1}^k$ of (e_n) is 10-equivalent to the unit vector basis of l_∞^k, then there is an equivalent unconditionally monotone norm $|\cdot|$ on X satisfying for all $x \in X$, $\|x\| \leqslant |x| \leqslant 3\|x\|$ and $|\sum x_n| \geqslant \left(\sum |x_n|^p\right)^{1/p}$ whenever $(\operatorname{supp} x_n)$ is pairwise disjoint.*

Proof. Define $|x| = \sup\left(\sum \|P_{E(n)}x\|^p\right)^{1/p}$, where the sup is taken over all sequences $(E(n))$ of pairwise disjoint finite sets of integers. We omit the routine verification that $|\cdot|$ has the desired properties.

Remark III.6. It follows from Remark III.2 that given $\varepsilon > 0$, the norm $|\cdot|$ in Corollary III.5 may be chosen to satisfy $\|x\| \leqslant |x| \leqslant (1+\varepsilon)\|x\|$ for all $x \in X$.

The final preliminary lemma is crucial for the proof of our main theorem. Roughly the lemma states the intuitively evident fact that if an unconditional basis admits a lower l_p estimate as in the conclusion of Corollary III.5, then $0 \leqslant x \leqslant y$ and $\|x\|$ close to $\|y\|$ implies that $\|y-x\|$ is small.

LEMMA III.7. *Suppose $\tau > 0$ and $1 < p < \infty$. Then there exists $\delta = \delta(\tau, p) > 0$ so that if (e_n) is an unconditionally monotone basis for some space X in which $\|\sum x_n\| \geqslant \left(\sum \|x_n\|^p\right)^{1/p}$ for any disjointly supported sequence (x_n), then X satisfies the following conditions.*

If x and y are in X, $0 \leqslant x \leqslant y$, $1-\delta \leqslant \|x\| \leqslant \|y\| \leqslant 1$, and E is defined to be the set $\{n\colon e_n^*(y) \leqslant (1+\tau)e_n^*(x)\}$, then

(A) $\|P_E x\| \geqslant 1-\tau$ *and*

(B) $\|y-x\| \leqslant \tau + [1-(1-\tau)^p]^{1/p}$.

Proof. Choose $\delta > 0$ so that $0 < \delta < \tau(\tau-\delta)$ and suppose x, y are as in the hypothesis.

Assume that $\|P_E x\| < 1-\tau$. Pick a norm one functional $x^* \in X^*$ so that $x^*(x) = \|x\|$. Then

$$(1-\delta) \leqslant x^*(x) = x^*(P_E x) + x^*(P_{\sim E}x) < (1-\tau) + x^*(P_{\sim E}x),$$

so that

$$(\tau-\delta) < x^*(P_{\sim E}x).$$

But then

$$\begin{aligned} 1 \geqslant x^*(y) &= x^*(P_E y + P_{\sim E}y) \geqslant x^*(P_E x) + (1+\tau)x^*(P_{\sim E}x) \\ &= x^*(x) + \tau x^*(P_{\sim E}x) > \|x\| + \tau(\tau-\delta) \geqslant (1-\delta) + \tau(\tau-\delta). \end{aligned}$$

Thus $\tau(\tau-\delta) < \delta$, which is a contradiction. This establishes (A). From (A) we have that $\|P_E y\| \geqslant 1-\tau$ and hence

$$\|P_{\sim E}y\| \leqslant [1-(1-\tau)^p]^{1/p}.$$

Therefore we have

$$\begin{aligned} \|y-x\| &\leqslant \|P_E y - P_E x\| + \|P_{\sim E}y - P_{\sim E}x\| \\ &\leqslant \tau + \|P_{\sim E}y\| \leqslant \tau + [1-(1-\tau)^p]^{1/p}. \end{aligned}$$

This proves (B) and completes the proof.

IV. Main results. The main theorem of this paper is an easy consequence of Theorem IV.1 and the renorming of Corollary III.5. The argument for Theorem IV. 1 is actually just a modification of the proof of Theorem 3.2 in Enflo and Rosenthal's paper [6]. The special properties possessed by L_1 and used by Enflo and Rosenthal have analogues in spaces with unconditional basis, if the basis satisfies a lower l_p estimate as in the hypothesis of Theorem IV.1. A major modification of the technique of [6] is necessary mainly because we know a priori from Proposition II.2 only that non-superreflexive spaces contain sequences $(x_i)_{i=1}^n$ of unit vectors with $\|\sum_{i=1}^n \varepsilon_i x_i\| \approx n$ for (ε_i) admissible. Thus, the Enflo–Rosenthal hypothesis of Lemma 3.1 in [6] that the average of $\|\sum_{i=1}^n \varepsilon_i f_i\|$ over *all* choices (ε_i) of signs is close to n does not suit our purpose. However, the elementary Lemma II.4 allows us to work with admissible choices of signs only.

THEOREM IV.1. *Suppose $p < \infty$, m is an integer, and $0 < \Delta < 3^{-1}$. Then there exist $\delta > 0$ and an integer k so that if (e_n) is an unconditionally monotone basis for some space X and $\|\sum x_n\| \geqslant \left(\sum \|x_n\|^p\right)^{1/p}$ for every disjointly supported sequence (x_n) in X, then the following is true: Given any sequence $(f_i)_{i=1}^k$ of unit vectors in X satisfying*

(i) $\quad k(1-\delta) < \left\|\sum_{i=1}^k \varepsilon_i f_i\right\|$ *for every admissible choice (ε_i) of signs,* *and*

(ii) *there is* $x^* \in X^*$ *with* $\|x^*\| = 1$ *and* $1-\Delta < x^*(f_i)$ *for* $1 \leqslant i \leqslant k$,

then there exist a subsequence $(f_{n_i})_{i=1}^m$ *of* $(f_i)_{i=1}^k$ *and disjoint sets* $(E(n_i))_{i=1}^m$ *of integers with* $\|f_{n_i} - P_{E(n_i)} f_{n_i}\| < \Delta$.

Consequently, $(f_{n_i})_{i=1}^m$ *is* $(1-3\Delta)^{-1}$ *equivalent to the unit vector basis for* l_1^m, *and there is a projection* P *from* X *onto* $[f_{n_i}]_{i=1}^m$ *with* $\|P\| \leqslant (1-\Delta)(1-2\Delta)^{-1}(1-3\Delta)^{-1}$.

Proof. A trivial approximation argument shows that we may assume that $\dim X < \infty$, say $\dim X = n$.

Choose $\tau_1 > 0$ so that

(iii) $$\tau_1 + [1-(1-\tau_1)^p]^{1/p} < \Delta$$

and let $\delta_1 = \delta(\tau_1, p)$ from Lemma III.7.

Now pick $\alpha > 0$ and the integer k to satisfy

(iv) $$1+\alpha - \frac{1-\alpha}{1+\alpha} < \delta_1,$$

(v) $$k > \frac{m}{\alpha}\left[1+\alpha - \frac{1-\alpha}{1+\alpha}\right].$$

Choose $\tau > 0$ so that

(vi) $$k[1-(1-\tau)^p]^{1/p} < \alpha/2,$$

and finally pick $\delta > 0$ to satisfy

(vii) $$\delta < \min\{\alpha/2, \delta(\tau, p)\}.$$

Write $f_i = \sum_{j=1}^n a_j^i e_j$, let $f = 1/k \sum_{j=1}^n \sum_{i=1}^k |a_j^i| e_j$ and set $g_\varepsilon = 1/k \sum_{j=1}^n \left|\sum_{i=1}^k \varepsilon_i a_j^i\right| e_j$ for each admissible choice $\varepsilon = (\varepsilon_i)_{i=1}^k$ of signs.

Now for each admissible ε, $0 \leqslant g_\varepsilon \leqslant f$ and $1-\delta \leqslant \|g_\varepsilon\| \leqslant \|f\| \leqslant 1$ by (i), so

(viii) $$\|f - g_\varepsilon\| \leqslant \tau + [1-(1-\tau)^p]^{1/p} \quad \text{by (vii)}.$$

Let $E = \{j\colon \sum_{i=1}^k |a_j^i| < (1+\alpha)\max_i |a_j^i|\}$. We may assume (by perturbing the a_j^i's slightly) that $\max_i |a_j^i|$ is achieved for only one i, $1 \leqslant i \leqslant k$. We want to show that

(ix) $$(1-\alpha) \leqslant \|P_E f\|.$$

Using Lemma II.4, pick for each $j \in \sim E$ an admissible choice $\varepsilon = \varepsilon(j)$ of signs so that

$$(1+\alpha)\left|\sum_{i=1}^k \varepsilon_i a_j^i\right| \leqslant \sum_{i=1}^k |a_j^i|.$$

Set $F(\varepsilon) = \{j \in \sim E\colon \varepsilon = \varepsilon(j)\}$. Now if (ix) fails, then

$$(1-\delta) \leqslant \|f\| \leqslant \|P_E f\| + \|P_{\sim E} f\| \leqslant (1-\alpha) + \|P_{\sim E} f\|$$

so $\alpha/2 \leqslant (\alpha - \delta) \leqslant \|P_{\sim E} f\|$. However, by applying Lemma III.7 with $y = f$ and $x = g_\varepsilon$, we have, for each admissible ε, that $(1-\tau) \leqslant \|P_{\sim F(\varepsilon)} f\|$ and hence $\|P_{F(\varepsilon)} f\| \leqslant [1-(1-\tau)^p]^{1/p}$. But then

$$\|P_{\sim E} f\| = \Big\|\sum_\varepsilon P_{F(\varepsilon)} f\Big\| \leqslant \sum_\varepsilon \|P_{F(\varepsilon)} f\| \leqslant k[1-(1-\tau)^p]^{1/p} < \alpha/2.$$

This contradiction yields (ix).

The rest of the proof is analogous to the Enflo–Rosenthal argument mentioned above. For $1 \leqslant i \leqslant k$, let $E(i) = \{j \in E\colon \max_l |a_j^l| = |a_j^i|\}$. Recall that we have assumed that the $E(i)$'s are pairwise disjoint. Also, from (ix) we have

$$k(1-\alpha) \leqslant k\|P_E f\| = \Big\|k \sum_{i=1}^k P_{E(i)} f\Big\| = \Big\|\sum_{i=1}^k \sum_{j \in E(i)} \sum_{l=1}^k |a_j^l| e_j\Big\|$$

$$\leqslant \Big\|\sum_{i=1}^k \sum_{j \in E(i)} (1+\alpha)|a_j^i| e_j\Big\| \leqslant (1+\alpha) \sum_{i=1}^k \Big\|\sum_{j \in E(i)} |a_j^i| e_j\Big\|$$

$$= (1+\alpha) \sum_{i=1}^k \Big\|\sum_{j \in E(i)} a_j^i e_j\Big\| = (1+\alpha) \sum_{i=1}^k \|P_{E(i)} f_i\|.$$

That is,

(x) $$k\frac{1-\alpha}{1+\alpha} \leqslant \sum_{i=1}^k \|P_{E(i)} f_i\|.$$

Since

$$\|P_{E(i)} f_i\| \leqslant \|f_i\| \leqslant 1 \quad \text{for } 1 \leqslant i \leqslant k,$$

we have from (v) and (x) that there are integers $1 \leqslant n_1 < n_2 < \ldots < n_m \leqslant k$ so that $\frac{1-\alpha}{1+\alpha} - \alpha \leqslant \|P_{E(n_i)} f_{n_i}\|$ for $1 \leqslant i \leqslant m$. Thus from (iv), (iii), and Lemma III.7 it follows that

$$\|f_{n_i} - P_{E(n_i)} f_{n_i}\| < \Delta.$$

To see the final conclusions, note that

$$x^*(P_{E(n_i)} f_{n_i}) \geqslant 1 - 2\Delta \quad \text{for} \quad 1 \leqslant i \leqslant m,$$

and hence (since $(P_{E(n_i)} f_{n_i})$ is disjointly supported) $(P_{E(n_i)} f_{n_i})$ is $(1-2\Delta)^{-1}$ equivalent to the unit vector basis for l_1^m. Further, we have from Proposition II.6 that there is a projection from X onto $[P_{E(n_i)} f_{n_i}]$ of norm $\leqslant (1-2\Delta)^{-1}$. Thus the last conclusions follow from Lemma II.7.

We come now to the main result of the paper.

THEOREM IV.2. *Assume that $(X, |\cdot|)$ has local unconditional structure, Y is a non-super-reflexive subspace of X, and there exist $K>1$ and an integer q so that $d(E, l_\infty^q) \geqslant K$ for each subspace E of X. Then there are subspaces E_n of Y and projections P_n from Y onto E_n with $|P_n|\cdot d(E_n, l_1^n) \to 1$. Further, there is a constant A so that, for any $\varepsilon>0$, $M<\infty$, and integer q, there exists an integer $N=N(\varepsilon, M, q)$ such that if $E\subset X$ and $d(E, l_1^N)\leqslant M$, then there are a subspace F of E and projection $P\colon X\to F$ satisfying $|P|\leqslant A$ and $d(F, l_1^q)\leqslant 1+\varepsilon$.*

Proof. Let (E_a) be a family of finite dimensional subspaces of X, directed by inclusion, whose union is X and such that each E_a has a basis $(e_i^a)_{i=1}^{d_a}$ with unconditional constant $\leqslant \lambda < \infty$. For each a, let $|\cdot|_a$ be an unconditionally monotone norm on E_a satisfying $|x| \leqslant |x|_a \leqslant \lambda |x|$ for $x\in E_a$.

From the hypothesis on X and Proposition II.5, there is an integer k so that $d\big((E, |\cdot|_a), l_\infty^k\big) \geqslant 10$ for any a and any subspace E of E_a. We then have from Corollary III.5 that there are $p=p(k)<\infty$ and unconditionally monotone norms $\|\ \|_a$ on E_a satisfying $|x|_a \leqslant \|x\|_a \leqslant 3|x|_a$ for all $x\in E_a$ and $\|\sum x_n\|_a \geqslant \big(\sum \|x_n\|_a^p\big)^{1/p}$ whenever $(x_n)\subset E_a$, (x_n) disjointly supported relative to $(e_n^a)_{n=1}^{d_a}$.

We next employ, for convenience, a Lindenstrauss compactness argument. Define $T_a\colon X\to$ Reals by

$$T_a x = \begin{cases} \|x\|_a, & \text{if } x\in E_a,\\ 0, & \text{if } x\notin E_a.\end{cases}$$

By passing to a subnet of (T_a), we assume $\|x\| \equiv \lim_a T_a x$ exists for each $x\in X$. Obviously $\|\cdot\|$ is a norm on X and $|x|\leqslant \|x\| \leqslant 3\lambda|x|$ for $x\in X$.

Now fix an integer m and $\varepsilon>0$. Choose $0<\Delta<3^{-1}$ so that $(1-\Delta)(1-2\Delta)^{-1}(1-3\Delta)^{-1}<1+\varepsilon$. Let δ and k be as given from Theorem IV.1 with this choice of m, Δ, and with p from above. Since $(Y, \|\cdot\|)$ is not super-reflexive, there exist by Proposition II.2 $(f_i)_{i=1}^k\subset Y$ satisfying (i) and (ii) in the statement of Theorem IV.1. For a sufficiently large we have that also

$$(\tilde f_i)_{i=1}^k \equiv (\|f_i\|_a^{-1} f_i)_{i=1}^k$$

satisfies (i) and (ii) of Theorem IV.1 with $X\equiv E_a$, $\|\cdot\| \equiv \|\cdot\|_a$, and $f_i\equiv \tilde f_i$.

Hence by Theorem IV.1 there exist, for sufficiently large a, subspaces F_a of $[f_i]$ and projections $P_a\colon E_a \overset{\text{onto}}{\to} F_a$ with $d(F_a, l_1^m)<1+\varepsilon$ and $\|P_a\|<1+\varepsilon$. Define $\tilde P_a\colon X\to F_a$ by

$$\tilde P_a x = \begin{cases} P_a x, & \text{if } x\in E_a,\\ 0, & \text{if } x\notin E_a.\end{cases}$$

Again by passing to a subnet, we may assume that $Px = \lim_a \tilde P_a x$ exists for all $x\in X$. Since $F_a\subset [f_i]$, $d(F_a, l_1^m)<1+\varepsilon$, and $\dim[f_i]<\infty$, we have that P is a projection, $PX\subset [f_i]$, $d(PX, l_1^m)\leqslant 1+\varepsilon$, and of course $\|P\|\leqslant 1+\varepsilon$. Note that even $PX = [f_{i_j}]_{j=1}^m$ for some $1\leqslant i_1<i_2<\ldots\leqslant k$.

Returning to the original norm on X, we have that $d\big((PX, |\cdot|), l_1^m\big) \leqslant 3\lambda(1+\varepsilon)$. Thus by Proposition II.5, for each integer n and $\varepsilon>0$, there is $E\subset Y$ with $d(E, l_1^n)<1+\varepsilon$.

Now suppose $M<\infty$, $\varepsilon>0$, and an integer m are given. Using Proposition II.5, we easily reduce to the case where $M<1+\varepsilon$, so assume that $M<1+\varepsilon$. Choose an integer r so that $(3\lambda M)^{2^{-r}}<(1-\delta)^{-1}$, where as before Δ satisfies $(1-\Delta)(1-2\Delta)^{-1}(1-3\Delta)^{-1}<1+\varepsilon$, while $\delta<\Delta$ and k come from Theorem IV.1 with this choice of m and Δ. Set $N=k^{2^r}$ and suppose that $(x_i)_{i=1}^N$ in $(X, |\cdot|)$ are unit vectors which are M-equivalent to the unit vector basis in l_1^m. Then in $(X, \|\cdot\|)$, $(\|x_a\|^{-1}x_i)_{i=1}^N$ is $3\lambda M$ equivalent to the unit vector basis of l_1^N and hence by Proposition II.5 there is a $\|\cdot\|$-normalized block basic sequence $(f_i)_{i=1}^k$ of (x_i) which is, in $(X, \|\cdot\|)$, $(1-\delta)^{-1}$ equivalent to the usual l_1^k basis. Thus $(f_i)_{i=1}^k$ satisfies (i) and (ii) of Theorem IV.1, hence by the first part of the proof, there exist $1\leqslant i_1 < i_2 < \ldots \leqslant i_m \leqslant k$ so that $[f_{i_j}]_{j=1}^m$ is, in $(X, \|\cdot\|)$, $1+\varepsilon$ complemented. But then $[f_{i_j}]_{j=1}^m$ is, in $(X, |\cdot|)$, $3\lambda(1+\varepsilon)$ complemented and $(|f_{i_j}|^{-1}f_{i_j})_{j=1}^m$ is M-equivalent to the usual l_1^m basis. Since $M<1+\varepsilon$, this completes the proof of the final conclusion of the theorem with the constant $A=3\lambda(1+\varepsilon)$.

To see the first conclusion, note that we have shown that Y contains a sequence (F_n) of subspaces which are A-complemented in Y and $d(F_n, l_1^n)<1$. Thus $Y^*\supset G_n$, with $d(G_n, l_\infty^n)\to 1$. Whence the first conclusion follows by duality. This completes the proof.

For the constant A in Theorem IV.2 we took (for any $\varepsilon>0$) $3(1+\varepsilon)\mathscr{U}(X)$. By Remark III.6 and the proof of Theorem IV.2, $A=(1+\varepsilon)\mathscr{U}(X)$ (for any $\varepsilon>0$) is also a permissible choice for the constant. Thus we can restate Theorem IV.2 as follows.

COROLLARY IV.3. *Suppose X has l.u.st. Then either*

(1) *X is super-reflexive, or*

(2) *$X\supset E_n$ with $d(E_n, l_\infty^n)\to 1$, or*

(3) *$X\supset E_n$ with $d(E_n, l_1^n)\to 1$ and E_n is $1+1/n$ complemented in X.*

Further, suppose neither (1) nor (2) holds. Then given any $\varepsilon>0$, $M<\infty$, and integer n, there exists an integer N so that if $E\subset X$ and $d(E, l_1^N)\leqslant M$, then $E\supset F$ so that $d(F, l_1^n)\leqslant 1+\varepsilon$ and F is $(1+\varepsilon)\mathscr{U}(X)$ complemented in X.

Remark. After this paper was submitted, Lior Tzafriri proved a nice result related to Corollary IV.3.

If X has an unconditional basis, then X contains uniformly complemented copies of l_p^n for some p, $p = 1, 2$ or ∞.

It is not known whether the hypothesis on X can be weakened to "X has l.u.st." (Unless, of course, X is not super-reflexive, in which case Tzafriri's theorem is a consequence of Corollary IV.3.)

Finally we apply this local version of James' theorem to the problem of Grothendieck's mentioned in the introduction. For this we need the following known proposition. In the proof we use $L_0(X, Y)$ to denote the finite rank operators from X to Y and $N(T)$ is the nuclear norm of T.

PROPOSITION IV.4. *Suppose X and Y are infinite dimensional Banach spaces. Assume that there are finite dimensional subspaces E_n of X, bases $(e_i^n)_{i=1}^{d(n)}$ for E_n, and projections P_n from X onto E_n so that $\sup\|P_n\| = \lambda < \infty$ and $\sup \mathscr{U}(e_i^n)_{i=1}^{d(n)} < \infty$. Then there are compact, non-nuclear operators from Y to X and from X to Y.*

Proof. Tseitlin [25] announced this when X is the range space, so we treat the case when X is the domain. (Actually, either case follows from the other by a duality argument. We wish to indicate a direct proof of one case because Tseitlin's paper gives no details.)

If $\limsup_{n\to\infty} d(E_n, l_1^{d(n)}) = \infty$, then by a result of Lindenstrauss and Pełczyński [19], there are operators $T_n\colon E_n \to Y$ with $\|T_n\| = 1$ and $\sup \pi_1(T_n) = \infty$. (Here $\pi_1(T)$ is the absolutely summing norm of T.) Thus $\|T_nP_n\| \leqslant \lambda$ but $N(T_nP_n)$ is unbounded as $n\to\infty$. This means that the nuclear norm is not equivalent to the operator norm on $L_0(X, Y)$, and hence there is a compact, non-nuclear operator from X to Y.

If $d(E_n, l_1^n) \leqslant M$ for each n, then $d(P_n^* X^*, l_\infty^n) \leqslant M\lambda$ for each n. Let $F_n \subseteq Y^*$, $\dim F_n = n$. By the universality property of l_∞^n, there exist integers $k(1), k(2), \ldots$ and operators $T_n\colon F_n \to P_{k(n)}^* X^*$ with $\|T_n\| = 1$, $\|T_n^{-1}\| < \lambda M$. The T_n's can be extended to operators $\tilde{T}_n\colon Y^* \to P_{k(n)}^* X^*$ satisfying $\sup\|\tilde{T}_n\| < \infty$. In fact, these extensions (by local reflexivity [17], or direct verification) can be taken weak* continuous, say $\tilde{T}_n = S_n^*$. Now $N(S_n^*) \to \infty$ (since the $S^*_{n|F_n}$'s are uniformly bounded isomorphisms with uniformly bounded inverses), thus also $N(S_n) \to \infty$. Hence the operator norm is not equivalent to the nuclear norm on $L_0(X, Y)$, whence there is a compact, non-nuclear operator from X to Y.

THEOREM IV.5. *Let X and Y be infinite dimensional Banach spaces. Suppose that X satisfies either*

(1) *X has l.u.st. or*

(2) *X is isomorphic to a subspace of a space which has l.u.st. and which contains no sequence E_n with $d(E_n, l_\infty^n) \to 1$.*

Then there are compact, non-nuclear operators from X to Y and from Y to X.

Proof. Assume (1). If X is super-reflexive, then the conclusion is a special case of the main result of [3]. If X is not super-reflexive, then the conclusion follows Theorem IV.2 and Proposition IV.4.

(2) follows in a similar manner from the results of [3] and Theorem IV.2.

Since $C(K)$ spaces obviously have l.u.st., every Banach space is isometric to a subspace of a space which has l.u.st. The results proved here lead to the question:

PROBLEM IV.6. *Suppose X does not contain a sequence (E_n) of subspaces with $d(E_n, l_\infty^n) \to 1$. Is X isomorphic to a subspace of some space Y such that Y has l.u.st. and Y does not contain a sequence (E_n) with $d(E_n, l_\infty^n) \to 1$?*

An affirmative answer to this embedding problem would (by the results herein) solve the Grothendieck problem, as well as show that every non-super-reflexive space X contains l_∞^n uniformly or contains l_1^n uniformly and uniformly complemented. After this paper was submitted for publication, T. Figiel and I proved that the answer to Problem IV.6 is affirmative for spaces X which are complemented in a Banach lattice.

Added in proof. R. C. James (*A non-reflexive Banach space which is uniformly non octahedral*) has answered Problem IV.6 in the negative.

References

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), pp. 151–164.

[2] A. Brunel and L. Sucheston, *On J-convexity and some ergodic super-properties of Banach spaces* (to appear).

[3] W. J. Davis and W. B. Johnson, *Compact, non-nuclear operators*, Studia Math. 51 (1974), pp. 81–85.

[4] E. Dubinsky, A. Pełczyński, and H. P. Rosenthal, *On Banach spaces X for which $\pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$*, Studia Math. 44 (1972), pp. 617–648.

[5] P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel J. Math. 13 (1972), pp. 281–288.

[6] — and H. P. Rosenthal, *Some results concerning $L(\mu)$ spaces*, J. Functional Anal. 14 (1973), pp. 325–348.

[7] T. Figiel, *Factorization of compact operators and applications to the approximation problem*, Studia Math. 45 (1973), pp. 191–210.

[8] D. P. Giesy, *On a convexity condition in normed linear spaces*, Trans. Amer. Math. Soc. 125 (1966), pp. 114–146.

[9] D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite dimensional Banach spaces*, Israel J. Math. 9 (1971), pp. 346–361.

[10] Y. Gordon and D.R. Lewis, *Absolutely summing maps and local unconditional structures*, (to appear).

[11] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc. 16 (1955).

[12] V. I. Gurarii and N. I. Gurarii, *On bases in uniformly convex and uniformly smooth Banach spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), pp. 210–215.

[13] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) 52 (1950), pp. 518–527.

[14] — *Uniformly non-square Banach spaces*, Ann. of Math. 80 (1964), pp. 542–550.

[15] — *Some self-dual properties of normed linear spaces*, Annals of Math. Studies 69 (1972), pp. 159–175.

[16] — *Super-reflexive spaces with bases*, Pacific J. Math. 41 (1972), pp. 409–419.

[17] W. B. Johnson, H. P. Rosenthal, and M. Zippin, *On bases, finite dimensional decompositions, and weaker structures in Banach spaces*, Israel J. Math 9 (1971), pp. 488–506.

[18] M. I. Kadec and M. G. Snobar, *Certain functionals on the Minkowski compactum*, Mat. Zametki 10 (1971), pp. 453–457 (in Russian).

[19] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications*, Studia Math. 29 (1968), pp. 275–326.

[20] — and H. P. Rosenthal, *The $\mathscr{L}_p$ spaces*, Israel J. Math. 7 (1969), pp. 325–349.

[21] B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Société Mathématique de France (1974).

[22] A. Persson and A. Pietsch, *p-nucleare und p-integrale Abbildungen in Banachräumen*, Studia Math. 33 (1969), pp. 19–62.

[23] H. P. Rosenthal, *On subspaces of L_p*, Ann. of Math. 97 (1973), pp. 344–373.

[24] J. J. Schäffer and K. Sundaresan, *Reflexivity and the girth of spheres*, Math. Ann. 184 (1970), pp. 163–168.

[25] I. I. Tseitlin, *On a particular case of the existence of a compact linear operator which is not nuclear*, Funk. Anal. i Pril. 6 (1973), p 102.

[26] L. Tzafriri, *Reflexivity in Banach lattices and their subspaces*, Journal of Functional Anal. 10 (1972), pp. 1–18.

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Weighted norm inequalities for maximal functions and singular integrals

by

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Abstract. We present simplified proofs of the weighted-norm inequalities of R. Hunt, B. Muckenhoupt and R. Wheeden, concerning singular integrals and maximal functions. The inequalities in question are

$$\int_{R^n} |Tf(x)|^p \omega(x)\,dx = C \int_{R^n} |f(x)|^p \omega(x)\,dx,$$

where T denotes either a singular integral operator, or the maximal function of Hardy and Littlewood, and ω satisifes appropriate (necessary and sufficient) conditions.

§ 1. This note is concerned with the problem of identifying those weight functions $\omega(x)$ on R^1 for which the Hilbert transform $Tf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)\,dy}{x-y}$ is bounded on $L^p(\omega(x)\,dx)$, that is

$$\int_{R^1} |Tf(x)|^p \omega(x)\,dx \leqslant C \int_{R^1} |f(x)|^p \omega(x)\,dx \quad \text{for all } f. \tag{1}$$

Until recently, the only significant partial result known was that of Helson and Szegö [6]: Inequality (1) holds for $p = 2$ if and only if $\omega = e^{b_1 + Tb_2}$ for functions $b_1, b_2 \in L^\infty$ with $\|b_2\|_\infty < \pi/2$. Unfortunately, there is no obvious way to tell whether a given ω can be so represented, so that even for L^2, the problem of inequality (1) remained open. Attempts to generalize the Helson–Szegö theorem to L^p ($p \neq 2$) were only partly successful.

Surprisingly, there is a simple necessary and sufficient condition for inequality (1) to hold. It was B. Muckenhoupt who made the key discovery, by studying the analogue of (1) for the maximal function

$$f^*(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy \quad \text{in } R^n.$$

(Here, Q denotes a cube with sides parallel to the axes.)