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On Toeplitz sections in FK-spaces

by

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Abstract. Let $T = (t_{mn})$ be a regular row-finite matrix. Let x be any sequence. Define $T_m x = \sum_{n=1}^{\infty} t_{mn} \sum_{k=1}^{n} x_k e^k$, where e^k is the sequence having 1 in the k-th position and 0's elsewhere. Let E be an FK-space containing the finite sequences and let E' be its dual. E has T-AK if $\lim_m T_m x = x$ for all $x \in E$ (i.e. the e^k's form a Toeplitz basis for E). E has T-FAK if $\lim_m f(T_m x)$ exists for all $x \in E$ and $f \in E'$. E has T-AB if the set $\{T_m x\colon m = 1, 2, \ldots\}$ is bounded in E for each $x \in E$. We introduce a new class of FK-spaces and show that every FK-space E having T-AK, T-FAK, or T-AB can be imbedded in a member of this class. We then use properties of this class to give a partial answer to the following general problem. Let T and T' be any given regular row-finite matrices. If E has T-AK (T-FAK) (T-AB), must E have T′-AK (T′-FAK) (T′-AB)? We introduce a new class of matrices for which this problem is solvable and show that we can expect quite different results for matrices not in this class.

1. Introduction. A *K-space* is a locally convex linear topological space of sequences in which the coordinate functionals are continuous. An *FK-space* is a complete metrizable K-space. Let $x = (x_k)$ be any sequence. Define the n-th section of x to be the sequence $\overset{n}{x} = \sum_{k=1}^{n} x_k e^k$, where e^k denotes the sequence having 1 in the k-th position and 0's elsewhere. We make the following generalization of sections. Let $T = (t_{mn})$ be a regular row-finite matrix. Define the m-th T-section of x to be the finite sequence $T_m x = \sum_{n=1}^{\infty} t_{mn} \overset{n}{x}$. Note that $T_m x = \overset{m}{x}$ if T is the identity matrix. Unless otherwise stated, E will denote an FK-space which contains E^{∞}, the set of finite sequences. We denote the set of continuous linear functionals on E by E'. We shall be concerned with FK-spaces with the following properties.

1.1. DEFINITION.
(i) *E has T-AK if* $\lim_m T_m x = x$ *for all* $x \in E$.
(ii) *E has T-FAK if* $\lim_m f(T_m x)$ *exists for all* $f \in E'$ *and* $x \in E$.
(iii) *E has T-AB if* $\{T_m x\colon m = 1, 2, \ldots\}$ *is bounded in E for all* $x \in E$.

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Clearly (i) implies (ii) and (ii) implies (iii). Denote by $E_{\text{T-AK}}$ the subset $\{x \in E\colon \lim_m T_m x = x\}$. In the case when T is the identity matrix, we denote properties (i)–(iii) above by AK, FAK and AB respectively. Spaces having the AK, FAK and AB properties have been studied by several authors. See, for example, the works of Zeller [15], and Garling [5], [6]. The above properties have been studied in a recent work of Buntinas [2], in the case when T is the Cesàro $(c, 1)$ matrix. Note that if E has T-AK for some T, then the e^k's form a Toeplitz basis for E. Toeplitz bases were originally introduced in [7] and [8].

In Section 3 we introduce a notion of duality which is relevant to the study of T-sections. This duality is analogous to that developed by Garling in [6] for ordinary sections. In Section 4 we introduce a new class of FK-spaces, and we show that we can suitably imbed any FK-space having T-AK, T-FAK or T-AB in a member of this class. This class is analogous to a class introduced by Garling, [6], in the case when T is the identity matrix. In Section 5 we use this imbedding to give a partial answer to the following general problem. Let T and T' be any two regular row-finite matrices. If E has T-AK (T-FAK) (T-AB), must E have T$'$-AK (T$'$-FAK) (T$'$-AB)?

2. Preliminary ideas and results. Let s denote the set of all sequences. s is an FK-space with the topology given by the seminorms $\{p_j\colon j = 1, 2, \ldots\}$, where $p_j(x) = |x_j|$. Let c denote the set of convergent sequences and let m denote the set of bounded sequences. Throughout, $T = (t_{mn})$ will be a regular row-finite matrix. It is well known that T is regular if and only if the following conditions hold:

(1) $$\sup_m \sum_{n=1}^{\infty} |t_{mn}| < \infty;$$

(2) $$\lim_m \sum_{n=1}^{\infty} t_{mn} = 1;$$

and

(3) $$\lim_m t_{mn} = 0 \quad \text{for } n = 1, 2, \ldots$$

For a sequence $y = (y_n)$ we define the sequence $Ty = ((Ty)_m)$, where $(Ty)_m = \sum_{n=1}^{\infty} t_{mn} y_n$. Let $c_T = \{y \in s\colon Ty \in c\}$ and $m_T = \{y \in s\colon Ty \in m\}$. It is known that c_T and m_T are FK-spaces with the topology given by the seminorms $\{p_j\colon j = 1, 2, \ldots\}$ and p, where $p(y) = \sup_m |(Ty)_m|$. (See [12], Section 12.4.) If $a = (a_k)$ and $x = (x_k)$ are any sequences, we denote by ax the sequence $(a_k \cdot x_k)$.

2.1. Theorem. *Suppose E has T-AB. Then $E_{T\text{-}AK} = \bar{E}^{\infty}$.*

Proof. The proof is a standard equicontinuity argument and is omitted. (cf. [7], Theorem 1$'$; [15], Satz 3.3 and [5], Proposition 1).

3. Duality properties.

3.1. Definition. *Let E be any sequence space.*

$$E^{\beta(T)} = \{y \in s\colon \lim_m \sum_{k=1}^{\infty} y_k (T_m x)_k \text{ exists for all } x \in E\}.$$

$$E^{\gamma(T)} = \{y \in s\colon \sup_m \Big|\sum_{k=1}^{\infty} y_k (T_m x)_k\Big| < \infty \text{ for all } x \in E\}.$$

$E^{\beta(T)}$ and $E^{\gamma(T)}$ are called the $\beta(T)$ and $\gamma(T)$ duals of E respectively.

3.2. Theorem. *Suppose E has T-AK. Then $E' = E^{\beta(T)} = E^{\gamma(T)}$.*

Proof. Since

$$f(x) = \lim_m f(T_m x) = \lim_m \sum_{k=1}^{\infty} x_k f(e^k) \sum_{n=k}^{\infty} t_{mn} = \lim_m \sum_{k=1}^{\infty} f(e^k)\,(T_m x)_k,$$

we can identify each $f \in E'$ with $(f(e^k)) \in E^{\beta(T)}$. That $E = E^{\beta(T)}$ will then follow from the Banach–Steinhaus theorem.

Clearly $E^{\beta(T)} \subseteq E^{\gamma(T)}$. Suppose $y \in E^{\gamma(T)}$. Define $f_m(x) = \sum_{k=1}^{\infty} y_k (T_m x)_k$ for $m = 1, 2, \ldots$ Since $y \in E^{\gamma(T)}$, $\sup_m |f_m(x)| < \infty$ for each $x \in E$. Thus the set $\{f_m\colon m = 1, 2, \ldots\}$ is weak* bounded in E'. It follows from [12], Section 13.3, Problem 20, that the set $\{f_m\colon m = 1, 2, \ldots\}$ is weak* relatively compact. Thus there is a sequence of integers (m_i) and a continuous linear functional f such that $\lim_i f_{m_i}(x) = f(x)$ for all $x \in E$. But $f(x) = \lim_m \sum_{k=1}^{\infty} f(e^k)(T_m x)_k$ and $f(e^k) = \lim_i f_{m_i}(e^k) = y_k$. Thus $y \in E^{\beta(T)}$.

3.3. Theorem (cf. [11], Theorem 3).

(i) *E has T-FAK if and only if $(E_{T\text{-}AK})^{\beta(T)} = E^{\beta(T)}$.*

(ii) *E has T-AB if and only if $(E_{T\text{-}AK})^{\gamma(T)} = E^{\gamma(T)}$.*

Proof. We will only prove (ii) as the proof of (i) is similar. Suppose $(E_{\text{T-AK}})^{\gamma(T)} = E^{\gamma(T)}$. Let $f \in E'$. Then $(f(e^k)) \in (E_{\text{T-AK}})^{\beta(T)} \subseteq (E_{\text{T-AK}})^{\gamma(T)} = E^{\gamma(T)}$. Thus $\sup_m |f(T_m x)| = \sup_m \Big|\sum_{k=1}^{\infty} f(e^k)(T_m x)_k\Big| < \infty$ for all $x \in E$. Thus E has T-AB by the Banach–Mackey theorem.

Conversely, suppose that E has T-AB. By Theorem 2.1, $E_{\text{T-AK}}$ is an FK-space with the topology induced by E. Let $y \in (E_{\text{T-AK}})^{\gamma(T)}$. By Theorem 3.2, y defines a continuous linear functional on $E_{\text{T-AK}}$ which has an extension, f, to all of E. Since E has T-AB, $\sup_m |f(T_m x)| = \sup_m \Big|\sum_{k=1}^{\infty} f(e^k)(T_m x)_k\Big|$

$= \sup_m \left|\sum_{k=1}^{\infty} y_k(T_m x)_k\right| < \infty$ for all $x \in E$. Thus $y \in E^{\gamma(T)}$. Since the opposite inclusion always holds, $E^{\gamma(T)} = (E_{\text{T-AK}})^{\gamma(T)}$.

3.4. Corollary. *Suppose E has T-FAK. Then $E^{\beta(T)} = E^{\gamma(T)}$.*

Very recently, Martin Buntinas [3], has obtained results which are similar to those of this section.

4. The spaces $E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$.

4.1. Definition. *Let $\mathscr{A}$ be a countable class of subsets of s. For each $A \in \mathscr{A}$ define the extended real-valued seminorm $p_{A(T)}$ on s by*

$$P_{A(T)}(x) = \sup_{\substack{m \\ a \in A}} \left|\sum_{k=1}^{\infty} a_k (T_m x)_k\right|.$$

Let $F_{\mathscr{A}(T)} = \{x \in s\colon p_{A(T)}(x) < \infty$ for all $A \in \mathscr{A}\}$ and $E_{\mathscr{A}(T)} = \{x \in F_{\mathscr{A}(T)}\colon \lim_m \sum_{k=1}^{\infty} a_k (T_m x)_k$ exists for all $A \in \bigcup_{\mathscr{A}} A\}$.

If $\mathscr{A}$ is a single set A, we denote $F_{\mathscr{A}(T)}$ by $F_{A(T)}$ and $E_{\mathscr{A}(T)}$ by $F_{A(T)}$. We will denote the sequence $(1, 1, 1, \ldots)$ by e. If $\mathscr{A} = \{e\}$, we denote $F_{\mathscr{A}(T)}$ by $F_{e(T)}$ and $E_{\mathscr{A}(T)}$ by $E_{e(T)}$.

4.2. Theorem. *$E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$ are FK-spaces with the topology generated by the seminorms $\{p_{A(T)}\colon A \in \mathscr{A}\}$ and $\{p_j\colon j = 1, 2, \ldots\}$. In addition, $E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$ contain E^{∞} if and only if each $A \in \mathscr{A}$ is coordinatewise bounded.*

Proof. Since p_j is a continuous seminorm for each j, $E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$ are K-spaces. Since $\mathscr{A}$ is a countable class of sets, $E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$ are metrizable. It remains to show that $E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$ are complete.

Define the continuous linear functional $f_m^a(x) = \sum_{k=1}^{\infty} a_k (T_m x)_k$ for each $a \in A$ and $m = 1, 2, \ldots$

Since $\{x \in s\colon p_{A(T)}(x) \leqslant \varepsilon\} = \bigcap_{m=1}^{\infty} \bigcap_{a \in A} \{x \in s;\ |f_m^a(x)| \leqslant \varepsilon\}$, it follows that $p_{A(T)}$ is lower semicontinuous on s. The completeness of $F_{\mathscr{A}(T)}$ now follows from Garling's completeness theorem ([5], Theorem 1). Now for each $m = 1, 2, \ldots$ and $a \in A$, $|f_m^a(x)| \leqslant p_{A(T)}(x)$. Thus the set $\{x \in F_{\mathscr{A}(T)}\colon \lim_m f_m^a(x)$ exists$\}$ is closed in $F_{\mathscr{A}(T)}$ by [12], Section 7.6, Theorem 4. Since $E_{\mathscr{A}(T)} = \bigcap_{A \in \mathscr{A}} \bigcap_{a \in A} \{x \in F_{\mathscr{A}(T)}\colon \lim_m f_m^a(x)$ exists$\}$, we have that $E_{\mathscr{A}(T)}$ is closed in $F_{\mathscr{A}(T)}$ and hence $E_{\mathscr{A}(T)}$ is complete.

Since $\sup_{a \in A} |a_i| \leqslant p_{A(T)}(e^i) \leqslant \sup_{\substack{m \\ a \in A}} \sum_{n=1}^{\infty} |t_{mn}|\,|a_i|$ and $\lim_m \sum_{k=1}^{\infty} a_k (T_m e^i)_k = \lim_m \sum_{n=i}^{\infty} t_{mn} a_i = a_i$, we have the second statement.

Unless otherwise stated, we shall assume that each $A \in \mathscr{A}$ is coordinatewise bounded.

4.3. Proposition. *$E_{\mathscr{A}(T)} = E_{\mathscr{A}(T)}^{\beta(T)\beta(T)}$ and $F_{\mathscr{A}(T)} = F_{\mathscr{A}(T)}^{\gamma(T)\gamma(T)}$.*

Proof. We first prove the proposition for spaces of the form $E_{A(T)}$. Let $B = \{x \in E_{A(T)}\colon p_{A(T)}(x) \leqslant 1\}$. Then $E_{B(T)}$ is an FK-space by Theorem 4.2. Note that B need not be coordinatewise bounded. We claim that $E_{B(T)}$ is a BK-space with norm $p_{B(T)}$. To show this we must find $M_j > 0$ such that $|y_j| \leqslant M_j p_{B(T)}(y)$ for all $y \in E_{B(T)}$. First suppose that $p_{A(T)}(e^j) \neq 0$. Let $M_j = P_{A(T)}(e^j)$. Then, since $e^j/M_j \in B$ and T is regular, $p_{B(T)}(y) \geqslant |y_j/M_j|$. If $p_{A(T)}(e^j) = 0$, let $M_j = 1$. Then for each $\lambda \in R$, $\lambda e^j \in B$. Then $\infty > p_{B(T)}(y) \geqslant \sup_{\lambda \in R} |\lambda y_j|$ and so $y_j = 0$. The claim is thus proven. Let $C = \{y \in E^{\beta(T)}\colon p_{B(T)}(y) \leqslant 1\}$. Since $A \subseteq C$, $E_{C(T)} \subseteq E_{A(T)}$. Suppose $x \in E_{B(T)}^{\beta(T)}$. Since $E_{B(T)}$ is a BK-space, $\sup_m \left|\sum_{k=1}^{\infty} x_k (T_m y)_k\right| \leqslant M p_{B(T)}(y)$ for some $M > 0$ and hence $x \in E_{C(T)}$. Now we have that $E_{B(T)} \subseteq E_{A(T)}^{\beta(T)}$ and thus $E_{A(T)}^{\beta(T)\beta(T)} \subseteq E_{B(T)}^{\beta(T)}$. We now have $E_{A(T)} \supseteq E_{C(T)} \supseteq E_{B(T)}^{\beta(T)} \supseteq E_{A(T)}^{\beta(T)\beta(T)} \supseteq E_{A(T)}$ and thus $E_{A(T)} = E_{A(T)}^{\beta(T)\beta(T)}$. Since the theorem is true for $E_{A(T)}$, it is then true for $E_{\mathscr{A}(T)} = \bigcap_{\mathscr{A}} E_{A(T)}$.

The proof of the second part is similar.

4.4. Theorem. (i) *Suppose E has T-AK. Then E can be expressed as a closed subspace of $E_{\mathscr{A}(T)}$ for some family $\mathscr{A}$. For this family $\mathscr{A}$, $E_{\mathscr{A}(T)}$ has T-FAK, $F_{\mathscr{A}(T)}$ has T-AB, $E_{\mathscr{A}(T)} = E^{\beta(T)\beta(T)}$, and $F_{\mathscr{A}(T)} = E^{\gamma(T)\gamma(T)}$.*

(ii) *Suppose E has T-FAK (T-AB). Then E can be expressed as a subspace of $E_{\mathscr{A}(T)}(F_{\mathscr{A}(T)})$ for some family $\mathscr{A}$. For this family $\mathscr{A}$, $E_{\mathscr{A}(T)}$ has T-FAK ($F_{\mathscr{A}(T)}$ has T-AB), $E_{T\text{-}AK} = (E_{\mathscr{A}(T)})_{T\text{-}AK}$ $(E_{T\text{-}AK} = (F_{\mathscr{A}(T)})_{T\text{-}AK})$, and $E_{\mathscr{A}(T)} = E^{\beta(T)\beta(T)}$ $(F_{\mathscr{A}(T)} = E^{\gamma(T)\gamma(T)})$.*

Proof. (i) Suppose E has T-AK. Let $\{q_j\colon j = 1, 2, \ldots\}$ be the seminorms which define E's topology. Let $U_j = \{x \in E\colon q_j(x) \leqslant 1\}$ and $A^j = \{a \in E^{\beta(T)}\colon \sup_{x \in U_j} \left|\lim_m \sum_{k=1}^{\infty} a_k (T_m x)_k\right| \leqslant 1\}$. Let $\mathscr{A} = \{A^j\colon j = 1, 2, \ldots\}$.

We have that $q_j(x) = \sup_{a \in A} \left|\lim_m \sum_{k=1}^{\infty} a_k (T_m x)_k\right|$ for all $x \in E$ by the Hahn–Banach theorem ([12], Section 4.4, Corollary 3). Thus

$$p_{A^j(T)}(x) = \sup_{\substack{m \\ a \in A^j}} \left|\sum_{k=1}^{\infty} a_k (T_m x)_k\right| = \sup_{\substack{m \\ a \in A^j}} \left|\lim_i \sum_{r=1}^{\infty} t_{ir} \sum_{k=1}^{r} a_k (T_m x)_k\right|$$

$$= \sup_{\substack{m \\ a \in A^j}} \left|\lim_i \sum_{k=1}^{\infty} a_k \big(T_i (T_m x)\big)_k\right| = \sup_m q_j(T_m x).$$

Since $\sup_m q_j(T_m x) = p_{A^j(T)}(x) < \infty$ for all j, we have $E \subseteq F_{\mathscr{A}(T)}$. Since $\bigcup_{j=1}^{\infty} A^j \subseteq E^{\beta(T)}$ we have $E \subseteq E_{\mathscr{A}(T)}$. Thus by [12], Section 11.3, Corollary 1, E's FK topology is stronger then the topology induced by

$E_{\mathscr{A}(T)}$. Since E has T-AK, $q_j(x) \leqslant p_{A}j_{(T)}(x)$ for all j, and hence E's FK topology is equivalent to that induced by $E_{\mathscr{A}(T)}$. Thus E is closed in $E_{\mathscr{A}(T)}$ and $E = (E_{\mathscr{A}(T)})_{\text{T-AK}} = (F_{\mathscr{A}(T)})_{\text{T-AK}}$.

It is clear from the definitions that $E^{\beta(T)} \supseteq E^{\beta(T)}_{\mathscr{A}(T)} \supseteq \operatorname{span} \bigcup_{j=1}^{\infty} A^j$. Suppose $y \in E^{\beta(T)}$. By Theorem 3.2 and [12], Section 12.1, fact ix, we have that $\left|\lim_m \sum_{k=1}^{\infty} y_k (T_m x)_k\right| \leqslant M \sum_{i=1}^{\infty} q_{j(i)}(x)$ for some $M > 0$ and some finite collection $\{j(i)\colon i = 1, 2, \ldots, p\}$ of integers. By Theorem 3.2 and [12], Section 4.4, Problem 30, we have that $y = \sum_{i=1}^{p} y^i$, where each $y^i \in E^{\beta(T)}$ is such that $\left|\lim_m \sum_{k=1}^{\infty} y^i_k (T_m x)_k\right| \leqslant M q_{j(i)}(x)$ for all $x \in E$. Thus $y \in \operatorname{span} \bigcup_{j=1}^{\infty} A^j$ and $E^{\beta(T)}_{\mathscr{A}(T)} = E^{\beta(T)}_{\mathscr{A}(T)} = \operatorname{span} \bigcup_{j=1}^{\infty} A^j$. Thus $E^{\beta(T)\beta(T)} = E^{\beta(T)\beta(T)}_{\mathscr{A}(T)} = E_{\mathscr{A}(T)}$ by Proposition 4.3. We also have that $F^{\gamma(T)}_{\mathscr{A}(T)} \supseteq \operatorname{span} \bigcup_{j=1}^{\infty} A^j = E^{\beta(T)} = E^{\gamma(T)} \supseteq F^{\gamma(T)}_{\mathscr{A}(T)}$. Thus $F^{\gamma(T)}_{\mathscr{A}(T)} = E^{\gamma(T)}$ and $E^{\gamma(T)\gamma(T)} = F^{\gamma(T)\gamma(T)}_{\mathscr{A}(T)} = F_{\mathscr{A}(T)}$. By Theorem 3.3 we have that $E_{\mathscr{A}(T)}$ has T-FAK and $F_{\mathscr{A}(T)}$ has T-AB. This concludes the proof of (i).

To prove (ii), suppose E has T-FAK (T-AB). Then $E_{\text{T-AK}}$ is an FK-space by Theorem 2.1. One can then observe that the spaces $E_{\mathscr{A}(T)}$ and $F_{\mathscr{A}(T)}$ constructed from $E_{\text{T-AK}}$ as in part (i) have the desired properties.

Remarks: (i) One should compare this result with those of Garling ([6], Theorem 5) and Ruckle ([10], Theorem 3.1) in the case when T is the identity matrix. A related result has been obtained by Buntinas ([3], Theorems 6 and 7). (ii) A space having T-FAK or T-AB need not be closed in the space $F_{\mathscr{A}(T)}$ of the preceding theorem. For example, let T be the identity matrix and let E be an FK-space such that $c_0 \subseteq E \subseteq m$ and yet E is not closed in m. An example of such a space, E, is found in [9], where a space F is given such that $F \cap c_0 = 0$ and $E = F + c_0$ is dense in m. One can easily show that $E_{\text{T-AK}} = c_0$, $c_0^{\beta(T)} = E^{\beta(T)} = l$, and $E^{\beta(T)\beta(T)} = E^{\gamma(T)\gamma(T)} = m$. E has T-FAK, by Theorem 3.3.

5. Applications.

5.1. Lemma. *Let $\mathscr{A}$ be any countable class of coordinatewise bounded subsets of s and let $x \in F_{\mathscr{A}(T)}$. Then the set $\{T'_i x\colon i = 1, 2, \ldots\}$ is bounded in $F_{\mathscr{A}(T)}$ if and only if the set*

$$A_x = \left\{ y^{(a,m)} \in s\colon y_j^{(a,m)} = \sum_{k=1}^{j} a_k (T_m x)_k,\ a \in A,\ m = 1, 2, \ldots \right\}$$

is bounded in $c_{T'}$ for every $A \in \mathscr{A}$.

Proof. Note that since T is row-finite, each set A_x is contained in $c_{T'}$. Recall that the topology of $c_{T'}$ is given by the seminorms p and $\{p_j\colon j = 1, 2, \ldots\}$. Suppose $\{T'_i x\colon i = 1, 2, \ldots\}$ is bounded in $F_{\mathscr{A}(T)}$.

Then for all $A \in \mathscr{A}$,

$$\infty > \sup_i p_{A(T)}(T'_i x) = \sup_{\substack{i,m\\ a\in A}} \left| \sum_{k=1}^{\infty} a_k \big(T_m (T'_i x)\big)_k \right| = \sup_{\substack{i,m\\ a\in A}} \left| \sum_{k=1}^{\infty} a_k \big(T'_i (T_m x)\big)_k \right|$$

$$= \sup_{\substack{i,m\\ a\in A}} \left| \sum_{j=1}^{\infty} t'_{ij} \sum_{k=1}^{j} a_k (T_m x)_k \right| = \sup_{\substack{m\\ a\in A}} p\left(y^{(a,m)}\right).$$

Also

$$\sup_{\substack{m\\ a\in A}} p_j\left(y^{(a,m)}\right) \leqslant \sup_{\substack{m\\ a\in A}} \left| \sum_{k=1}^{j} a_k x_k \right| \sum_{n=1}^{\infty} |t_{mn}| < \infty$$

since each $A \in \mathscr{A}$ is coordinatewise bounded and T is regular. Thus A_x is bounded in $c_{T'}$ for each $A \in \mathscr{A}$.

Conversely, suppose each A_x is bounded in $c_{T'}$ for each $A \in \mathscr{A}$. By reversing the above argument, we see that $\sup_i p_{A(T)}(T'_i x) < \infty$ for each $A \in \mathscr{A}$. Also $\sup_i p_k(T'_i x) \leqslant |x_k| \sup_i \left|\sum_{j=1}^{\infty} t'_{ij}\right| < \infty$. Thus the set $\{T'_i x\colon i = 1, 2, \ldots\}$ is bounded in $F_{\mathscr{A}(T)}$.

5.2. Theorem. *Suppose E has T-AK (T-FAK) (T-AB) and $c_T \subseteq c_{T'}$. Then E has T′-AK (T′-FAK) (T′-AB).*

Proof. Suppose E has T-AB. Imbed E in $F_{\mathscr{A}(T)}$ as in Theorem 4.4. We have for each $x \in E$ and $A \in \mathscr{A}$, A_x is bounded in c_T by Lemma 5.1. By [12], Section 11.3, Corollary 1, the topology of $c_{T'}$ is weaker than the topology of c_T. Thus A_x is bounded in $c_{T'}$ and so the set $\{T'_i x\colon i = 1, 2, \ldots\}$ is bounded in $F_{\mathscr{A}(T)}$. Since the set $\{T'_i x\colon i = 1, 2, \ldots\}$ is contained in E^∞, we have that the set is bounded in $(F_{\mathscr{A}(T)})_{\text{T-AK}} = E_{\text{T-AK}}$. Thus $\{T'_i x\colon i = 1, 2, \ldots\}$ is bounded in E for each $x \in E$.

Suppose E has T-FAK. Then, since $c_T \subseteq c_{T'}$, $\lim_m f(T_m x) = \lim_m \sum_{n=1}^{\infty} t_{mn} \sum_{k=1}^{n} x_k f(e^k) = \lim_i \sum_{j=1}^{\infty} t'_{ij} \sum_{k=1}^{j} x_k f(e^k) = \lim_i f(T'_i x)$ for all $x \in E$ and $f \in E'$. Thus E has T-FAK.

Suppose E has T-AK. It then has T-AB and hence T′-AB. Since $E = E^\infty$, the result follows from Theorem 2.1.

5.3 Lemma. *If $\left(\sum_{k=1}^{j} a_k x_k\right) \in c_T$ and $\left\{\left(\sum_{k=1}^{j} a_k (T_m x)_k\right)\colon m = 1, 2, \ldots\right\}$ is bounded in $c_{T'}$, then $\left(\sum_{k=1}^{j} a_k x_k\right) \in \bar{c}^{cT}$.*

Proof. Let $s = (s_{mn})$, where $s_{mn} = 1$ for $m \leqslant n$ and $s_{mn} = 0$ otherwise. One can then observe that the second hypothesis holds if and only if $\{b^m ax\colon m = 1, 2, \ldots, b^m_k = \sum_{n=k}^{\infty} t_{mn}\}$ is bounded in c_{TS} and that the conclusion holds if and only if $ax \in \overline{E^\infty}^{c_{TS}}$. The result follows from [4], Theorem 3.3.

It has been shown by Zeller [16] that given any regular (row-finite) matrix T, there is a regular (row-finite) matrix $\overline{T}$ such that $c_{\overline{T}} = \bar{c}^{cT}$. We will use this notation in the following theorem.

5.4. THEOREM. *E has T-AK (T-FAK) (T-AB) if and only if E has $\overline{T}$-AK ($\overline{T}$-FAK) ($\overline{T}$-AB).*

Proof. Suppose E has T-AB. Imbed E in $F_{\mathscr{A}(T)}$ as in Theorem 4.4. We have for each $x \in E$ and $A \in \mathscr{A}$, A_x is bounded in c_T. Since $c_{\overline{T}}$ is closed in c_T, we have that A_x is bounded in $c_{\overline{T}}$. Arguing as in the proof of Theorem 5.2, we have that the set $\{\overline{T}_i x\colon i = 1, 2, \ldots\}$ is bounded in E for each $x \in E$. Thus E has $\overline{T}$-AB.

Suppose E has T-FAK. Imbed E in $E_{\mathscr{A}(T)}$ as in Theorem 4.4. From Theorem 3.3 and the proof of Theorem 4.4, we have that $E^{\beta(T)} = \operatorname{span} \bigcup_{\mathscr{A}} A$. Let $f \in E'$. Then $\big(f(e^k)\big) \in \operatorname{span} \bigcup_{\mathscr{A}} A$. By Lemma 5.1, $\{\big(\sum_{k=1}^{j} f(e^k)(T_m x)_k\big)\colon m = 1, 2, \ldots\}$ is bounded in c_T. By Lemma 5.3, $\big(\sum_{k=1}^{j} f(e^k) x_k\big) \in \bar{c}^{cT} = c_{\overline{T}}$. Thus $\lim_m f(\overline{T}_m x)$ exists for all $f \in E'$ and E has T-FAK.

Suppose E has T-AK. Then E has T-AB and hence T-AB. Since E^∞ is dense in E, the result follows from Theorem 2.1.

The converse follows from Theorem 5.2.

5.5. DEFINITION. *A regular row-finite matrix is said to be of type S if and only $E_{e(T)}$ has T-AB.*

The following proposition shows that if T satisfies the well-known "mean value property", then T is of type S. See [1] and [13]. The $(C, 1)$ matrix and the identity matrix have this property.

5.6. PROPOSITION. *If c_T has AB, then T is of type S.*

Proof. For $T = (t_{mn})$, we define $T' = (t'_{ij})$, where $t'_{i1} = 0$ and $t'_{ij+1} = t_{ij}$. T' is clearly regular and row-finite. It is a straightforward computation to verify that

$$\sum_{n=1}^{\infty} t_{mn} \sum_{k=1}^{n} (T_i x)_k = \sum_{n=1}^{\infty} t_{mn} (T_i y)_n + \sum_{j=1}^{\infty} t_{ij} (T'_m y)_j,$$

where $y = (y_j)$ is such that $y_j = \sum_{k=1}^{j} x_k$. Since c_T has AB, it has T-AB and T′-AB by Theorem 5.2. Thus for $x \in E_{e(T)'}$

$$\sup_i p_{e(T)}(T_i x) \leqslant \sup_{m,i} \Big| \sum_{n=1}^{\infty} t_{mn} (T_i y)_n \Big| + \sup_{m,i} \Big| \sum_{j=1}^{\infty} t_{ij} (T'_m y)_j \Big| < \infty.$$

5.7. THEOREM. (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) $\Leftrightarrow$ (iv):

(i) *T is of type S.*

(ii) *$E_{e(T)}$ has T-AK.*

(iii) *$F_{e(T)}$ has T-AB.*

(iv) *$F_{\mathscr{A}(T)}$ has T-AB for every coordinatewise bounded family $\mathscr{A}$ of subsets of s.*

Proof. Suppose T is of type S. Let S be the matrix in Lemma 5.3. Observe $E_{e(T)} = c_{TS}$. The result follows from [4], Theorem 3.3 and Theorem 2.1.

Suppose $F_{e(T)}$ has T-AB. Then $\{T_i\colon i = 1, 2, \ldots\}$ is an equicontinuous set of operators on $F_{e(T)}$. Thus there exists $M > 0$ and a set of integers $\{j(q)\colon q = 1, 2, \ldots, p\}$ such that $\sup_i p_{e(T)}(T_i x) \leqslant M\big[p_{e(T)}(x) + \sum_{q=1}^{p} |x_{j(q)}|\big]$.

Suppose $x \in F_{\mathscr{A}(T)}$. Then for each $A \in \mathscr{A}$, we have

$$\sup_i p_{A(T)}(T_i x) = \sup_{\substack{i\\ a \in A}} p_{e(T)}(T_i a x)$$

$$\leqslant M\Big[\sup_{a \in A} p_{e(T)}(ax) + \sum_{q=1}^{p} |a_{j(q)} x_{j(q)}|\Big] \leqslant M\Big[p_{A(T)}(x) + \sum_{q=1}^{p} |a_{j(q)} x_{j(q)}|\Big] < \infty$$

since $x \in F_{\mathscr{A}(T)}$ and A is coordinatewise bounded. Thus $F_{\mathscr{A}(T)}$ has T-AB.

That (iv) implies (i) is obvious.

Remarks. (i) If T is of type S, it follows from Lemma 5.2 that T is perfect (i.e., c is dense in c_T). Thus any non-perfect T is not of type S. Example 5.9 is perfect but not of type S. (ii) If T is not of type S, it is still possible for a space of the form $F_{\mathscr{A}(T)}$ to have T-AB. Any FK-space having AB will have T-AB by Theorem 5.2. The construction in Theorem 4.4 will yield the desired space $F_{\mathscr{A}(T)}$.

5.8. PROPOSITION. *Let T be of type S. If $c_T \nsubseteq c_{T'}$, then $E_{e(T)}$ has T-AK but does not have T'-AB.*

Proof. The first statement follows from Theorem 5.7.

We may choose $y \in c_T / m_{T'}$, otherwise $m_{T'} \supseteq c_T$ and since T is perfect, $c_{T'} \supseteq c_T$. Define $x = (x_k)$ as follows: $x_1 = y_1$ and $x_k = y_k - y_{k-1}$. Then $x \in E_{e(T)}$. Let $f \in E'_{e(T)}$ be defined by $f(x) = \lim_m \sum_{k=1}^{\infty} (T_m x)_k$. Now

$$\sup_i |f(T'_i x)| = \sup_i \Big| \sum_{j=1}^{\infty} t'_{ij} \sum_{k=1}^{j} x_k f(e^k) \Big| = \sup_i \Big| \sum_{j=1}^{\infty} t'_{ij} y_j \Big| = \infty$$

since $y \notin m_{T'}$. Thus $\{T'_i x\colon i = 1, 2, \ldots\}$ is not bounded in $E_{e(T)}$.

EXAMPLE. Define the matrix $T = (t_{mn})$ by the following equations: $t_{11} = t_{m-1,m} = t_{mn} = 1/2$ for $m = 2, 3, \ldots$, $t_{mn} = 0$ otherwise. It is shown in [14], Theorem 8 that T is perfect. Also c_T has $(C, 1)$-AB by [14], Lemma 1. It follows from Proposition 5.8 and the following proposition that T is not of type S.

5.9. PROPOSITION. *Let T be the matrix of Example 5.9. E has T-AK (T-AB) if and only if E has AK (AB).*

Proof. Suppose E has T-AB. Imbed E in $F_{\mathscr{A}(T)}$ as in Theorem 4.5. By Lemma 5.1 we have that the set A_x is bounded in c_T for each $A \in \mathscr{A}$ and $x \in E$. Thus

$$\infty > \sup_{\substack{m,i\\ a\in\mathcal{A}}} \Big|\sum_{j=1}^{i} t_{ij}\sum_{k=1}^{i-1} a_k(T_m x)_k\Big| \geqslant \sup_{\substack{i>2\\ a\in\mathcal{A}}}\Big|\sum_{j=1}^{i} t_{ij}\sum_{k=1}^{j} a_k(T_{i-2}x)_k\Big|$$

$$= \sup_{\substack{i>2\\ a\in\mathcal{A}}}\Big|(1/2)\sum_{k=1}^{i-2}a_kx_k+(1/2)\sum_{k=1}^{i-1}a_kx_k\Big|.$$

Also,

$$\infty > \sup_{\substack{i>1\\ a\in\mathcal{A}}}\Big|\sum_{j=1}^{i} t_{ij}\sum_{k=1}^{j} a_k(T_i x)_k\Big|$$

$$\geqslant \sup_{\substack{i>1\\ a\in\mathcal{A}}}\Big|(1/2)\sum_{k=1}^{i-1}a_kx_k\Big| - \sup_{\substack{i>1\\ a\in\mathcal{A}}}\Big|(1/4)\sum_{k=1}^{i-1}a_kx_k+(1/4)\sum_{k=1}^{i}a_kx_k\Big|.$$

We have from the inequality immediately above that the second term is finite, and thus $\sup_{\substack{i\\ a\in\mathcal{A}}}\Big|\sum_{k=1}^{i} a_kx_k\Big| < \infty$. Also

$$\sup_{\substack{m,j\\ a\in\mathcal{A}}}\Big|\sum_{k=1}^{j} a_k(T_m x)_k\Big| \leqslant \sup_{\substack{i,m\\ a\in\mathcal{A}}}\sum_{n=1}^{\infty}|t_{mn}|\Big|\sum_{k=1}^{i}a_kx_k\Big| < \infty.$$

Thus A_x is bounded in $c_I = c$. By Lemma 5.1, the set $\{\overset{m}{x}\colon m = 1, 2, \ldots\}$ is bounded in $F_{\mathscr{A}(T)}$ and hence in E. Thus E has AB. It follows from theorem 2.1 that if E has T-AK, then E has AK.

Theorems 5.2 and 5.4 provide a partial answer to the question posed at the beginning of this paper. Proposition 5.8 then gives a complete answer in the case of type S matrices. Theorem 5.4 and Proposition 5.9 indicate the kind of results one might expect for matrices which are not of type S.

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