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Beurling algebras on locally compact groups, tensor products, and multipliers

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Abstract. Let G, H, and K be locally compact groups and $\theta\colon K\to G$ and $\psi\colon K\to H$ be continuous homomorphisms. A $(\theta, p;\ \psi, q)$-multiplier is a bounded linear transformation T of $L^p(G)$ into $L^q(H)$ such that $T\circ \bar{L}_{\theta(z)} = L_{\psi(z)}\circ T$ for all z in K, where L_x is the left translation by x operator. Via tensor product theory, a representation of the Banach space of $(\theta, p;\ \psi, q)$-multipliers can be obtained by identifying the topological tensor product $L^p(G)\oplus_K L^{-q}(H)$. A fundamental step in this analysis is the representation of the tensor product $L^1(G)\otimes_K \bar{L}^{-1}(H)$ as $L^1(G\otimes_K H)$, where $G\otimes_K H$ is a locally compact homogeneous space (carrying a quasi-invariant measure) canonically related to G; H, K, θ and ψ. More generally, it is shown here that $L^1_\omega(G)\otimes_{L^1_\zeta(K)} L^1_\eta(H) \cong L^1_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H)$, where ω, η, and ζ are weight functions on G, H, and K defining the Beurling algebras $L^1_\omega(G)$, $L^1_\eta()$ and $L^1_\zeta(K)$. The analysis is effected by obtaining an extension of the isomorphism $L^1_\omega(G)/J^1_\omega(G, H) \cong L^1_{\dot\omega}(G/H)$ of Reiter (for closed *normal* subgroups H of G) to permit arbitrary closed subgroups H of G.

If G is a locally compact group, $L^p(G)$, for $1\leqslant p\leqslant\infty$, denotes the usual Lebesgue space with respect to left Haar measure on G. For each $x\in G$, L_x denotes the left translation operator on $L^p(G)$ given by $L_xf(y) = f(x^{-1}y)$ for $f\in L^p(G)$ and $y\in G$. Let G, H, and K be locally compact groups, and let $\theta\colon K\to G$ and $\psi\colon K\to H$ be continuous group homomorphisms. Let $1\leqslant p, q\leqslant\infty$. A $(\theta, p;\ \psi, q)$-*multiplier* is a bounded linear transformation T from $L^p(G)$ into $L^q(H)$ such that $T\circ L_{\theta(z)} = L_{\psi(z)}\circ T$ for all $z\in K$. In this context the "multiplier problem" is to characterize the space $\mathrm{Hom}_K(L^p(G), L^q(H))$ of $(\theta, p;\ \psi, q)$-multipliers of $L^p(G)$ into $L^q(H)$. When $G = H = K$ and $\theta = \psi = \mathrm{id}_G$, the identity map on G, we recapture the classical multiplier problem of characterizing the bounded linear transformations of $L^p(G)$ into $L^q(G)$ which commute with left translation by the elements of G. When $1\leqslant p\leqslant\infty$ and $1\leqslant q<\infty$, and $\dfrac{1}{q}+\dfrac{1}{q'} = 1$, the theory of tensor products of Banach modules introduced by Rieffel in [12] shows that

$$\mathrm{Hom}_K\big(L^p(G), L^{q'}(H)\big) \cong \big(L^p(G)\otimes_K \bar{L}^q(H)\big)^*, \tag{0.1}$$

where the isomorphism is linear and isometric, $(\cdot)^*$ denotes the Banach space dual, and, where $L^p(G)$ and $\bar{L}^q(H) = L^q(H)$ are the left and right Banach K-modules, respectively, under the actions: $z\cdot g(\cdot) = L_{\theta(z)}g(\cdot)$ and $z\cdot h(\cdot) = L_{\psi(z)^{-1}}h(\cdot)$ for $z\in K$, $g(\cdot)\in L^p(G)$ and $h(\cdot)\in L^q(H)$. A suitable function space representation of the K-module tensor product, $L^p(G)\otimes_K \bar{L}^q(H)$, would yield a representation theorem for the $(\theta, p; \psi, q)$-multipliers analogous to that obtained for the classical multipliers by Figà-Talamanca [1], Figà-Talamanca and Gaudry [2], and Rieffel [14].

The specific concern of this paper is with the case $p = 1$ and $q' = \infty$, for then in view of relation (0.1), attention is directed to the tensor product, $L^1(G)\otimes_K \bar{L}^1(H)$, and the problem is to characterize this Banach space. In [10] the author has characterized the tensor product, $L^1(G)\otimes_{L^1(K)}L^1(H)$, in *all* instances of algebra actions of $L^1(K)$ on $L^1(G)$ and $L^1(H)$ for locally compact Abelian groups G, H, and K (cf. Theorem 6.5 [10]).

In this paper we extend the analysis in [4], [12], and [10] to arbitrary locally compact groups G, H, and K, and continuous homomorphisms $\theta: K\to G$ and $\psi: K\to H$. In fact, it is shown that if the Beurling algebras $L^1_\omega(G)$ and $\bar{L}^1_\eta(H)$ are left and right Banach $L^1_\zeta(K)$-modules under the induced actions from θ and ψ, then the tensor product $L^1_\omega(G)\otimes_{L^1_\zeta(K)}\bar{L}^1_\eta(H)$ is isometrically isomorphic to a weighted L^1-space on a homogeneous space $G\otimes_K H$ carrying a quasi-invariant measure; the paper also contains an extension of this to vector-valued Beurling spaces.

1. Beurling algebras. Throughout this section G will denote a locally compact group with left Haar measure dx. A *weight function* on G is an upper semicontinuous function $\omega: G\to R^+$ such that (i) ω is bounded away from 0, (ii) $\omega(xy)\leqslant \omega(x)\omega(y)$ for all $x, y\in G$.

The Beurling algebra on G with weight function ω, denoted $L^1_\omega(G)$, is the subalgebra of $L^1(G)$ consisting of those f such that $f\omega\in L^1(G)$, and $L^1_\omega(G)$ is a Banach algebra under the norm $\|f\|_{1,\omega} = \int_G |f(x)|\,\omega(x)\,dx$, $f\in L^1_\omega(G)$. These algebras were introduced by Reiter in [15, § 3].

Let H be a closed subgroup of G, $d\xi$ the left Haar measure on H, G/H the homogeneous space of left cosets of H in G, π_H the canonical projection of $G\to G/H$, and Δ_G and Δ_H the modular functions on G and H. According to [16], Chap. 8, § 1,2, there is a quasi-invariant positive measure $d_q\dot{x}$ on G/H corresponding to a strictly positive continuous function q on G satisfying the functional equation

$$q(x\xi) = q(x)\frac{\Delta_H(\xi)}{\Delta_G(\xi)} \quad \text{for all } x\in G \text{ and } \xi\in H, \tag{1.1}$$

such that the measures dx, $d\xi$, $d_q\dot{x}$ are canonically related, i.e.,

$$\int_G f(x)\,dx = \int_{G/H}\left\{\int_H \frac{f(x\xi)}{q(x\xi)}\,d\xi\right\} d_q\dot{x}, \quad f\in L^1(G). \tag{1.2}$$

The mapping $T_{H,q}$ defined by $T_{H,q}f(\dot{x}) = \int_H f(x\xi)/q(x\xi)\,d\xi$, $\dot{x} = \pi_H(x)$, $f\in L^1(G)$, is a linear contraction of $L^1(G)$ onto $L^1(G/H)$.

If ω is a weight function on G, then $\dot{\omega}(\dot{x}) = \inf_{\xi\in H}\omega(x\xi)$, $\dot{x} = \pi_H(x)$, is clearly an upper semicontinuous positive function on G/H with values bounded away from 0. Note that $\dot{\omega}$ has no submultiplicative properties unless H is normal. However, one can still form the weighted Lebesgue space and Banach space

$$L^1_{\dot{\omega}}(G/H) = \Big\{h\in L^1(G/H; d_q\dot{x})\,\Big|\, \|h\|_{1,\dot{\omega}} = \int_{G/H}|h(\dot{x})|\,\dot{\omega}(\dot{x})\,d_q\dot{x} < \infty\Big\}.$$

In this case there obtains the following generalization of Reiter ([15], § 5, and cf. [16], Chap. 3, § 7.4).

(1.1) Lemma. *If H is a closed subgroup of G, then $T_{H,q}$ maps $L^1_\omega(G)$ onto $L^1_{\dot{\omega}}(G/H)$. In fact, if $J^1_\omega(G, H)$ is the kernel of the restriction of $T_{H,q}$ to $L^1_\omega(G)$, then*

$$L^1_{\dot{\omega}}(G/H) \cong L^1_\omega(G)/J^1_\omega(G, H)$$

and this isomorphism is not only algebraic but isometric (the right-hand side being provided with the ordinary quotient norm).

Proof. The elegant proof of Reiter ([16], Chap. 3, § 7.6–7.9) for closed normal subgroups can be carried over with appropriate modifications; we point out the less obvious changes. The lemma is first proved for G countable at infinity exactly as in § 7.6 and § 7.7, Chap. 3 of [16], by merely changing T_H to $T_{H,q}$ and observing that the necessary Proposition 4.9, Chap. 3, [16], can be extended to closed subgroups with a minor modification in the proof given there (cf. the extended formula of Mackey–Bruhat ([16], eq. (2), p. 164 and lines (−7) to (−4), p. 165) is applied to $q(\cdot)\chi_{M_0}(\cdot)$ and $q(\cdot)\chi_M(\cdot)$).

If G_* is any open subgroup of G, and if H_* is the open subgroup $H\cap G_*$ in H, then Δ_{G_*} and Δ_{H_*} are simply the restrictions of Δ_G and Δ_H to G_* and H_*, respectively. Consequently, the restriction q_* of q to G_* satisfies the functional equation

$$q_*(x\xi) = q_*(x)\frac{\Delta_{H_*}(\xi)}{\Delta_{G_*}(\xi)} \quad \text{for all } x\in G_* \text{ and } \xi\in H_*,$$

and hence with the restricted Haar measures on G_* and H_* there corresponds a quasi-invariant measure $d_{q_*}\dot{x}$ on G_*/H_*. Let τ denote the topological isomorphism $\tau: G_*/H_* \cong \pi_H(G_*)$, $\tau(x/H_*) = \pi_H(x)$, $x\in G_*$. Then the restriction of $d_q\dot{x}$ to $\pi_H(G_*)$ is precisely the positive measure induced on $\pi_H(G_*)$ by τ and the measure $d_{q_*}\dot{x}$ on G_*/H_*, i.e.,

$$\int_{G_*/H_*} h\circ\tau\, d_{q_*}\dot{x} = \int_{\pi_H(G_*)} h\, d_q\dot{x}, \quad h\in \mathscr{K}(\pi_H(G_*)). \tag{1.3}$$

Indeed, let $h \in \mathscr{K}(\pi_H(G_*))$ and let h_1 denote the natural extension of h to a function on G/H. According to Chap. 8, § 2.3, and Chap. 3, § 4.2, of [16], there is an $f \in \mathscr{K}(G_*)$ such that $T_{H_*,q_*}f = h \circ \tau \in \mathscr{K}(G_*/H_*)$. Let f_1 be the natural extension of f to a function on G. Now, from [16], Chap. 3, § 7.8 (i), we have $T_{H,q}f_1(x/H) = T_{H_*,q_*}f(x/H_*) = h(\tau(x)) = h(x/H)$, for all $x \in G_*$ and hence $T_{H,q}f_1(x/H) = h(x/H)$ for all $x \in G_*H$. Since $\operatorname{supp}(f_1) = \operatorname{supp}(f) \subseteq G_*$, we have $T_{H,q}f_1(x/H) = 0 = h_1(x/H)$ for all $x \in G \setminus (G_*H)$, so that $T_{H,q}f_1 = h_1$. Applying the Mackey–Bruhat formula (1.2), we obtain $\int_{\pi_H(G_*)} h\,d_q\dot{x} = \int_{G/H} h_1\,d_q\dot{x} = \int_G f_1\,dx = \int_{G_*} f\,dx = \int_{G_*/H_*} T_{H_*,q_*}f\,d_{q_*}\dot{x} = \int_{G_*/H_*} h \circ \tau\, d_{q_*}\dot{x}$.

In proving the result for general G the analysis in Chap. 3, § 7.9 of [16] can be continued without change down to line (19). Following line (19) and in the notation of § 7.9, Chap. 3 [16] we have from the fact that the lemma holds for groups countable at infinity,

$$d_f = \inf_{G_*} \int_{G_*/H_*} |T_{H_*,q_*}f_*(\dot{x})|\,\dot{\omega}_*(\dot{x})\,d_q\dot{x}.$$

Since $T_{H_*,q_*}f_*(\dot{x}) = T_{H,q}f(\tau(\dot{x}))$ for all $\dot{x} \in G_*/H_*$, we have by applying relation (1.3),

$$d_f = \inf_{G^*} \int_{\pi_H(G_*)} |T_{H,q}f(\dot{x})|\,\dot{\omega}_*(\tau^{-1}(\dot{x}))\,d_q\dot{x}.$$

Now, noting that $\dot{\omega}_*(\tau^{-1}(\dot{x})) = \inf_{\xi \in H_*} \omega(x\xi)$, $\dot{x} = \pi_H(x)$, $x \in G_*$, the remainder of the proof that $d_f = \|T_{H,q}f\|_{1,\dot{\omega}}$ proceeds as in the proof following line (20) of [16], § 7.9, Chap. 3, with the replacement there of T_H by $T_{H,q}$. This completes the proof.

In the lemma that follows we obtain a characterization of the subspace $J^1_\omega(G, H)$ for general closed subgroups H of G, extending the characterization of Reiter ([16], Chap. 3, § 6.4) for closed normal subgroups.

(1.2) Lemma. *Let H_0 be a subgroup of G with closure H and let D be a norm dense subset of $L^1_\omega(G)$. Then $J^1_\omega(G, H)$ is the closed linear subspace in $L^1_\omega(G)$ spanned by all elements of the form $A_\eta f - f$, $\eta \in H_0$, $f \in D$, where $A_\eta f(x) = f(x\eta)\Delta_G(\eta)$, $x \in G$.*

Proof. First, suppose $H_0 = H$ and $D = \mathscr{K}(G)$. Since the proof for this case is only a modification of the proof in § 6.4, Chap. 3 of [16], we merely sketch it. Let J denote the closed linear subspace generated. $J \subseteq J^1_\omega(G, H)$ since $T_{H,q}A_\eta f = T_{H,q}f$ for all $\eta \in H$ and $f \in \mathscr{K}(G)$ by relation (1.1). Conversely, since J is left invariant it suffices (§ 7.2, § 7.3 (Remark 1), and § 6.3 of Chap. 3 [16]) to show every continuous $\phi \in L^\infty_\omega(G)$ (Def. 7.3, Chap. 3 [16]) that is orthogonal to J is also orthogonal to $J^1_\omega(G, H)$. Let $R'_\eta\phi(x) = \phi(x\eta^{-1})$; then since $0 = \langle A_\eta f - f, \phi\rangle = \langle f, R'_\eta\phi - \phi\rangle$ for all $f \in \mathscr{K}(G)$, we have by the continuity of ϕ that $\phi(x\eta) = \phi(x)$ for all $x \in G$, $\eta \in H$.

Thus, $\phi = \phi' \circ \pi_H$, where ϕ' is a continuous function in $L^\infty_{\dot{\omega}}(G/H)$. An application of the formula of Mackey–Bruhat (1.2) shows that ϕ is orthogonal to $J^1_\omega(G, H)$. Therefore, $J^1_\omega(G, H) \subseteq J$.

Now, we treat the general case for H_0 and D as described in the lemma. Let J denote the closed linear subspace generated. Let $f \in D$ and $\eta \in H_0$. For $g \in \mathscr{K}(G)$ we have

$$\|(A_\eta f - f) - (A_\eta g - g)\|_{1,\omega} \leqslant (\omega(\eta^{-1}) + 1)\|f - g\|_{1,\omega}$$

and thereby from the density of $\mathscr{K}(G)$ in $L^1_\omega(G)$ and the first part of the proof we have $J \subseteq J^1_\omega(G, H)$. On the other hand, let $g \in \mathscr{K}(G)$ and $\xi \in H$. Given $\varepsilon > 0$, choose $f \in D$ so that $\|g - f\|_{1,\omega} < (1 + \omega(\xi^{-1}))^{-1}\varepsilon/2$. Since $x \to A_x f$ is strongly continuous at the identity of G (cf. § 7.2 (3ρ), Chap. 3 [16]) and H_0 is dense in H, there is an η in H_0 such that $\|f - A_{\xi^{-1}\eta}f\|_{1,\omega} \leqslant \omega(\xi^{-1})\varepsilon/2$. Then we have

$$\|(A_\xi g - g) - (A_\eta f - f)\|_{1,\omega} \leqslant \|A_\xi(g - f)\|_{1,\omega} + \|A_\xi f - A_\eta f\|_{1,\omega} + \|g - f\|_{1,\omega}. \tag{1.4}$$

The sum of the first and third terms in the right-hand side of (1.4) is bounded above by $(\omega(\xi^{-1}) + 1)\|g - f\|_{1,\omega} < \varepsilon/2$. As for the second term we have by the choice of η in H_0,

$$\|A_\xi f - A_\eta f\|_{1,\omega} = \|A_\xi(f - A_{\xi^{-1}\eta}f)\|_{1,\omega} \leqslant \omega(\xi^{-1})\|f - A_{\xi^{-1}\eta}f\|_{1,\omega} < \varepsilon/2.$$

Since ε is arbitrary, $A_\xi g - g \in J$, $g \in \mathscr{K}(G)$, $\xi \in H$, and by the first part of the proof, $J^1_\omega(G, H) \subseteq J$. The proof is complete.

(1.3) Corollary. *Let H_0 and D be as in Lemma 1.2. If S is a generating set for H_0 and D is right H_0-translation invariant, then $J^1(G, H)$ is the closed linear subspace in $L^1(G)$ spanned by all elements of the form $A_s f - f$ and $A_{s^{-1}}f - f$ for $s \in S$ and $f \in D$.*

Proof. Let J denote the closed linear subspace generated. By Lemma 1.2 we need only show $J^1_\omega(G, H) \subseteq J$. Let $f \in D$ and $\eta \in H_0$. Since S generates H_0, we can write $\eta = t_1 t_2 \ldots t_n$, where $t_i = s_i^{\lambda_i}$, $s_i \in S$, and λ_i is $+1$ or -1 for $i = 1, \ldots, n$. Setting $t_{j+1} \ldots t_n = \xi_{j+1}$ for $j = 1, \ldots, n-1$, we have

$$A_\eta f - f = (A_{t_n}f - f) + \sum_{j=1}^{n-1} \big(A_{t_j}(A_{\xi_{j+1}}f) - (A_{\xi_{j+1}}f)\big).$$

Since $\xi_{j+1} \in H_0$ and D is right H_0-translation invariant, i.e. $A_{\xi_{j+1}}f \in D$, we have $A_\eta f - f \in J$. Thus, by Lemma 1.2, $J^1_\omega(G, H) \subseteq J$.

We conclude this section with an extension of Theorem 7.10, Chap. 3 of Reiter [16].

(1.4) Corollary. *Let ϕ be a complex-valued function on G/H, H being a closed subgroup of G. Then ϕ is in $L^\infty_{\dot{\omega}}(G/H)$ if and only if $\phi \circ \pi_H$ is in $L^\infty_\omega(G)$ and in either case*

$$\|\phi\|_{\infty,\dot{\omega}} = \|\phi \circ \pi_H\|_{\infty,\omega}.$$

2. Tensor products of Beurling algebras. The main tools in the remainder of the paper are Banach modules, their Banach space tensor products, and their elementary properties. In this regard, the reader is urged to consult the paper [13] of Rieffel.

Let G, H, and K be locally compact groups, and let $\theta: K \to G$ and $\psi: K \to H$ be continuous homomorphisms. $M(K)$ denotes the Banach algebra of all complex-valued regular Borel measures on K. $L^1(G)$ is a left Banach $M(K)$-module under the action $(\mu, g) \to \mu *_\theta g$, where $\mu *_\theta g(x) = \int_K g(\theta(z)^{-1}x)\,d\mu(z)$, and $L^1(H)$ is a right Banach $M(K)$-module under the action $(\mu, h) \to \mu^\sim *_\psi h$, where $\mu^\sim$ in $M(K)$ is defined by the relation $\langle \mu^\sim, f\rangle = \langle \mu, f^*\rangle$, f in $C_0(K)$, where $f^*(z) = f(z^{-1})$. The map $\mu \to \mu^\sim$ of $M(K)$ into $M(K)$ is a real adjoint operation on the algebra $M(K)$ with $f^\sim(z) = f^*(z)/\Delta_K(z)$ for f in $L^1(K) \subset M(K)$. We will indicate the fact that $L^1(H)$ is a right module under this action by writing $\bar{L}^1(H)$.

Now, suppose ω, η, and ζ are weight functions on G, H, and K, respectively, and let $L^1_\omega(G)$, $L^1_\eta(H)$, and $L^1_\zeta(K)$ denote the corresponding Beurling (group) algebras on G, H, and K. Let $M_\zeta(K)$ denote the Beurling (measure) algebra on K corresponding to the weight function ζ, i. e., $M_\zeta(K)$ is the subalgebra consisting of all μ in $M(K)$ such that $\|\mu\|_\zeta \equiv \int_K \zeta\, d|\mu|$ is finite. $M_\zeta(K)$ is a Banach algebra under the norm $\|\ \|_\zeta$, and $L^1_\zeta(K)$ is the closed ideal in $M_\zeta(K)$ consisting of those measures in $M_\zeta(K)$ absolutely continuous with respect to left Haar measure on K.

(2.1) Lemma. (1) *$L^1_\omega(G)$ is a left Banach $L^1_\zeta(K)$-module under the action $(f, g) \to f *_\theta g$ if and only if there is a constant $M \geqslant 0$ such that $\omega(\theta(z)x) \leqslant M\zeta(z)\omega(x)$ for locally a. e. (z, x) in $K \times G$.*

(2) *$L^1_\eta(H)$ is a right Banach $L^1_\zeta(K)$-module under the action $(f, h) \to f^\sim *_\psi h$ if and only if there is a constant $M \geqslant 0$ such that $\eta(\psi(z)^{-1}y) \leqslant M\zeta(z)\eta(y)$ for locally a. e. (z, y) in $K \times H$.*

The proof of Lemma 2.1 is routine and is omitted.

(2.2) Remark. Although the condition on ω, ζ, and θ in Lemma 2.1 is clearly sufficient for $L^1_\omega(G)$ to be a left Banach module over $L^1_\zeta(K)$ under the action $(f, g) \to f *_\theta g$, it is not in general sufficient for $L^1_\omega(G)$ to be a left Banach module over $M_\zeta(K)$ under the action $(\mu, g) \to \mu *_\theta g$. But rather, a stronger condition, such as: for a. e. x in G, $\omega(\theta(z)x) \leqslant M\zeta(z)\omega(x)$, is needed. Furthermore, note that if continuous homomorphisms θ and ψ and weight functions ω and η are given, then there is *always* a weight function ζ on K such that $\omega(\theta(z)x) \leqslant \zeta(z)\omega(x)$, $\eta(\psi(z)^{-1}y) \leqslant \zeta(z)\eta(y)$, v. z., let $\zeta(z) = \max\big(\omega(\theta(z)), \eta(\psi(z)^{-1})\big)$, for z in K.

Let Q denote the closed subgroup in $G \times H$ and closure of the subgroup $\{(\theta(z), \psi(z)): z \in K\}$. Let $G \otimes_K H$ denote the locally compact homogeneous space, $(G \times H)/Q$, of left cosets of Q in $G \times H$. Equip $G \times H$ with the product Haar measure $dx \otimes dy$ and let $d(u, v)$ denote the left Haar measure on Q. Let $d_q(x\dot{,} y)$ denote the quasi-invariant positive measure on $G \otimes_K H$ so that $dx \otimes dy$, $d(u, v)$, and $d_q(x\dot{,} y)$ are canonically related. Let $\omega^* \wedge \eta^*$ denote the weight function on $G \times H$ defined by $\omega^* \wedge \eta^*(x, y) = \omega^*(x)\eta^*(y) = \omega(x^{-1})\eta(y^{-1})$; let $\omega^* \otimes_\zeta \eta^*$ denote the quotient weight function on $G \otimes_K H$ given by $\omega^* \otimes_\zeta \eta^*(x\dot{,} y) = \inf_{(u,v) \in Q} \omega^* \wedge \eta^*(xu, yv)$, for $(x\dot{,} y) \in G \otimes_K H$. $L^1_{\omega^* \otimes_\zeta \eta^*}(G \otimes_K H)$ denotes the weighted Lebesgue space corresponding to $\omega^* \otimes_\zeta \eta^*$ and $d_q(x\dot{,} y)$.

With the above notations and definitions we are prepared to state the primary result of this section.

(2.2) Theorem. *Let Λ_ζ be a subset of $M_\zeta(K)$ such that $L^1_\omega(G)$ and $\bar{L}^1_\eta(H)$ are left and right Banach Λ_ζ-modules under the actions $(\mu, g) \to \mu *_\theta g$ and $(\mu, h) \to \mu^\sim *_\psi h$, respectively, and such that*

(i) *$\delta_z * \Lambda_\zeta \subseteq \Lambda_\zeta$ for all z in K, and*

(ii) *for each g in $\mathscr{K}(G)$, h in $\mathscr{K}(H)$, and $\varepsilon > 0$, there exists a μ in Λ_ζ such that $\|g - \mu *_\theta g\|_{1,\omega} < \varepsilon$ and $\|h - \mu^\sim *_\psi h\|_{1,\eta} < \varepsilon$.*

Then there obtains the natural isometric isomorphism

$$L^1_\omega(G) \otimes_{\Lambda_\zeta} \bar{L}^1_\eta(H) \cong L^1_{\omega^* \otimes_\zeta \eta^*}(G \otimes_K H),$$

where the element $g \otimes h$ corresponds to $T_{Q,q}(g^\sim \wedge h^\sim)$.

(2.3) Corollary. *If $L^1_\omega(G)$ and $\bar{L}^1_\eta(H)$ are left and right Banach $L^1_\zeta(K)$-modules under the actions induced by θ and ψ (cf. Lemma 2.1), then*

$$L^1_\omega(G) \otimes_{L^1_\zeta(K)} \bar{L}^1_\eta(H) \cong L^1_{\omega^* \otimes_\zeta \eta^*}(G \otimes_K H),$$

where the isomorphism is linear and isometric.

Proof of Corollary. By Theorem 2.2 it is only necessary to show that $\Lambda_\zeta \equiv L^1_\zeta(K)$ satisfies conditions (i) and (ii). Condition (i) is obvious and if g in $\mathscr{K}(G)$, h in $\mathscr{K}(H)$, and $\varepsilon > 0$ are given, choose neighborhoods of the identity, U in G and V in H, such that $\|L_x g - g\|_{1,\omega} < \varepsilon$ for x in U and $\|L_y h - h\|_{1,\eta} < \varepsilon$ for y in V. Let N be a symmetric neighborhood of the identity in K such that $\theta(N) \subseteq U$ and $\psi(N) \subseteq V$. If $f \in \mathscr{K}^+(K)$, $\|f\|_1 = 1$, and $\operatorname{supp}(f) \subset N$, then we have

$$\|f *_\theta g - g\|_{1,\omega} \leqslant \int_N \|L_{\theta(z)} g - g\|_{1,\omega} f(z)\,dz < \varepsilon,$$

$$\|f^\sim *_\psi h - h\|_{1,\eta} \leqslant \int_N \|L_{\psi(z)} h - h\|_{1,\eta} f^\sim(z)\,dz < \varepsilon,$$

and so condition (ii) is satisfied.

Proof of Theorem 2.2. Let R denote the isometric isomorphism, $L^1_\omega(G) \otimes_\gamma L^1_\eta(H) \cong L^1_{\omega^*}(G) \otimes_\gamma L_{\eta^*}(H)$, implemented by the map $R(g \otimes h)$

$= g^{\sim}\otimes h^{\sim}$. Let S denote the Grothendieck [6, p. 90] and Johnson [9] isomorphism, $L^1_{\omega^*}(G)\otimes_\gamma L^1_{\eta^*}(H) \cong L^1_{\omega^*\wedge\eta^*}(G\times H)$ (cf. [5], Remark 3, p. 304), implemented by the correspondence $S(g_1\otimes h_1) = g_1\wedge h_1$, and finally let T denote the composite isometric isomorphism $S\circ R$. By Lemma 1.1, there obtains the isometric isomorphism

$$L^1_{\omega^*\wedge\eta^*}(G\times H)/J^1_{\omega^*\wedge\eta^*}(G\times H, Q) \cong L^1_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H),$$

with the left-hand side provided with the ordinary quotient norm. The proof of Theorem 4.1 is complete once we have shown that $T(K_0) = J^1_{\omega^*\wedge\eta^*}(G\times H, Q) \equiv J$, where K_0 is the closed linear subspace in $L^1_\omega(G)\otimes_\gamma L^1_\eta(H)$ generated by all elements of the form

$$(\mu *_\theta g)\otimes h - g\otimes(\mu^{\sim} *_\psi h), \qquad \mu\in\Lambda_\zeta,\ g\in L^1_\omega(G),\ h\in L^1_\eta(H).$$

We first show that $T(K_0)\subseteq J$. Let $\mu\in\Lambda_\zeta$, $g\in\mathscr{K}(G)$, and $h\in\mathscr{K}(H)$. Then for all $(\dot{x,y}) = (x,y)/Q\in G\otimes_K H$, we have

$$T_{Q,q}\big((\mu *_\theta g)^{\sim}\wedge h^{\sim}\big)(\dot{x,y})$$
$$= \int_Q\int_K \frac{g\big(\theta(z^{-1})u^{-1}x^{-1}\big)h^{\sim}(yv)}{q(xu, yv)}\,\Delta_G\big((xu)^{-1}\big)\,d\mu(z)\,d(u,v).$$

Now, by interchanging the order of integration, making the change of variables $Q\ni(u,v)\rightsquigarrow\big(u\theta(z^{-1}), v\psi(z^{-1})\big)\in Q$, and taking account for the modular function on Q, we have

$$T_{Q,q}\big((\mu *_\theta g)^{\sim}\wedge h^{\sim}\big)(\dot{x,y})$$
$$= \int_K\int_Q \frac{g\big((xu)^{-1}\big)h^{\sim}\big(yv\psi(z^{-1})\big)}{q\big(xu\theta(z^{-1}), yv\psi(z^{-1})\big)}\,\frac{\Delta_Q\big(\theta(z^{-1}),\psi(z^{-1})\big)}{\Delta_G(xu)\,\Delta_G\big(\theta(z^{-1})\big)}\,d(u,v)\,d\mu(z).$$

Since $q\big(xu\theta(z^{-1}), yv\psi(z^{-1})\big) = q(xu,yv)\Big(\Delta_Q\big(\theta(z^{-1}),\psi(z^{-1})\big)\Big)/\Delta_{G\times H}\big(\theta(z^{-1}),\psi(z^{-1})\big)$ and since $\Delta_{G\times H}(x,y) = \Delta_G(x)\,\Delta_H(y)$ in general, we have

$$T_{Q,q}\big((\mu *_\theta g)^{\sim}\wedge h^{\sim}\big)(\dot{x,y}) = \int_Q\int_K \frac{g^{\sim}(xu)\,h\big(\psi(z)(yv)^{-1}\big)}{q(xu,yv)}\,\Delta_H\big((yv)^{-1}\big)\,d\mu(z)\,d(u,v)$$
$$= \int_Q \frac{g^{\sim}(xu)\,\mu^{\sim} *_\psi h\big((yv)^{-1}\big)}{q(xu,yv)}\,\Delta_H\big((yv)^{-1}\big)\,d(u,v)$$
$$= T_{Q,q}\big(g^{\sim}\wedge(\mu^{\sim} *_\psi h)^{\sim}\big)(\dot{x,y}).$$

Thus, $T_{Q,q}\Big(T\big((\mu *_\theta g)\otimes h - g\otimes(\mu^{\sim} *_\psi h)\big)\Big) = 0$ for all μ in Λ_ζ, g in $\mathscr{K}(G)$, h in $\mathscr{K}(H)$; by a density argument it holds for all g in $L^1_\omega(G)$ and h in $L^1_\eta(H)$. It follows by the linearity and continuity of T that $T(K_0)\subseteq J$. To prove the reverse inclusion, let D be the norm dense subset of $L^1_{\omega^*\wedge\eta^*}(G\times H)$ consisting of all functions which are finite linear combinations of functions

of the form $g\wedge h$ for $g\in\mathscr{K}(G)$, $h\in\mathscr{K}(H)$. Noting that Q is the closure of the subgroup $\{(\theta(z),\psi(z))\,|\,z\in K\}$ in $G\times H$, we have by Lemma 1.2 that $J = J^1_{\omega^*\wedge\eta^*}(G\times H, Q)$ is the closed linear subspace in $L^1_{\omega^*\wedge\eta^*}(G\times H)$ generated by all elements of the form $A_{(\theta(z),\psi(z))}k - k$ for z in K and k in D. But, since $A_{(\theta(z),\psi(z))}g\wedge h = (A_{\theta(z)}g)\wedge(A_{\psi(z)}h)$, we have from the definition of D and linearity of $A_{(\cdot,\cdot)}$ that J is the closed linear subspace in $L^1_{\omega^*\wedge\eta^*}(G\times H)$ spanned by all elements of the form

$$(A_{\theta(z)}g)\wedge(A_{\psi(z)}h) - g\wedge h, \qquad g\in\mathscr{K}(G),\ h\in\mathscr{K}(H).$$

Now, since $\big(L_{\theta(z)}(g^{\sim})\big)^{\sim} = A_{\theta(z)}g$ and $\big(L_{\psi(z)}(h^{\sim})\big)^{\sim} = A_{\psi(z)}h$, it is easily seen that $T(\tau) = (A_{\theta(z)}g)\wedge(A_{\psi(z)}h) - g\wedge h$, where

$$\tau = L_{\theta(z)}(g^{\sim})\otimes L_{\psi(z)}(h^{\sim}) - g^{\sim}\otimes h^{\sim}\in L^1_\omega(G)\otimes_\gamma L^1_\eta(H).$$

We show $\tau\in K_0$, for then it follows from the isomorphic and isometric property of T that $J\subseteq T(K_0)$. Let $\varepsilon > 0$ be given, and choose $\mu\in\Lambda_\zeta$ so that $\|g^{\sim} - \mu *_\theta g^{\sim}\|_{1,\omega} < \varepsilon$ and $\|\mu^{\sim} *_\psi h^{\sim} - h^{\sim}\|_{1,\eta} < \varepsilon$. Now,

$$\tau = \big(L_{\theta(z)}(g^{\sim} - \mu *_\theta g^{\sim})\otimes L_{\psi(z)}h^{\sim}\big) +$$
$$+\big(L_{\theta(z)}(\mu *_\theta g^{\sim})\otimes L_{\psi(z)}h^{\sim} - g^{\sim}\otimes\mu^{\sim} *_\psi h^{\sim}\big) + \big(g^{\sim}\otimes(\mu^{\sim} *_\psi h^{\sim} - h^{\sim})\big).$$

The sum of the first and last terms in this expansion of τ are bounded in norm by $\varepsilon\big(\omega(\theta(z))\|L_{\psi(z)}h^{\sim}\|_{1,\eta} + \|g^{\sim}\|_{1,\omega}\big)$. From the fact that in general $L_{\theta(z')}g' = \delta_{z'} *_\theta g'$, we have

$$L_{\theta(z)}(\mu *_\theta g^{\sim}) = \delta_z *_\theta(\mu *_\theta g^{\sim}) = (\delta_z * \mu) *_\theta g^{\sim}$$

and

$$(\delta_z * \mu)^{\sim} *_\psi (L_{\psi(z)}h^{\sim}) = (\mu^{\sim} * \delta_{z^{-1}}) *_\psi (\delta_z *_\psi h^{\sim}) = \mu^{\sim} *_\psi h^{\sim}.$$

Applying these relations to the middle term in the above expansion of τ, we see that this middle term is equal to

$$\big((\delta_z * \mu) *_\theta g^{\sim}\big)\otimes L_{\psi(z)}h^{\sim} - g^{\sim}\otimes\big((\delta_z * \mu)^{\sim} *_\psi (L_{\psi(z)}h^{\sim})\big)$$

and since $\delta_z * \mu$ is in Λ_ζ by hypothesis, it is in K_0. Since $\varepsilon > 0$ is arbitrary, we have τ in K_0. The proof is complete.

(2.4) Remark. The set of all left translates of a two-sided approximate unit in $L^1_\zeta(K)$ is an example of a subset Λ_ζ in $M_\zeta(K)$ with minimal structure satisfying conditions (i) and (ii) of Theorem 2.2 (and of course, provided conditions (1) and (2) of Lemma 2.1 on ω, η, ζ, θ, and ψ are satisfied). Furthermore, if $\omega(\theta(\cdot)) \leqslant M\zeta(\cdot)$ and $\eta(\psi(\cdot)^{-1}) \leqslant M\zeta(\cdot)$, then $\Lambda_\zeta = K$ and $\Lambda_\zeta = M_\zeta(K)$ satisfy the hypotheses (i) and (ii) of Theorem 2.2. On the other hand, we remark that Theorem 3.14 of [12] is not directly applicable since in general $L^1_\omega(G)$ and $L^1_\eta(H)$ are not *uniformly bounded* Banach K-modules under the left and right actions induced by $\theta\colon K\to G$ and $\psi\colon K\to H$: $(k, g)\to\delta_k *_\theta g$ and $(k, h)\to\delta_k^{\sim} *_\psi h = \delta_{k^{-1}} *_\psi h$.

(2.6) Remark. Consider the (classical) case when $G = H = K$, $\theta = \psi = \mathrm{id}_G$, and $\omega = \eta = \zeta \equiv 1$. Then $Q = \mathrm{diag}(G\times G)$ and $G\otimes_G G = (G\times G)/Q$. The mapping $\tau((x,y)/Q) = xy^{-1}\in G$ for $(x,y)/Q \in G\otimes_G G$ is a topological isomorphism of $G\otimes_G G$ onto G and where $\tau^{-1}(x) = (x,e)/Q$, $x\in G$. The mapping $\varrho(x) = (x,x)$, $x\in G$, is a topological and group isomorphism of G onto Q; let $d(u,u)$ be the Haar measure on Q induced by the adjoint map of ϱ and the Haar measure dx on G, i.e.

$$(2.1)\qquad \int_G R\circ\varrho(x)\,dx = \int_Q R(u,u)\,d(u,u) \quad \text{for all } R\in L^1(Q).$$

Define $r(x,y) = q(x,y) \equiv \Delta_G(y^{-1})$ for $(x,y)\in G\times G$. Then $r(x,y)$ satisfies the relations $r(xx', yy') = r(x,y)r(x',y')$, $r(u,u) = \Delta_Q(u,u)/\Delta_{G\times G}(u,u)$, so that we can take the measure $d_r(x,\dot y) = d_q(x,\dot y)$ on $G\otimes_G G$ to be a relatively invariant positive measure ([16], Chap. 8, § 1.4). Applying relation (2.1) to $T_{Q,r}(g\wedge h^{\sim})(\tau^{-1}(\cdot))$ one can show that

$$(2.2)\qquad T_{Q,r}(g\wedge h^{\sim})(\tau^{-1}(\cdot)) = g*h(\cdot),\qquad g,h\in L^1(G),$$

and a further computation will show

$$\int_{G\otimes_G G} T_{Q,r}(g\wedge h^{\sim})(x,\dot y)\,d_r(x,\dot y) = \int_G g*h(x)\,dx,\qquad g,h\in L^1(G).$$

It then is seen that $d_r(x,\dot y)$ is the positive measure on $G\otimes_G G$ induced by the adjoint of τ and the Haar measure dx on G. Thus, the adjoint of τ induces the algebraic and isometric isomorphism

$$\tau^*: L^p(G;dx) \cong L^p(G\otimes_G G; d_r(x,\dot y))$$

for all $1\leqslant p\leqslant\infty$; in particular, when $p=1$, $L^1(G)\cong L^1(G\otimes_G G)$. Of course, the fact that $L^1(G)\cong L^1(G)\otimes_{L^1(G)}\bar L^1(G)$ is directly derivable from Theorem (4.4) of [12]. Relation (2.2) above along with Theorem 2.2 suggests a method for handling the $(\theta,p;\psi,q)$-multiplier problem mentioned in the introduction while at the same time recapturing the classical representations in [1], [2], and [13]. We pursue these notions in a subsequent paper [11].

3. $(\theta, 1;\psi,\infty)$-multipliers. In this section we assume that either

(1) $\eta(\psi(z)^{-1}y)\leqslant M\zeta(z)\eta(y)$ for a.e. (z,y) in $K\times H$

or

(1)′ $\eta(\psi(z)^{-1})\leqslant M\zeta(z)$ for locally a.e. z in K.

In either case, if f is in $L^1_\zeta(K)$ and φ is in $L^\infty_\eta(H)$, then $f*_\psi\varphi(y) = \int_K f(z)\varphi(\psi(z)^{-1}y)\,dz$ defines a continuous function on H and $\|f*_\psi\varphi\|_{\infty,\eta} \leqslant M\|f\|_{1,\zeta}\|\varphi\|_{\infty,\eta}$, where $L^\infty_\eta(H)$ is the Banach space dual of $L^1_\eta(H)$ ([16], Chap. 3, § 7.3) with the dual pairing given by

$$\langle h,\varphi\rangle = \int_H h(y)\varphi(y)\,dy.$$

Under this action, $L^\infty_\eta(H)$ is a left Banach $L^1_\zeta(K)$-module, and in view of the relation

$$\langle f^{\sim}*_\psi h,\varphi\rangle = \langle h, f*_\psi\varphi\rangle,$$

f in $L^1_\zeta(K)$, h in $\bar L^1_\eta(H)$, and φ in $L^\infty_\eta(H)$, this left $L^1_\zeta(K)$-module action is the adjoint action of $L^1_\zeta(K)$ on $L^\infty_\eta(H)$ induced by the right action of $L^1_\zeta(K)$ on $\bar L^1_\eta(H)$. Further, note that $\delta_z*_\psi\varphi$, for z in K, is well defined with $\|\delta_z*_\psi\varphi\|_{\infty,\eta}\leqslant M\|\varphi\|_{\infty,\eta}\eta(\psi(z)^{-1})$, and $(z,\varphi)\to\delta_z*_\psi\varphi$ makes $L^\infty_\eta(H)$ into a left Banach K-module and it is the adjoint action of that induced by the right action of K on $\bar L^1_\eta(H)$.

Let $L^\infty_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H)$ denote the Banach space dual of $L^1_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H)$ and the space of complex valued $d_q(x,\dot y)$-measurable functions Φ on $G\otimes_K H$ such that

$$\|\Phi\|_{\infty,\omega^*\otimes_\zeta\eta^*} = \text{ess-sup}\left(|\Phi(x,\dot y)|/\omega^*\otimes_\zeta\eta^*(x,\dot y)\right) < \infty.$$

Combining Theorem 2.2 and [13], Corollary 2.13 we obtain a characterization of the space of $(\theta,1;\psi,\infty)$-multipliers for Beurling algebras on arbitrary locally compact groups.

(3.1) Corollary. *There obtain the isometric isomorphisms*

$$\mathrm{Hom}_K(L^1_\omega(G), L^\infty_\eta(H)) = \mathrm{Hom}_{L^1_\zeta(K)}(L^1_\omega(G), L^\infty_\eta(H)) \cong L^\infty_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H),$$

where the multiplier T corresponds to the linear functional Φ so that

$$\langle h, Tg\rangle = \int_{G\otimes_K H} T_{H,q}(g^{\sim}\wedge h^{\sim})\,\Phi\,d_q(x,\dot y)$$

for all h in $L^1_\eta(H)$ and g in $L^1_\omega(G)$.

4. Extensions to vector valued Beurling spaces. Notations are assumed to be as in Section 2. If V and W are Banach spaces, then $L^1_\omega(G,V)$ and $L^1_\eta(H,W)$ denote the usual Lebesgue–Bochner spaces of vector valued functions. If A is a Banach algebra then $L^1_\zeta(K,A)$ is a Banach algebra with vector valued convolution as multiplication (see [9]). If V and W are left and right Banach A-modules for a Banach algebra A, and if ω,η,ζ,θ and ψ satisfy conditions (1) and (2) of Lemma 2.1, then $L^1_\omega(G,V)$ and $L^1_\eta(H,W)\equiv\bar L^1_\eta(H,W)$ are left and right Banach $L^1_\zeta(K,A)$-modules under the actions:

$$(f,g)\to f*_\theta g(x) = (B)\int_K f(z)\cdot g(\theta(z)^{-1}x)\,dz,$$

$$(f,h)\to f^{\sim}*_\psi h,\quad \text{where } f^{\sim}(z) = f(z^{-1})/\Delta_K(z),$$

with $\|f*_\theta g\|_{1,\omega}\leqslant \mathrm{Const}\|f\|_{1,\zeta}\|g\|_{1,\omega}$ and $\|f^{\sim}*_\psi h\|_{1,\eta}\leqslant\mathrm{Const}\|f\|_{1,\zeta}\|h\|_{1,\eta}$. The verification that these actions are Banach module actions is tedious but presents no unexpected difficulties and is left to the reader.

(4.1) THEOREM. *If A is a Banach algebra with a bounded two-sided approximate unit, and V and W are essential left and right Banach A-modules, then*

$$L^1_\omega(G, V)\otimes_{L^1_\zeta(K,A)} \bar{L}^1_\eta(H, W) \cong L_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H, V\otimes_A W),$$

where the isomorphism is algebraic and isometric, and where the element $(g(\cdot)v)\otimes(h(\cdot)w)$ corresponds to the element $T_{Q,q}(g^{\sim}\wedge h^{\sim})(\cdot)v\otimes w$ for all g in $L^1_\omega(G)$, h in $\bar{L}^1_\eta(H)$, v in V and w in W.

A proof of Theorem 5.1 can be obtained through the following sequence of lemmas. For the sake of brevity in presentation we omit their proofs.

(4.2) LEMMA. *Let A' and A be Banach algebras, and let V' and V be left (resp. right) Banach A' and A-modules, repectively. Then $V'\otimes_\gamma V$ is a left (resp. right) Banach $A'\otimes_\gamma A$-module under the action induced by*

$$(a'\otimes a)\cdot(v'\otimes v) = (a'\cdot v')\otimes(a\cdot v),$$

for $a'\in A'$, $a\in A$, $v'\in V'$, and $v\in V$.

(4.3) LEMMA. *Let V and W be left and right Banach A-modules. If $L^1_\omega(G)\otimes_\gamma V$ and $\bar{L}^1_\eta(H)\otimes_\gamma W$ are regarded as left and right Banach $L^1_\zeta(K)\otimes_\gamma A$-modules under the action described in Lemma 4.2, then the isomorphisms $L^1_\omega(G, V)\cong L^1_\omega(G)\otimes_\gamma V$ and $\bar{L}^1_\eta(H, W)\cong \bar{L}^1_\eta(H)\otimes_\gamma W$ intertwine the module actions by $L^1_\zeta(K, A)$ and $L^1_\zeta(K)\otimes_\gamma A$.*

(4.4) LEMMA. *Let A' and A be Banach algebras with bounded two-sided approximate units, and let V', V and W', W be assential left and right Banach A', A-modules, respectively. Then there obtains the natural isometric isomorphism*

$$(V'\otimes_\gamma V)\otimes_{(A'\otimes_\gamma A)}(W'\otimes_\gamma W)\cong(V'\otimes_{A'}W')\otimes_\gamma(V\otimes_A W),$$

where the element $(v'\otimes v)\otimes(w'\otimes w)$ corresponds to the element $(v'\otimes w')\otimes(v\otimes w)$.

We have found the hypothesis of Lemma 4.4 necessary in order to show the canonical epimorphism $(V'\otimes_\gamma V)\otimes_\gamma(W'\otimes_\gamma W)\to(V'\otimes_{A'}W')\otimes_\gamma(V\otimes_A W)$ has kernel identical to the reducing subspace of $(V'\otimes_\gamma V)\otimes_\gamma(W'\otimes_\gamma W)$ defining the tensor module $(V'\otimes_\gamma V)\otimes_{(A'\otimes_\gamma A)}(W'\otimes_\gamma W)$.

A proof of Theorem 4.1 can now be obtained from Lemmas 4.3 and 4.4 and Corollary 2.3, v. z., we have

$$\begin{aligned} L^1_\omega(G, V)\otimes_{L^1_\zeta(K,A)}\bar{L}^1_\eta(H, W) &\cong \big(L^1_\omega(G)\otimes_\gamma V\big)\otimes_{(L^1_\zeta(K)\otimes_\zeta A)}\big(\bar{L}^1_\eta(H)\otimes_\gamma W\big)\\ &\cong\big(L^1_\omega(G)\otimes_{L^1_\zeta(K)}\bar{L}^1_\eta(H)\big)\otimes_\gamma(V\otimes_A W)\\ &\cong L^1_{\omega^*\otimes_\zeta\eta^*}(G\otimes_K H, V\otimes_A W).\end{aligned}$$

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