d'une façon immédiate, c.-à-d. sans faire usage de l'équation du type:

$$\frac{d^2}{dt^2}x(t) = A\left(x(t), \frac{d}{dt}x(t)\right)$$

(ce qu'on fait d'habitude), et qui peut être dépourvue de sens dans le cas des espaces plus généraux.

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Received December 9, 1972 (733)

STUDIA MATHEMATICA, T. LII. (1974)

On topological, Lipschitz, and uniform classification of LF-spaces

by

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Abstract. It is proved that all essentially infinite-dimensional LF-spaces (i.e. LF-spaces which are not isomorphic to the strict inductive limit of finite-dimensional spaces) are topologically equivalent. The problem of Lipschitz and uniform classification is also studied. Some partial results are obtained.

Introduction. The problem of the topological classification of Fréchet spaces was investigated by several authors (see for ex. [1], [2], [3], [8]). There exists a conjecture that all infinite-dimensional Fréchet spaces with the same density character are homeomorphic to each other. In 1966, Kadec verified this conjecture for separable Banach spaces, and then Anderson extended this result to the class of separable Fréchet spaces.

In Section 2 of this paper we establish a theorem of the Kadec–Anderson type for separable LF-spaces. It should be noted that the methods presented in that section enable us to solve completely the problem of topological classification of LF-spaces with the density character less than $\aleph$, under the hypothesis that the Kadec–Anderson theorem extends to Fréchet spaces with density character $\aleph'$, for all $\aleph' < \aleph$.

The problem of the uniform classification of Fréchet spaces has not been solved yet (even in the separable case). Lindenstrauss [11] has proved that there exists a continuum of Banach spaces which are different with respect to uniform homeomorphisms. There is a conjecture that in the class of separable Fréchet spaces the following fact holds:

(C) *A space X is uniformly homeomorphic with a space Y if and only if X is isomorphic to Y.*

The validity of this conjecture in the case where X is a Hilbert space has been established by Enflo [6]. Similar results for $X = s$ (the space of all scalar sequences) and for $X = H\times s$ have been obtained by the author [12], [16]. Paper [12] contains also some invariants of uniform homeomorphisms in the class of Fréchet spaces.

In Section 4 we study the problem of uniform classification of LF-spaces. We show that in the class of separable LF-spaces the conjecture (C) is true under the assumption that X is isomorphic to a subspace of a space of the form $\sum H_i \times s_i$. Also, we prove that the linear dimension is an invariant of uniform homeomorphisms in the class of real Montel LF-spaces.

In Section 3 we study Lipschitz mappings in LF-spaces. The results obtained in this section are used in the proofs of theorems in Section 4.

Remark. In this paper we study spaces over the field of reals only; however, the majority of the results of the paper extend trivially to the complex case.

I am very much indebted to Dr H. Toruńczyk, who pointed to me an unpublished result of his (Lemma 2.9) which plays an important role in the proof of the main result of Section 2, and who made essential simplifications of the proofs in this paper.

1. Preliminaries. A locally convex vector space $(X, \mathcal{T})$ is said to be an LF-*space* iff there exists a sequence $\{(X_n, \mathcal{T}_n): n \in N\}$ of Fréchet spaces such that the following conditions are fulfilled:

1. $X_1 \subsetneqq X_2 \subsetneqq X_3 \subsetneqq \ldots \subset \bigcup_{i=1}^{\infty} X_i = X$,

2. for every $n \in N$ the topology $\mathcal{T}_{n+1}$ restricted to X_n coincides with the topology $\mathcal{T}_n$,

3. the topology $\mathcal{T}$ is the locally convex inductive limit topology on X generated by the identical embeddings of $(X_n, \mathcal{T}_n)$ into X for $n \in N$.

If this is the case then we write

$$X = \operatorname{ind\,lim} X_n. \tag{1}$$

Obviously, if $(X, \mathcal{T})$ is an LF-space then the representation of X in the form (1) is not unique.

Let X be an LF-space, $X = \operatorname{ind\,lim} X_n$. It can be proved that

a convex subset U of X is open in X if and only if $U \cap X_n$ is an open subset of $(X_n, \mathcal{T}_n)$ for every $n \in N$, a subset B of X is bounded in X if and only if there exists an n_0 such that $B \subset X_{n_0}$ and B is bounded in $(X_{n_0}, \mathcal{T}_{n_0})$.

For any two topological spaces (A, ϱ_1) and (B, ϱ_2) we shall denote by $A \times B$ the Cartesian product of A and B endowed with the product topology. The Cartesian product of a family $\{A_\vartheta, \varrho_\vartheta: \vartheta \in \Theta\}$ of topological spaces will also be denoted by $\prod_{\vartheta \in \Theta} A_\vartheta$.

Let X_n be a Fréchet space for $n = 1, 2, \ldots$ By $\sum_{i \in N} X_i$, or briefly $\sum X_i$, we shall denote the LF-space of all eventually zero sequences $(x_1, x_2, \ldots)$, $x_i \in X_i$ for $i = 1, 2, \ldots$, endowed with the topology of the locally convex inductive limit generated by the canonical embeddings $(x_1, x_2, \ldots, x_n) \to (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ of finite products.

In the sequel we shall need the following fact:

PROPOSITION 1.1. *Let $X = \sum X_i$ be an LF-space. Then the sets of the form*

$$\Big\{(x_1, x_2, \ldots) \in \sum X_i:\ x_i \in U_i \text{ for } i \in N\Big\}, \tag{2}$$

where U_i is an arbitrary convex neighbourhood of the origin in $(X_i, \mathcal{T}_i)$ for $i \in N$, constitute a base of neighbourhoods of the origin in X.

Proof. It follows from the definition of the topology of $\sum_{i \in N} X_i$ that the sets (2) are convex neighbourhoods of the origin in X. Conversely, we shall prove that every convex neighbourhood of the origin in X contains a set of the form (2). To this end let U be a convex neighbourhood of the origin in X. Put

$$\begin{aligned}\tilde{V}_i &= U \cap (\{0\} \times \{0\} \times \ldots \times \{0\} \times X_i \times \{0\} \times \{0\} \times \ldots)\\ &= \{0\} \times \{0\} \times \ldots \times \{0\} \times V_i \times \{0\} \times \ldots\end{aligned}$$

for $i \in N$ and define

$$W_n = \Big\{(x_1, x_2, \ldots) \in \sum_{i \in n} X_i:\ x_i \in 2^{-i} V_i \text{ for } i = 1, 2, \ldots, n \text{ and } x_i = 0 \text{ for } i > n\Big\}$$

for $n \in N$. Then for every $n \in N$ we have

$$U \supset \sum_{i \in 1}^{n} 2^{-i} U \supset \sum_{i=1}^{n} 2^{-i} \tilde{V}_i \supset W_n.$$

Put

$$W = \bigcup_{n=1}^{\infty} W_n.$$

We have $W \subset U$. It is easy to see that the set W is of the form (2) (with $U_i = 2^{-i} V_i$). ■

Let $X = \operatorname{ind\,lim} X_i$ be an arbitrary LF-space. One can also consider on X the topology $\mathcal{T}'$ of the inductive limit taken in the category of topological spaces. In the sequel, the space $(X, \mathcal{T}')$ will be denoted by $\varinjlim X_n$, and will be called a tLF-*space*. The tLF-spaces possess some nice properties which the LF-spaces need not have. It easily follows from the definition of the inductive limit topology in the category of topological spaces that

(*) *if $X = \varinjlim X_n$, then the following conditions are equivalent:*

(i) *a mapping f defined on a subset A of X is continuous,*

(ii) *$f|_{A \cap X_n}$ is continuous for every $n \in N$,*

(iii) *f is sequentially continuous.*

On the other hand, one can prove

THEOREM 1.2. *Let $X = \operatorname{ind\,lim} X_n$. Then conditions* (i), (ii) *and* (iii) *are equivalent if and only if all X_n's are finite-dimensional.*

Thus the topologies of $\operatorname{ind\,lim} X_n$ and $\underrightarrow{\lim} X_n$ are, in general, different. It follows from Theorem 1.2 that these topologies coincide if and only if X_n is finite-dimensional for each n.

Now, let $(X, \mathscr{T})$ be a locally convex topological vector space. A set $\{P_\vartheta\colon \vartheta \in \Theta\}$ of continuous pseudonorms on X is said to be a *system of pseudonorms inducing the topology on* X iff the family of sets $\{x \in X\colon P_\vartheta(x) < \alpha\}$ for $\vartheta \in \Theta$ and $\alpha > 0$ forms a base of neighbourhoods of the origin for the topology $\mathscr{T}$.

By $\operatorname{dens} X$ we denote the density character of the space $(X, \mathscr{T})$, i.e. the smallest cardinal number $\aleph$ for which there exists a dense subset A of X with $\bar{\bar{A}} = \aleph$.

Let $(X, \mathscr{T})$ and $(Y, \mathscr{T})$ be locally convex topological vector spaces. A one-to-one mapping f from X onto Y is said to be a *homeomorphism* between $(X, \mathscr{T})$ and $(Y, \mathscr{T})$ iff both f and f^{-1} are continuous. If in addition f is linear, then f is said to be an *isomorphism*. A homeomorphism f is said to be a *uniform homeomorphism* iff both f and f^{-1} are uniformly continuous. A one-to-one mapping f from a subset A of X into Y is said to be a *homeomorphic embedding* of a subset A of X into Y iff both f and f^{-1} are continuous. In the same way we define an *isomorphic embedding* of a linear subspace X_0 of X into Y or a *uniform embedding* of a subset A of X into Y.

In the sequel we shall identify isomorphic spaces. Thus, for example, the notation $X = Y$ means that X is isomorphic to Y.

We shall consider only vector spaces over the field R of reals.

2. Topological classification. Let R_i be a copy of the real line for $i = 1, 2, \ldots$ We have

THEOREM 2.1. *An* LF-*space X is homeomorphic with the space $\sum R_i$ if and only if it is isomorphic with $\sum R_i$.*

Proof. Indeed, let $X = \operatorname{ind\,lim} X_i$ be an LF-space and let $\dim X_{i_0} = \infty$ for some $i_0 \in N$. It can easily be proved that then there exists a compact convex infinite-dimensional subset C contained in $X_{n_0} \subset X$. On the other hand, every compact subset contained in $\sum R_i$ is finite-dimensional. Hence if X is homeomorphic with $\sum R_i$ then $\dim X_i < \infty$ for every $i \in N$, which means that the space X is the inductive limit of finite-dimensional spaces. But this implies that X is isomorphic with $\sum R_i$. ∎

In the sequel, by an *essentially infinite-dimensional* LF-*space* we shall mean any LF-space which is not isomorphic to $\sum R_i$. This terminology is justified by Theorem 2.1.

Let H be the infinite-dimensional separable Hilbert space. The space $H \times \sum R_i$ is an LF-space: $H \times \sum R_i = \operatorname{ind\,lim} H \times \prod_{i \leqslant n} R_i$.

The following theorem holds:

THEOREM 2.2. *Let $X = \operatorname{ind\,lim} X_i$ be an essentially infinite-dimensional separable* LF-*space. Then X is homeomorphic with $H \times \sum R_i$, provided that $\dim X_{i+1}/X_i < \infty$ for all but finitely many $i \in N$.*

Proof. Let n_0 be such that $\dim X_{i+1}/X_i < \infty$ for all $i \geqslant n_0$. Observe that then X_{n_0} is an infinite-dimensional Fréchet space. Indeed, if this is not the case, then $X = \operatorname{ind\,lim} X_i = \operatorname{ind\,lim} X_{n_0+i}$ is an inductive limit of finite-dimensional Fréchet spaces and hence X is not an essentially infinite-dimensional LF-space, which contradicts the assumption of the theorem.

Denote by Y_i the space X_{i+1}/X_i for every $i \geqslant n_0$. Then we have

$$X = \operatorname*{ind\,lim}_i X_i = \operatorname*{ind\,lim}_i X_{n_0+i} = \operatorname*{ind\,lim}_i X_{n_0} \times Y_{n_0+1} \times \ldots \times Y_{n_0+i}$$
$$= X_{n_0} \times \sum_{i > n_0} Y_i,$$

since, owing to the way we have chosen n_0, we have $\dim Y_i < \infty$ for $i \geqslant n_0$. Thus the space $\sum_{i>n_0} Y_i$ is isomorphic to the space $\sum R_i$. Hence

$$X = X_{n_0} \times \sum R_i.$$

By the Kadec–Anderson theorem ([8]) we infer that the space X_{n_0} is homeomorphic with H, but this implies that the space $X = X_{n_0} \times \sum R_i$ is homeomorphic with the space $H \times \sum R_i$. ∎

THEOREM 2.3. *Let $X = \operatorname{ind\,lim} X_i$ be an essentially infinite-dimensional* LF-*space, where for every $i \in N$ the space X_i is a reflexive Banach space. Then the space X is homeomorphic with the space $H_0 \times \sum R_i$, where H_0 denotes the infinite-dimensional Hilbert space with the same density character as X, provided that $\dim X_{i+1}/X_i < \infty$ for all but finitely many i's.*

Proof. The proof is similar to the proof of Theorem 2.2. The only difference is that we use the Bessaga theorem [3] on the topological equivalence of all infinite-dimensional reflexive Banach spaces with the same density character, instead of the Kadec–Anderson theorem. ∎

Now, let X be a Fréchet space and let Y be a fixed closed subspace of X. For every $x \in X$ denote

$$x^{\sim} = \{p \in X\colon p - x \in Y\} \subset X \tag{3}$$

In the sequel we shall identify the sets of the form (3) with the elements of the quotient space X/Y.

In the sequel we shall need several lemmas.

LEMMA 2.4. *Let X be a Fréchet space and let Y be its closed linear subspace. Then there exists a continuous selection f of the set-valued mapping*

$$I: X/Y \ni x^{\sim} \to x^{\sim} \subset X,$$

where I denotes the inverse mapping to the quotient mapping

$$Q: X \ni x \to x^{\sim} \in X/Y.$$

This means that there exists a continuous mapping

$$f: X/Y \to X$$

such that $f(x^{\sim}) = p \in x^{\sim}$ for every $x^{\sim} \in X/Y$.

Proof. The lemma is an easy consequence of the Michael theorem on selection [17]. ■

LEMMA 2.5. *Let $X = \operatorname{indlim} X_n$ be an LF-spaces. Then X is homeomorphic with the space $Y = \sum Y_i$, where $Y_1 = X_1$ and $Y_i = X_i/X_{i-1}$ for $i \geqslant 2$.*

Proof. Let $I_n: X_n/X_{n-1} \to X_n$ be the inverse mapping of the quotient mapping $Q_n: X_n \to X_n/X_{n-1}$ for $n \in N$ (we put additionally $X_0 = \{0\}$) and let

$$f_n: Y_n = X_n/X_{n-1} \to X_n$$

be a continuous selection for I_n (Lemma 2.4). Without loss of generality we may assume that $f_n(0^{\sim}) = 0$ for $n \in N$.

Define a mapping h from $\sum Y_i$ into $\operatorname{indlim} X_i$ by the formula

$$h(y_1, y_2, \ldots) = \sum_{i \in N} f_i(y_i) \quad \text{for} \quad (y_1, y_2, \ldots) \in \sum Y_i. \tag{4}$$

Observe that

$$h(Y_1 \times Y_2 \times \ldots \times Y_n \times \{0\} \times \{0\} \times \ldots) = X_n \text{ for } n \in N. \tag{5}$$

Indeed, equality (5) is satisfied for $n = 1$. Assume that we have proved equality (5) for $k = 1, 2, \ldots, n-1$. By (4),

$$h(Y_1 \times Y_2 \times \ldots \times Y_n \times \{0\} \times \{0\} \times \ldots) \subset X_n.$$

On the other hand, assume that $x_n \in X_n$. Then we have

$$x_n = \big(x_n - f_n Q_n(x_n)\big) + f_n Q_n(x_n),$$

where $x_n - f_n Q_n(x_n) \in X_{n-1}$ and $Q_n(x_n) \in Y_n$. According to our assumption there exists

$$(y_1, y_2, \ldots, y_{n-1}, 0, 0, \ldots) \in Y_1 \times Y_2 \times \ldots \times Y_{n-1} \times \{0\} \times \{0\} \times \ldots$$

such that

$$h(y_1, y_2, \ldots, y_{n-1}, 0, 0, \ldots) = x_n - f_n Q_n(x_n).$$

Hence

$$h(y_1, y_2, \ldots, y_{n-1}, Q_n(x_n), 0, 0, \ldots) = x_n,$$

which completes the proof of equality (5).

Moreover,

$$h_{|Y_1 \times Y_2 \times \ldots \times Y_n \times \{0\} \times \{0\} \times \ldots}$$

is a one-to-one mapping for every $n \in N$. We shall prove it by induction. For $n = 1$ it is obvious. Assume that we have proved it for $n = 1, 2, \ldots \ldots, n-1$ and let

$$(y_1, y_2, \ldots, y_n, 0, 0, \ldots), (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, 0, 0, \ldots) \in Y_1 \times Y_2 \times \ldots \times Y_n \times \{0\} \times \{0\} \times \ldots$$

be such that

$$h(y_1, y_2, \ldots, y_n, 0, 0, \ldots) = \sum_{i=1}^{n} f_i(y_i) = \sum_{i=1}^{n} f_i(\bar{y}_i) = h(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, 0, 0, \ldots).$$

Then

$$Q_n\Big(\sum_{i=1}^{n} f_i(y_i)\Big) = y_n = \bar{y}_n = Q_n\Big(\sum_{i=1}^{n} f_i(\bar{y}_i)\Big). \tag{6}$$

Hence

$$\sum_{i=1}^{n-1} f_i(y_i) = h(y_1, y_2, \ldots, y_{n-1}, 0, 0, \ldots) = h(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n-1}, 0, 0, \ldots) = \sum_{i=1}^{n-1} f_i(\bar{y}_i).$$

According to our assumption we have $y_i = \bar{y}_i$ for $i = 1, 2, \ldots, n-1$. This and (6) imply that $(y_1, y_2, \ldots, y_n, 0, 0, \ldots) = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, 0, 0, \ldots)$, which concludes the proof of the fact that h is a one-to-one mapping.

Now, it remains only to prove that h is a continuous and open mapping. Let $(\bar{y}_1, \bar{y}_2, \ldots) \in Y = \sum Y_i$ and let $h(\bar{y}_1, \bar{y}_2, \ldots) = x = \sum f_i(\bar{y}_i) \in X$. Let V be an arbitrary convex neighbourhood of the origin in X. Put $V_i = V \cap X_i$ for $i = 1, 2, \ldots$ The continuity of the selection f_i at the point $\bar{y}_i$ for $i \in N$ implies that there exist convex neighbourhoods $U_i \subset X_i/X_{i-1}$ of the points $\bar{y}_i$ for $i \in N$ such that $f_i(U_i) \subset 2^{-i} V_i + f_i(\bar{y}_i)$ for $i \in N$. Let U be a neighbourhood of the point $(\bar{y}_1, \bar{y}_2, \ldots)$ in Y of the form

$$U = \{(y_1, y_2, \ldots) \in Y: y_i \in U_i \text{ for } i \in N\}$$

(Prop. 1.1). Then $h(U) \subset V+x$. Indeed, let $(y_1, y_2, \ldots) \in U$. Then it follows from the definition of the neighbourhoods U_i that

$$h(y_1, y_2, \ldots) = \sum_{i \in N} f_i(y_i) \in \sum_{i \in N} \left(2^{-i} V_i + f_i(\bar{y}_i)\right)$$
$$= \sum_{i \in N} f_i(\bar{y}_i) + \sum_{i \in N} 2^{-i} V_i \subset h(\bar{y}_1, \bar{y}_2, \ldots) + V,$$

which concludes the proof of the continuity of h.

Now we shall show that the mapping h is open at the origin. Observe that $h(0) = 0$. Let U be an arbitrary neighbourhood of the origin in Y of the form

$$U = \{(y_1, y_2, \ldots) \in Y\colon\ y_i \in U_i \text{ for } i \in N\},$$

where each U_i is an arbitrary convex neighbourhood of the origin in $Y_i = X_i/X_{i-1}$.

Fix a sequence $\{c_n\colon n \in N\}$ of positive numbers such that $1 > c_n > c_1 \cdot c_2 \cdot \ldots \cdot c_n > \frac{1}{2}$ for every $n \in N$. For every $n \in N$ we shall construct a convex neighbourhood V_n of the origin in X_n such that the conditions

(i)$_n$ $h(U) \supset V_n$,

(ii)$_n$ $V_n \cap X_{n-1} \supset c_n V_{n-1}$

will be satisfied for $n \in N$.

Then

$$V = \bigcup_{n \in N} \frac{1}{2c_1 \cdot c_2 \cdot \ldots \cdot c_n} V_n$$

will be a convex neighbourhood of the origin in X (it suffices to observe that by (ii)$_n$

$$\frac{1}{2c_1 c_2 \ldots c_n} V_n \cap X_{n-1} \supset \frac{1}{2c_1 c_2 \ldots c_{n-1}} V_{n-1}$$

for $n \in N$) such that

$$V \subset \bigcup_{n \in N} V_n \subset h(U).$$

We shall construct the sequence $\{V_n\colon n \in N\}$ by induction. First, we put $V_1 = U_1$ and suppose that we have defined the sets $V_1, V_2, \ldots, V_{n-1}$ satisfying conditions (i)$_k$ and (ii)$_k$ for $k = 1, 2, \ldots, n-1$. Let W_0 be a convex neighbourhood of the origin in X_n such that

$$u(x_n) = x_n - f_n Q_n(x_n) \in (1-c_n) V_{n-1}$$

for $x_n \in W_0$ (the existence of such a neighbourhood of the origin follows from the fact that the correspondence

$$u\colon x_n \to x_n - f_n Q_n(x_n)$$

is a continuous mapping from X_n onto X_{n-1} with $u(0) = 0$). Then the set $W = c_n V_{n-1} + (1-c_n) W_0$ satisfies the conditions

$$X_{n-1} \cap W \supset c_n V_{n-1} \tag{7}$$

and

$$x_n - f_n Q_n(x_n) \in V_{n-1} \quad \text{for} \quad x_n \in W. \tag{8}$$

Indeed, to prove (8) it suffices to observe that every $x_n \in W$ admits a representation at the form $x_n = c_n x_{n-1} + z_n$, where $x_{n-1} \in V_{n-1} \subset X_{n-1}$ and $z_n \in W_0$. Hence, according to the fact that $Q_n(c_n x_{n-1} + z_n) = Q_n(z_n)$, we obtain

$$x_n - f_n Q_n(x_n) = c_n x_{n-1} + z_n - f_n Q_n(z_n) \in c_n V_{n-1} + (1-c_n) V_{n-1} \subset V_{n-1}.$$

Put $V_n = W \cap \{x_n \in X_n\colon Q_n(x_n) \in U_n\}$. Then V_n is a convex neighbourhood of the origin and it follows from (7) that V_n satisfies condition (ii)$_n$. In order to prove that V_n satisfies condition (i)$_n$ we fix an arbitrary $x_n \in X_n$ and represent x_n at the form

$$x_n = \left(x_n - f_n Q_n(x_n)\right) - f_n Q_n(x_n).$$

It follows from the definition of the set V_n that

$$x_n - f_n Q_n(x_n) \in V_{n-1} \subset h(U) \cap X_{n-1}.$$

Hence by (5) and (i)$_{n-1}$ we infer that there exist $y_i \in U_i$ for $i = 1, 2, \ldots, n-1$ such that

$$\sum_{i=1}^{n-1} f_i(y_i) = x_n - f_n Q_n(x_n).$$

Putting $y_n = Q_n(x_n)$ we obtain $x_n = \sum_{i=1}^{n} f_i(y_i)$, where $y_i \in U_i$ for $i = 1, 2, \ldots, n$. Hence $x_n \in f(U)$, which completes the proof of the fact that f is an open mapping at the origin. A similar argument shows that f is open at every point $y \in \sum Y_i$. ■

Lemma 2.6. *Let $Y = \sum Y_i$, where Y_i is an infinite-dimensional separable Fréchet space for $i \in N$. Then Y is homeomorphic with the space $\sum H_i$, where H_i denotes an infinite-dimensional separable Hilbert space for $i \in N$.*

Proof. It is enough to observe that by Proposition 1.1 the mapping f given by the formula

$$f(y_1, y_2, \ldots) = \left(f_1(y_1), f_2(y_2), \ldots\right)$$

for $(y_1, y_2, \ldots) \in Y$, where f_i is a Kadec–Anderson homeomorphism, $f_i\colon Y_i \xrightarrow[\text{onto}]{} H_i$, such that $f_i(0) = 0$ for $i \in N$, is the desired homeomorphism between spaces Y and $\sum H_i$. ■

LEMMA 2.7. *Let* $X = \operatorname{ind\,lim} X_i$ *be an essentially infinite-dimensional separable* LF-*space. Then* X *is homeomorphic with one of the following two spaces:*

with the space $H \times \sum R_i$, *where* H *is an infinite-dimensional separable Hilbert space,*

with the space $\sum H_i$, *where* H_i *is an infinite-dimensional separable Hilbert space for* $i \in N$.

Proof. If $\dim X_{i+1}/X_i < \infty$ for all but finitely many i's, then by Theorem 2.2 the space X is homeomorphic with $H \times \sum R_i$. Otherwise we may assume without loss of generality that there is a $k \in N$ such that the spaces $Y_1 = X_k$ and $Y_n = X_{k+n}/X_{k+n-1}$ are infinite-dimensional. Then, by Lemma 2.5, $X = \operatorname{ind\,lim}_i X_{k+i}$ is homeomorphic with $\sum Y_i$, and the latter space is, by Lemma 2.6, homeomorphic with $\sum H_i$. ∎

LEMMA 2.8. *Let* $X = \operatorname{ind\,lim} X_i$ *be an* LF-*space such that each* X_i *is a reflexive Banach space and* $\dim X_1 = \infty$. *Then the space* X *is homeomorphic with the space* $\sum H_i$, *where* H_1 *is an infinite-dimensional Hilbert space with the same density character as* X_1 *and* H_i *is for* $i \geqslant 2$ *either an* n*-dimensional Hilbert space if* $\dim X_i/X_{i-1} = n$ *or an infinite-dimensional Hilbert space with the same density character as* X_i/X_{i-1} *if* $\dim X_i/X_{i-1} = \infty$.

Proof. The proof of the lemma is analogous to the proof of Lemma 2.7. It is enough to observe that a quotient space of a reflexive Banach space is reflexive and to apply the theorem [3] on the topological equivalence of reflexive Banach spaces with the same density character instead of the Kadec–Anderson theorem. ∎

Observe that by Lemma 2.7 in order to prove that all essentially infinite-dimensional separable LF-spaces are homeomorphic it suffices to show that the spaces which appear in this lemma (namely, $H \times \sum R_i$ and $\sum H_i$) are homeomorphic. To prove this fact we shall need the following lemma of Toruńczyk.

LEMMA 2.9. *Let* $a\colon [0, \infty) \to [0, \infty)$ *be a continuous function with* $a^{-1}(0) = \{0\}$ *and let* H *be an infinite-dimensional Hilbert space. Then there exists a homeomorphism*

$$u\colon H \times H \xrightarrow[\text{onto}]{} H \times [0, \infty) = H \times R^+$$

satisfying the conditions

(i) $u(p_1, p_2) = \big(\tilde{u}(p_1, p_2), \|p_2\|\big)$ *for every* $(p_1, p_2) \in H \times H$,

(ii) $\|\tilde{u}(p_1, p_2) - p_1\| \leqslant a(\|p_2\|)$ *for every* $(p_1, p_2) \in H \times H$.

Proof (Toruńczyk, unpublished). In order to prove the theorem we shall need the following two facts:

(A) There exists a homeomorphism $\varphi\colon S \xrightarrow[\text{onto}]{} H$, where H is an infinite-dimensional Hilbert space and S is its unit sphere (Klee [9]).

(B) For every continuous function $a\colon (0, \infty) \xrightarrow[\text{onto}]{} (0, \infty)$ there exists a homeomorphism $v\colon H \times H \times (0, \infty) \xrightarrow[\text{onto}]{} H \times (0, \infty)$ such that $v(x_1, x_2, t) = \big(\tilde{v}(x_1, x_2, t), t\big)$ and $\|\tilde{v}(x_1, x_2, t) - x_1\| \leqslant a(t)$ for $(x_1, x_2, t) \in H \times H \times (0, \infty)$ (Toruńczyk [20], § 5).

Now we put for $(x_1, x_2) \in H \times H$

$$u(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if} \quad x_2 = 0, \\ \left(\tilde{v}\left(x_1, \varphi\left(\dfrac{x_2}{\|x_2\|}\right), \|x_2\|\right), \|x_2\|\right) & \text{otherwise,} \end{cases}$$

where φ is the homeomorphism of (A), and $\tilde{v}$ is the function of (B) for the given function a. It is obvious that $u_{|H \times (H \setminus \{0\})}$ is a homeomorphism and it follows from the conditions on v that u is a homeomorphism from $H \times H$ onto $H \times R^+$. ∎

Let us introduce the following notation. Let $\sum R_i^+$ denote the set of $(t_1, t_2, \ldots) \in \sum R_i$ such that $t_i \geqslant 0$ for $i \in N$. In the sequel we shall consider the set $\sum R_i^+$ as the subset of the LF-space $\sum R_i$. Put

$$R^{+,n} = \{(t_1, t_2, \ldots) \in \sum R_i^+ \colon t_i = 0 \text{ for } i > n\}$$

for $n \in N$. Let H_i, for $i \in N$, be a copy of the infinite-dimensional separable Hilbert space. Put

$$H^n = \Big\{(p_1, p_2, \ldots) \in \sum H_i \colon p_i = 0 \text{ for } i > n\Big\}$$

for $n \in N$. Up to the end of this section we shall identify the sets H^n and $R^{+,n}$ with the sets $H_1 \times H_2 \times \ldots \times H_n$ and $[0, \infty)^n$ for $n \in N$.

LEMMA 2.10. *The space* $\sum H_i$ *is homeomorphic with* $H_1 \times \sum R_i^+ \subset H_1 \times \sum R_i$.

Proof. Let $u(p_1, p_2) = \big(\tilde{u}(p_1, p_2), \|p_2\|\big)$ be a homeomorphism between $H_1 \times H_1$ and $H_1 \times R^+$ satisfying the condition

$$\|\tilde{u}(p_1, p_2) - p_1\| \leqslant \|p_2\|. \tag{9}$$

The existence of such homeomorphism follows from the previous lemma with $a(t) = t$ for $t \in R^+$. Next, for every $n \geqslant 2$, we let

$$\tilde{u}_n(p_1, p_2) = \tilde{u}\big(p_1, I_n(p_n)\big) \quad \text{for} \quad (p_1, p_n) \in H_1 \times H_n,$$

where I_n denotes an arbitrary isometric isomorphism of H_n onto H_1 for $n \geqslant 2$. For every $n \geqslant 2$ define

$$f_n(p) = \Big(\tilde{u}_n\Big(\tilde{u}_{n-1}\big(\ldots \tilde{u}_3\big(\tilde{u}_2(p_1, p_2), p_3\big), \ldots\big), p_n\Big), \|p_2\|, \|p_3\|, \ldots, \|p_n\|\Big) \tag{10}$$

for $p \in H^n$; $p = (p_1, p_2, \ldots, p_n)$. Observe that

$$f_{n+1|H^n} = f_n \quad \text{for} \quad n \geqslant 2.$$

(here and below we identify H^n with $H^n\times\{0\}\times\{0\}\times\ldots$). Indeed, let $p=(p_1,p_2,\ldots,p_n,0)\in H^{n+1}$, where $p_i\in H_i$ for $1\leqslant i\leqslant n$. Then by (10)

$$f_{n+1}(p)=\Big(\Big(\tilde u_{n+1}\Big(\tilde u_n\Big(\ldots\big(\tilde u_2(p_1,p_2),p_3\big),\ldots\Big),p_n\Big),0\Big),\|p_2\|,\|p_3\|,\ldots,\|p_n\|,0\Big).$$

By (9), $u(p,0)=p$ for every $p\in H_1$. Hence according to the definition of $\tilde u_n$ we have

$$\begin{aligned}f_{n+1}(p)&=\Big(\tilde u_n\Big(\tilde u_{n-1}\Big(\ldots\big(\tilde u_2(p_1,p_2),p_3\big),\ldots\Big),p_n\Big),\|p_2\|,\|p_3\|,\ldots,\|p_n\|,0\Big)\\&=f_n(\overline p,0),\end{aligned}$$

where $\overline p=(p_1,p_2,\ldots,p_n)\in H^n$. An easy consequence of the fact that u is a homeomorphism of $H_1\times H_1$ onto $H\times R^+$ and of the definition of the functions f_n for $n\geqslant 2$ is that f_n is a homeomorphism of H^n onto $H_1\times\underbrace{\times R^+\times R^+\times\ldots\times R^+}_{n-1\text{ times}}=H\times R^{+,n-1}$ for every $n\geqslant 2$.

Since, for every $n\geqslant 2$, f_{n+1} is an extension of the homeomorphism $f_n\colon H^n\xrightarrow[\text{onto}]{}H_1\times R^{+,n-1}$ to the homeomorphism acting from H^{n+1} onto $H_1\times R^{+,n}$ and since

$$\sum H_i=\bigcup_{i\in N}H^i\quad\text{and}\quad H_1\times\sum R_i=\bigcup_{i\in N}H_1\times R^{+,i}$$

we infer that the mapping f defined by the formula

$$f(p)=f_n(p)\quad\text{for}\quad p\in\sum H_i;\ p\in H^n$$

is a well-defined one-to-one mapping from $\sum H_i$ onto $H\times\sum R_i$.

It can easily be verified (by using (9) and (10) resp.) that the function f satisfies the conditions:

(i) if $x=(x_1,x_2,\ldots)\in\sum H_i$, $f(x)=(y,\|x_2\|,\|x_3\|,\ldots)$, then $\|y-x_1\|\leqslant\sum\limits_{i\geqslant 2}\|x_i\|$,

(ii) if $x=(x_1,x_2,\ldots)\in\sum H_i$, $x'=(x_1,x_2,\ldots,x_n,0,0,\ldots)\in\sum H_i$ for some $n\in N$, $f(x)=(y,\|x_2\|,\|x_3\|,\ldots)$, $f(x')=(y',\|x_2\|,\|x_3\|,\ldots)$, then $\|y-y'\|\leqslant\sum\limits_{i>n}\|x_i\|$.

The easy proof by induction with respect to $n=\inf\{j\in N\colon x\in H^j\}$ is left to the reader.

We shall prove that f is continuous. Let $\overline x=(\overline x_1,\overline x_2,\ldots)\in\sum H_i$ and let U be a neighbourhood of the point $f(\overline x)=(\overline y,\|\overline x_2\|,\|\overline x_3\|,\ldots)\in H_1\times\sum R_i^+$. We shall find a neighbourhood V of the point $\overline x$ in $\sum H_i$ such that $f(V)\subset U$.

By Proposition 1.1 there exists a sequence $\{a_i\colon i\in N\}$ of positive numbers such that

$$U\supset\Big\{(y,a_2,a_3,\ldots)\in H_1\times\sum R_i\colon\ \|y-\overline y\|<a_1\ \text{and}\ \big|a_i-\|\overline x_i\|\big|<a_i\ \text{for}\ i>2\Big\}=U_0.$$

Fix $n\in N$ such that $\overline x\in H^n$. It follows from the continuity of the function $f_{|H^n}$ that there exists an $\varepsilon>0$ such that for every $x=(x_1,x_2,\ldots,x_n,0,0,\ldots)\in H^n$ with $f(x)=(y,\|x_2\|,\|x_3\|,\ldots)$ we have

$$\|y-\overline y\|<2^{-1}a_1\tag{11}$$

provided that $\|x_i-\overline x_i\|<\varepsilon$ for $i=1,2,\ldots,n$.

Put

$$r_i=\min\{\varepsilon,a_i,2^{-i-1}a_1\}$$

for $i\in N$ and

$$V=\Big\{(x_1,x_2,\ldots)\in\sum H_i\colon\ \|x_i-\overline x_i\|<r_i\ \text{for}\ i\in N\Big\}.$$

Then $f(V)\subset U_0\subset U$. Indeed, let $x=(x_1,x_2,\ldots)\in V$ and let $f(x)=(y,\|x_2\|,\|x_3\|,\ldots)$. We have

$$\big|\|x_i\|-\|\overline x_i\|\big|<r_i\leqslant a_i$$

for $i=2,3,\ldots$ Hence it remains only to prove that

$$\|y-\overline y\|<a_1,$$

but the last inequality follows immediately from (ii) and (11), which concludes the proof of the continuity of f.

Now we shall prove that f is an open mapping. Let U be an arbitrary open subset of $\sum H_i$ and let $\overline x=(\overline x_1,\overline x_2,\ldots)\in U$. We shall show that $f(U)$ is a neighbourhood of the point $f(\overline x)=(\overline y,\|\overline x_2\|,\|\overline x_3\|,\ldots)$ in $H_1\times\sum R_i^+$. Fix $n\in N$ such that $\overline x\in H^n$ and take an open convex neighbourhood V of the point $\overline x$ in H^n and positive numbers β_i for $i>n$ such that

$$\begin{aligned}U\supset\Big\{(x_1,x_2,\ldots,x_n,x_{n+1},\ldots)\in\sum H_i\colon\ &(x_1,x_2,\ldots,x_n)\in V\\&\text{and}\ \|x_i\|<\beta_i\ \text{for}\ i>n\Big\}.\end{aligned}$$

Since $f_{|H^n}$ is a homeomorphism of H^n onto $H_1\times R^{+,n-1}$, we infer that there exists an $r>0$ such that the set

$$\begin{aligned}W=\Big\{(y,a_2,a_3,\ldots)\in H_1\times\sum R_i^+\colon\ &\|y-\overline y\|<r\ \text{and}\\&\big|a_i-\|x_i\|\big|<r\ \text{for}\ i=2,3,\ldots,n\ \text{and}\ a_i=0\ \text{for}\ i>n\Big\}\end{aligned}$$

is contained in $f(V)$. Put

$$\gamma_i=\min\{\beta_i,2^{-i-1}r\}$$

for $i>n$ and

$$\begin{aligned}\tilde W=\Big\{&(y,a_2,a_3,\ldots)\in H_1\times\sum R_i^+\colon\\&(y,a_2,a_3,\ldots,a_n,0,0,\ldots)\in\tfrac12\big(W+f(\overline x)\big)\ \text{and}\ a_i<\gamma_i\ \text{for}\ i>n\Big\}.\end{aligned}$$

Then $\tilde{W}$ is a neighbourhood of the point $f(\bar{x})$ in $H_1 \times \sum R_i^+$ and it remains only to prove that

$$f^{-1}(\tilde{W}) \subset U_0 = \Big\{(x_1, x_2, \ldots) \in \sum H_i \colon (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in V \text{ and } \|x_i\| < \gamma_i \text{ for } i > n\Big\}.$$

Assume the contrary. Then there exists an $x = (x_1, x_2, \ldots) \in f^{-1}(\tilde{W})$ with $x \notin U_0$. Hence either $\|x_i\| \geqslant \gamma_i$ for some $i > n$ or $\tilde{x} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \notin V$ and $\|x_i\| < \gamma_i$ for every $i > n$. Put $f(x) = (y, a_2, a_3, \ldots)$. If $\|x_{i_0}\| \geqslant \gamma_{i_0}$ for some $i_0 > n$, then by (i) we have $a_{i_0} > \gamma_{i_0}$, which contradicts the definition of the set $\tilde{W}$. On the other hand, in the case where $\tilde{x} \notin V$, since $W \subset f(V)$ and f is a one-to-one mapping, we infer that

$$f(\tilde{x}) = (\tilde{y}, a_2, a_3, \ldots, a_n, 0, 0, \ldots) \notin W. \tag{12}$$

It follows from condition (ii) that

$$\|y - \tilde{y}\| \leqslant \sum_{i>n} \|x_i\| < \tfrac{1}{2} r.$$

This and (12) imply $(y, a_2, a_3, \ldots, a_n, 0, 0, \ldots) \notin 2^{-1}\big(W + f(\bar{x})\big)$. By the definition of the set $\tilde{W}$ we have $(y, a_2, a_3, \ldots, a_n, 0, 0, \ldots) \in 2^{-1}\big(W + f(\bar{x})\big)$. We get a contradiction which proves the fact that f is an open mapping. ■

LEMMA 2.11. *The space* $\sum R_i$ *is homeomorphic with* $\sum R_i^+$.

Proof. The space $\sum R_i$ coincides with the space $\varinjlim R^n$. Hence by property $(*)$ of Section 1 we conclude that the mapping f defined on a subset A of $X = \sum R_i$ is continuous if and only if $f_{|A \cap R^n}$ is continuous for every $n \in N$.

Put for $t_1, t_2 \in R$

$$h_1(t_1, t_2) = \operatorname{Re}[(t_1 + it_2)^2] = t_1^2 - t_2^2$$

and

$$h_2(t_1, t_2) = \operatorname{Im}[(t_1 + it_2)^2] = 2t_1 t_2$$

and define for every $n \in N$ the mapping $g_n \colon \sum R_i \to \sum R_i$ by the equality

$$g_n(p) = g_n(a_1, a_2, \ldots) = \big(a_1, a_2, \ldots, a_{n-1}, h_1(a_n, a_{n+1}), h_2(a_n, a_{n+1}), a_{n+2}, a_{n+3}, \ldots\big)$$

for $p = (a_1, a_2, \ldots) \in \sum R_i$.

Observe that the mapping g_1 maps $\sum R_i^+$ onto $R_1 \times \sum_{i \geqslant 2} R_i^+ \subset \sum R_i$. It is easy to see that g_1 is a homeomorphism of $\sum R_i^+$ onto $R_1 \times \sum_{i \geqslant 2} R_i^+$. Also one can prove that for every $n \in N$ the mapping f_n defined by the equalities: $f_1 = g_1$; $f_k = g_k \circ f_{k-1}$ for $k = 2, 3, \ldots$ is a homeomorphism of $\sum R_i^+$ onto $R_1 \times R_2 \times \ldots \times R_n \times \sum_{i>n} R_i^+ \subset \sum R_i$.

Put

$$f(p) = \lim f_n(p) \quad \text{for} \quad p \in \sum R_i^+.$$

Since for $p = (a_1, a_2, \ldots, a_k, 0, 0, \ldots) \in \sum R_i$ we have $f(p) \in R_1 \times R_2 \times \ldots \times R_k \times \{0\} \times \{0\} \times \ldots$ and $g_n(p) = p$ for $n > k$, we infer that the limit $f_n(p)$ exists for every $p \in \sum R_i^+$.

Observe that $f_{|R^n} = f_n$ for every $n \in N$. This implies that f is continuous. Since f_n carries $\sum R_i^+$ onto $R_1 \times R_2 \times \ldots \times R_n \times \sum_{i>n} R_i^+$, we conclude that f is a one-to-one mapping and

$$f \colon \sum R_i^+ \xrightarrow[\text{onto}]{} \sum R_i.$$

A similar argument as before shows that the mapping $f^{-1} \colon \sum R_i \xrightarrow[\text{onto}]{} \sum R_i^+$ is continuous. Hence f is a homeomorphism between $\sum R_i^+$ and $\sum R_i$. ■

Now we can prove the main result of this section.

THEOREM 2.12. *Every two essentially infinite-dimensional separable* LF-*spaces are homeomorphic.*

Proof. It suffices to show that every essentially infinite-dimensional separable LF-space is homeomorphic with the space $H \times \sum R_i$.

Let X be an arbitrary separable essentially infinite-dimensional LF-space. It follows from Lemma 2.7 that either X is homeomorphic with the space $H \times \sum R_i$ or X is homeomorphic with $\sum H_i$. Hence it is enough to prove that the spaces $H \times \sum R_i$ and $\sum H_i$ are homeomorphic. Indeed, by Lemma 2.10, the space $\sum H_i$ is homeomorphic with $H \times \sum R_i^+$. On the other hand, according to the previous lemma, $\sum R_i^+$ is homeomorphic with $\sum R_i$. Hence the spaces $H \times \sum R_i$ and $\sum H_i$ are homeomorphic. ■

THEOREM 2.13. *Let* $X = \operatorname{ind\,lim} X_n$, *where* X_n *is a reflexive Banach space for* $n \in N$, *be an essentially infinite-dimensional* LF-*space. Then* X *is homeomorphic with the space* $H_0 \times \Sigma R_i$, *where* H_0 *is an infinite-dimensional Hilbert space with* $\operatorname{dens} H_0 = \operatorname{dens} X$, *provided that there exists an* $n_0 \in N$ *such that* $\operatorname{dens} X_{n_0} = \operatorname{dens} X$. *If it is not the case, then there exists an increasing sequence* $\{\aleph_i \colon i \in N\}$ *of cardinals with* $\aleph_i < \operatorname{dens} X$ *and* $\lim \aleph_i = \operatorname{dens} X$, *and the space* X *is homeomorphic with the space* $\sum H(\aleph_i)$ *where* $H(\aleph_i)$ *denotes the Hilbert space with the density character* $\aleph_i$ *for* $i \in N$.

Proof. The proof of the theorem is, by Lemma 2.8, similar to the proof of Theorem 2.12, so we omit it. ■

Remark. A Fréchet space X is said to be *topologically infinitely divisible* iff the space $X \times X \times X \times \ldots$ is homeomorphic with X. Recently, Toruńczyk has proved [21] that every topologically infinitely divisible

Fréchet space is homeomorphic with a Hilbert space with a suitable density character. This shows that the analogue of Theorem 2.13 is valid for LF-spaces of the form $\sum X_i$, where X_i is a topologically infinitely divisible Fréchet space for $i \in N$.

Remark. The following shows that non-separable LF-spaces with the same density character need not be homeomorphic. Let $\{\aleph_i: i \in N\}$ be an increasing sequence of infinite cardinals and let $\lim \aleph_i = \aleph$. Then (with the notation introduced above) the spaces $\sum H(\aleph_i)$ and $H(\aleph) \times \sum R_i$ are not homeomorphic; nevertheless they have the same density characters.

Note that the method presented in this section enables us to solve completely the problem of topological classification of LF-spaces, provided that the Kadec–Anderson theorem extends to Fréchet spaces with an arbitrary density character. More precisely, we have

THEOREM 2.14. *Assume that every two infinite-dimensional Fréchet spaces with the density character $\bar{\aleph}$ less than a fixed cardinal $\aleph$ are homeomorphic. Then every* LF-*space* $X = \operatorname{ind}\lim X_n$ *with* $\operatorname{dens} X = \aleph' < \aleph$ *is homeomorphic with the space* $H(\aleph') \times \sum R_i$, *provided that there exists an* $n_0 \in N$ *such that* $\operatorname{dens} X_{\kappa_0} = \operatorname{dens} X$. *If this is not the case, then* X *is homeomorphic with the space* $\sum H(\aleph_i')$, *where* $\{\aleph_i': i \in N\}$ *is an arbitrary strictly increasing sequence of cardinals with* $\lim \aleph_i' = \operatorname{dens} X$.

Proof. The first part of the theorem can be shown in the same manner as in the proof of Theorem 2.13. In order to prove the second part, note that, by the same argument as in the proof of Theorem 2.13, X is homeomorphic with the space $\sum H(\aleph_i)$, where $\aleph_i = \operatorname{dens} X_{n_i}$, $\aleph_i < \aleph_{i+1}$, for some increasing subsequence of positive integers $n_1, n_2, \ldots$ To complete the proof, it is sufficient to observe that the space $\sum H(\aleph_i)$ is isomorphic with every space of the form $\sum H(\aleph_i')$, provided that $\aleph_i' < \aleph_{i+1}'$ and $\lim \aleph_i' = \lim \aleph_i' = \operatorname{dens} X$. ∎

The following theorem is an easy consequence of Theorem 1.2.

THEOREM 2.15. *An* LF-*space* X *is homeomorphic with a* tLF-*space* Y *if and only if* $X = \sum R_i$ *and* $Y = \lim\limits_{\to} R^i$. *If this is the case, then* X *is isomorphic with* Y.

Nevertheless, we have

THEOREM 2.16. *Every two essentially infinite-dimensional separable* tLF-*spaces are homeomorphic.*

The proof of this theorem is similar to the proof of Theorem 2.12. Observe that, owing to the property (*) of Section 1, the proofs of some of the lemmas can be essentially simplified.

Also, one can prove analogues of Theorem 2.13 and Theorem 2.14.

3. Lipschitz classification. In this section we shall consider the problem of Lipschitz classification of LF-spaces. First of all, we shall recall some definitions (cf. [12] and [2]).

DEFINITION 3.1. A mapping F from a subset A of a locally convex vector space X into a locally convex vector space Y is said to *satisfy the first order Lipschitz condition* if and only if for every continuous pseudonorm P on Y there exist a continuous pseudonorm Q on X and a constant $K > 0$ such that

$$P\big(F(x)-F(z)\big) \leqslant KQ(x-z)$$

for every $x, z \in A$.

In the sequel, the mappings satisfying the first order Lipschitz condition will also be called *Lipschitz mappings*.

DEFINITION 3.2. A one-to-one mapping F from a subset A of a locally convex vector space X into a locally convex vector space Y is said to be a *Lipschitz embedding of A into Y* if and only if both F and F^{-1} satisfy the first order Lipschitz condition. If, moreover, F maps X onto Y then we say that F is a *Lipschitz homeomorphism*.

Let X and Y be locally convex vector spaces and let $\{Q_\vartheta: \vartheta \in \Theta\}$ and $\{P_\varphi: \varphi \in \Phi\}$ be arbitrary systems of continuous pseudonorms inducing the topologies on X and Y, respectively, where Θ and Φ are arbitrary sets of indices. Then a Lipschitz mapping $\bar{F}$ from a subset A of X into Y is said to *satisfy the Lipschitz condition with the same set of constants as another Lipschitz mapping F from A into Y with respect to the fixed systems of pseudonorms $\{Q_\vartheta: \vartheta \in \Theta\}$ and $\{P_\varphi: \varphi \in \Phi\}$* if and only if for every $\varphi_0 \in \Phi$, $\vartheta_0 \in \Theta$, $K > 0$ the inequality

$$P_{\varphi_0}\big(\bar{F}(x)-\bar{F}(z)\big) \leqslant KQ_{\vartheta_0}(x-z)$$

holds for every $x, z \in A$, provided that the inequality

$$P_{\varphi_0}\big(F(x)-F(z)\big) \leqslant KQ_{\vartheta_0}(x-z)$$

holds for all $x, z \in A$.

In a similar manner one can define $\bar{F}$ as a *Lipschitz embedding (homeomorphism) with the same set of constants as a Lipschitz embedding (homeomorphism) F with respect to the fixed systems of pseudonorms $\{Q_\vartheta: \vartheta \in \Theta\}$ and $\{P_\varphi: \varphi \in \Phi\}$.*

In the sequel we shall often omit the phrase "with respect to the fixed systems of pseudonorms $\{Q_\vartheta: \vartheta \in \Theta\}$ and $\{P_\varphi: \varphi \in \Phi\}$" provided that it does not lead to a misunderstanding.

In this section we shall need an infinite-dimensional version of the Rademacher Theorem [22] on differentiability of Lipschitz mappings. In order to formulate it we shall need the notion of a set of measure zero in a separable Fréchet space. Sudakov has proved in [19] that i X is an infinite-dimensional locally convex space then there is no Borel σ-finite measure on X satisfying the natural conditions: (a) $\mu(U) > 0$ for every open subset U of X and (b) for every measurable subset A of X, $\mu(A) = 0$ if and only if $\mu(A+x) = 0$ for every $x \in X$. Nevertheless, if X is a separable Fréchet space then one can define a σ-ideal of sets in X which behave like sets of Lebesgue measure zero in Euclidean spaces.

Definition 3.3 (Christensen [5]). A subset A of a separable Fréchet space X is said to be a *subset of measure zero in* X if and only if there exist a probability Borel measure μ on X and a Borel subset $\bar{A} \subset X$ such that $A \subset \bar{A}$ and

$$\mu(\bar{A}+x) = 0$$

for every $x \in X$.

Christensen has proved in [5] that the union of countably many sets of measure zero is also a set of measure zero. On the other hand, it is easy to see that if U is an open subset of X then U is not a subset of measure zero in X.

Let F be a mapping from a subset A of a linear locally convex space X into a linear locally convex space Y.

By the *derivative* of F at the point $a \in A$ in the direction $x \in X$ we mean

$$F'_x(a) = \lim_{\lambda\to 0} \frac{F(a+\lambda x) - F(a)}{\lambda}$$

provided that $a+\lambda x \in A$ for all scalars λ in a neighbourhood of zero, and that the limit above exists.

Now let U be an open subset in X. We shall say that F is *differentiable at a point* $a \in U$ iff

1. $F'_x(a)$ exists for every $x \in X$,
2. the mapping $(DF)_a(x) = F'_x(a)$ is a linear mapping from X into Y.

If this is the case then the mapping $(DF)_a\colon X \to Y$ is said to be the *differential of* F *at the point* a. It can easily be seen that if F is a Lipschitz mapping from a subset A in X into Y and if F is differentiable at a point $a \in A$, then the differential $(DF)_a$ is continuous. Moreover, $(DF)_a$ satisfies the Lipschitz condition with the same set of constants as F (with respect to arbitrarily chosen systems of continuous pseudonorms inducing topologies on X and Y). Similarly if F is a Lipschitz embedding of A into Y then the differential $(DF)_a$ is a Lipschitz embedding of X into Y with the same set of constants as F (with respect to arbitrarily chosen systems of continuous pseudonorms inducing topologies on X and Y).

Now we shall define a class of spaces with "nice" differential properties (see [14]).

Definition 3.4. A Fréchet space X is said to be a GF-*space* if and only if it satisfies the condition:

(G) *for every Lipschitz mapping F from interval $[0, 1]$ into X the Lebesgue measure of the set of $t \in [0, 1]$ such that $F'_1(t)$ exists is equal to* 1.

(Here we consider the interval $[0, 1]$ as a subset of the one-dimensional Banach space $(R, |\ |)$.)

The class of GF-spaces is quite rich. Namely, the following theorem ([14]) holds.

Theorem 3.5.

(i) *Every reflexive Banach space is a* GF-*space,*

(ii) *every Montel–Fréchet space is a* GF-*space,*

(iii) *the Cartesian product of a countable family of* GF-*spaces is a* GF-*space,*

(iv) *any closed subspace of a* GF-*space is a* GF-*space.*

The Infinite-Dimensional Rademacher Theorem, which we have mentioned before, reads as follows:

Theorem 3.6. *Let F be a Lipschitz mapping from an open subset U of a separable Fréchet space X into a* GF-*space Y. Then the set of x in U such that F is not differentiable at the point x is of measure zero in X.*

The proof of the theorem is exactly the same as the proof of Theorem 4.5 of [14].

Theorem 3.6 cannot be generalized to the Lipschitz mappings in LF-spaces.

Example 3.7. *The space $\sum R_i$ satisfies condition* (G) *of Definition* 3.4. *Nevertheless the mapping $F\colon \sum R_i \to \sum R_i$ defined by*

$$F(p) = (|a_1|, |a_2|, \ldots) \quad \textit{for} \quad p = (a_1, a_2, \ldots) \in \sum R_i$$

satisfies the first order Lipschitz condition but is not differentiable at any point $x \in \sum R_i$.

Proof. Obviously, F is a Lipschitz mapping. Let x be an arbitrary fixed point in $\sum R_i$. Then there exists a $k \in N$ such that $x = (a_1, a_2, \ldots, a_k, 0, 0, \ldots)$. Let $e_{k+1} = (0, 0, \ldots, 0, 1, 0, 0, \ldots) \in \sum R_i$ (the $k+1$-st coordinate is equal to 1). Then

$$\lim_{\lambda\to +0} \frac{F(x+\lambda e_{k+1}) - F(x)}{\lambda} = e_{k+1}$$

while

$$\lim_{\lambda\to -0} \frac{F(x+\lambda e_{k+1}) - F(x)}{\lambda} = -e_{k+1}.$$

Hence the mapping F is not differentiable at the point x.

Moreover, it can be proved that for every LF-space X there exists a Lipschitz mapping $F\colon X \to X$ which is not differentiable at any point $x \in X$.

Now we shall prove

THEOREM 3.8. *Let* $X = \operatorname{ind\,lim} X_n$ *be an* LF-*space, where* X_n *is a reflexive Banach space for* $n \in N$ *and let an* LF-*space* Y *be Lipschitz embeddable in* X. *Then* $Y = \operatorname{ind\,lim} Y_n$, *where* Y_n *is a reflexive Banach space for* $n \in N$.

Proof. Let F be a Lipschitz embedding of LF-space Y into the LF-space X and let $Y = \operatorname{ind\,lim} Y_n$. Fix $n_0 \in N$. We shall show that Y_{n_0} is a reflexive Banach space. Let

$$A_n = Y_{n_0} \cap F^{-1}(X_n)$$

for $n \in N$. Since the sets A_n are closed in Y_{n_0} for $n \in N$ and

$$\bigcup_{n \in N} A_n = Y_{n_0},$$

then by the standard Baire category argument we infer that there exists an n_1 such that A_{n_1} contains an open subset of Y_{n_0}. Let U be an open subset contained in A_{n_1}. It is easy to see that $\tilde{F} = F_{|U}$ is a Lipschitz embedding of U into X_{n_1}.

Observe that Y_{n_0} is a Banach space, because of the fact that F^{-1} maps bounded subsets onto bounded subsets and hence Y_{n_0} possesses a bounded open subset. The reflexivity in the category of Banach spaces is a separable property (i.e. a Banach space X is reflexive iff every separable closed subspace of X is reflexive). Therefore it suffices to prove that every separable closed subspace of Y_{n_0} is a reflexive Banach space. To this end, let Z be an arbitrary closed seperable subspace of Y_{n_0}. Fix $p \in U$ and put

$$\bar{F}(y) = \tilde{F}(y+p) \quad \text{for} \quad y \in Z \text{ and } p+y \in U.$$

It can easily be seen that $\bar{F}$ is a Lipschitz embedding of the set $\tilde{U} = Z \cap \cap (U-p)$ contained in Y_{n_0} into X_{n_1}. Obviously, $Z \cap (U-p)$ is an open non-empty subset of Z. Hence, by the Infinite-Dimensional Rademacher Theorem, there exists a $z_0 \in Z$ such that $(D\bar{F})_{z_0}$ exists. Since $\bar{F}$ is a Lipschitz embedding of an open subset $\tilde{U}$ of Z into X_{n_1}, we conclude that $(D\bar{F})_{z_0}$ is a linear Lipschitz embedding of Z into X_{n_1}. Hence the space Z is isomorphic with a closed subspace of the reflexive space X_{n_1}. This implies that Z is reflexive. Thus Y_{n_0} is a reflexive space. ■

In a similar manner one can prove the following

THEOREM 3.9. *Let* $X = \operatorname{ind\,lim} X_n$ *be an* LF-*space, where* X_n *is a* GF-*space, for* $n \in N$, *and let an* LF-*space* Y *be Lipschitz embeddable in* X. *Then* $Y = \operatorname{ind\,lim} Y_n$, *where* Y_n *is a* GF-*space for every* $n \in N$.

Proof. By Theorem 2.8 of [14] the property of "being a GF-space" is a separable property and every Fréchet space which is isomorphically embeddable in a GF-space is a GF-space. Hence to prove the theorem it suffices to repeat the same argument as in the proof of the previous theorem. ■

THEOREM 3.10. *Let* $X = \operatorname{ind\,lim} X_n$ *be a Montel* LF-*space and let* $Y = \operatorname{ind\,lim} Y_n$ *be an* LF-*space Lipschitz embeddable in* X. *Then* Y *is isomorphically embeddable in* X.

Proof. Let F be a Lipschitz embedding of the space Y into X. As in the proof of Theorem 3.8, one can show that for every $n \in N$ there exists an open subset U_n in Y_n such that $F(U_n) \subset X_{k(n)}$ for some $k(n) \in N$. On the other hand, it is easy to see that if $X = \operatorname{ind\,lim} X_n$ is a Montel LF-space then each space X_n is a Montel–Fréchet space. Hence Y is separable. Therefore Y_n are separable for every $n \in N$.

By the Infinite-Dimensional Rademacher Theorem, for every $n \in N$ there exists a $y_n \in U_n$ such that the mapping $F_{|U_n}$ is differentiable at the point y_n. Put

$$F_n = (DF_{|U_n})_{y_n}$$

for $n \in N$. Obviously, for every $k \in N$ the mapping F_n is a linear Lipschitz embedding of Y_k into $X_{n(k)}$ and, moreover, F_k regarded as a Lipschitz embedding of the subset Y_n of Y into X satisfies the Lipschitz condition with the same set of constants as F (with respect to arbitrary fixed systems of pseudonorms inducing the topologies on X and Y, respectively).

Let $\{z_k \colon k \in N\}$ be a sequence of points in Y satisfying the conditions:

(i) *for every* $n \in N$ *the set* $\{z_k \colon z_k \in Y_n\}$ *is dense in* Y_n,

(ii) $z_n \in Y_n$ *for every* $n \in N$,

(iii) $z_1 = 0$,

(iv) *the set* $\{z_k \colon k \in N\}$ *is a linear space over the field* Q *of rational numbers.*

Put $x_{n,k} = F_n(z_k)$ for $n \in N$ and $k \leqslant n$. Observe that for every $k \in N$ the set $\{x_{k,n} \colon n \geqslant k\}$ is bounded in X. Indeed, since the mapping F_n is linear for every $n \in N$, we have $F_n(z_1) = F_n(0) = 0$ for $n \in N$. This means that $x_{1,n} = 0$ for $n \in N$. Let P be an arbitrary continuous pseudonorm on X. By the definition of a Lipschitz mapping, there exist a continuous pseudonorm Q on Y and a constant $K > 0$ such that

$$P\big(F(y) - F(z)\big) \leqslant KQ(y-z)$$

for $y, z \in Y$. Since for every $n \in N$ the mapping F_n satisfies the Lipschitz condition with the same set of constants as F, we have

$$P(x_{k,n}) = P\big(F_n(z_k) - F_n(z_1)\big) \leqslant KQ(z_k)$$

for every $k \in N$. But this means that the set $\{x_{k,n} \colon n \geqslant k\}$ is bounded for every $k \in N$. It is well known that a subset A is bounded in an LF-space $X = \operatorname{ind\,lim} X_n$ if and only if there exists an $m \in N$ such that A is contained and bounded in the space X_m. Hence, for every $k \in N$ there exists an $m(k) \in N$ such that the set $\{x_{k,n}: n \geqslant k\}$ is contained and bounded in the space $X_{m(k)}$.

Now put $x_1 = 0$. Since the set $\{x_{2,n}: n \geqslant 2\}$ is contained and bounded in the Montel–Fréchet space $X_{n(2)}$, there exists a subsequence of positive integers $\{n_{2,i}: i \in N\}$ such that the sequence $\{x_{2,n_{2,i}}: i \in N\}$ is convergent. Put

$$x_2 = \lim_{i \to \infty} x_{2,n_{2,i}}.$$

Next, by the same argument we choose a subsequence $\{n_{3,i}: i \in N\}$ of the sequence $\{n_{2,i}: i \in N\}$ such that the sequence $\{x_{3,n_{3,i}}: i \in N\}$ is convergent, and we put

$$x_3 = \lim_{i \to \infty} x_{3,n_{3,i}}.$$

In a similar manner, for every $k \in N$ we define a subsequence $\{n_{k,i}: i \in N\}$ of positive integers such that the following conditions are fulfilled.

(v) $\{n_{k+1,i}: i \in N\} \subset \{n_{k,i}: i \in N\}$ *for every* $k \in N$,

(vi) *for every* $k \in N$ *the sequence* $\{x_{k,n_{k,i}}: i \in N\}$ *is convergent in* X.

Finally, we put

$$x_k = \lim_{i \to \infty} x_{k,n_{k,i}} \quad \text{for} \quad k \in N.$$

Now we put $n_i = n_{i,i}$ for every $i \in N$. Observe that for every $k \in N$

$$F_{n_i}(z_k) = x_{k,n_{i,i}} \to x_k \quad \text{as} \quad i \to \infty.$$

Define the mapping G of the subset $\{z_k: k \in N\}$ of the space Y into the space X by the formula

$$G(z_k) = x_k \quad \text{for} \quad k \in N.$$

Since for every $n \in N$ the mapping F_n considered as a Lipschitz embedding of the subset $\{z_k: k \leqslant n\}$ of the space Y into the space X satisfies the Lipschitz condition with the same set of constants as F, it can easily be proved that G is a Lipschitz embedding of the subset $\{z_k: k \in N\}$ of the space Y into the space X with the same set of constants as F. Hence G can be uniquely extended to a Lipschitz embedding of $\overline{\{z_k: k \in N\}} = Y$ into X. Let $\tilde{G}$ be this extension. Obviously, $\tilde{G}$ is a Lipschitz embedding of the space Y into the space X with the same set of constants as F.

Now we shall show that the mapping $\tilde{G}$ is linear. It follows immediately from (iv) and from the fact that F_n is a linear mapping for $n \in N$ that the mapping G is linear over the field Q of rationals. But this implies that $\tilde{G}$ is also linear.

In order to complete the proof of the theorem it is enough to observe that the mapping $\tilde{G}$ is a linear Lipschitz embedding of the space Y into the space X (and hence $\tilde{G}$ is an isomorphic embedding of Y into X). ■

COROLLARY 3.11. *If an* LF-*space* Y *is Lipschitz homeomorphic with a Montel* LF-*space* X, *then* X *and* Y *have the same linear dimension* (i.e. Y is isomorphic with a subspace of X and, conversely, X is isomorphic with a subspace of Y).

Note that in [14], Theorem 5.5 it is proved that a Fréchet space which is Lipschitz homeomorphic with a Montel–Fréchet space Y is isomorphic with Y.

COROLLARY 3.12. *Let* X *and* Y *be* LF-*spaces such that* Y *is Lipschitz embeddable in* X. *Then*

(i) *if* X *is a Montel space, then* Y *is a Montel space,*

(ii) *if* X *is a Schwartz space, then* Y *is a Schwartz space,*

(iii) *if* X *is a nuclear space, then* Y *is nuclear.*

Observe that the statement (i) also follows trivially from the fact that if F is a Lipschitz mapping then $F(B)$ is bounded for every bounded subset B.

THEOREM 3.13. *Let* H_i, *for* $i = 0, 1, 2, \ldots$, *be infinite-dimensional separable Hilbert spaces and let* s_i, *for* $i = 0, 1, 2, \ldots$, *be copies of the Fréchet space* s *of all real sequences. If* X *is one of the following spaces:*

$$\sum_{i \in N} s_i, \quad \sum_{i \in N} H_i, \quad s_0 \times \sum_{i \in N} R_i, \quad H_0 \times \sum_{i \in N} R_i, \quad \sum_{i \in N} H_i \times s_i,$$
$$H_0 \times s_0 \times \sum_{i \in N} R_i, \quad H_0 \times \sum_{i \in N} s_i, \quad s_0 \times \sum_{i \in N} H_i,$$

and $Y = \operatorname{ind\,lim} Y_n$ *is an* LF-*space which is Lipschitz embeddable in* X, *then* Y *is isomorphically embeddable in* X.

Proof. Since s is a Montel space, then the spaces $\sum s_i$ and $s_0 \times \sum R_i$ are Montel. Hence in the case where either $X = \sum s_i$ or $X = s_0 \times \sum R_i$ the theorem is a particular case of Theorem 3.10.

Now we shall prove the theorem in the case where $X = \sum H_i \times s_i$. The other cases can easily be proved in the same manner.

In the sequel, instead of the space $\sum H_i \times s_i$ we shall deal with the space $(\sum H_i) \times (\sum s_i)$, which is isomorphic to it. In order to simplify the notation we introduce the following convention: the points of the space $\sum H_i$ will be denoted by bold face $\boldsymbol{a}, \boldsymbol{b}, \ldots$ and, for example, $\boldsymbol{a} = (a_1, a_2, \ldots)$, where $a_i \in H_i$ for $i \in N$, and similarly the points of the space $\sum s_i$ will be denoted by bold face $\boldsymbol{x}, \boldsymbol{y}, \ldots$ and, for example, $\boldsymbol{x} = (x_1, x_2, \ldots)$, where $x_i \in s_i$ for $i \in N$.

Fix an arbitrary system of continuous pseudonorms $\{Q_\varphi\colon \varphi\in\Phi\}$ inducing the topology on Y.

Let $\mathfrak{N}$ be a set of all sequences of positive integers. For every pair $\mathfrak{k},\mathfrak{n}\in\mathfrak{N}$, $\mathfrak{k}=\{k_i\colon i\in N\}$, $\mathfrak{n}=\{n_i\colon i\in N\}$, we define a continuous pseudonorm $P_{\mathfrak{n},\mathfrak{k}}$ on the space X by the formula

$$P_{\mathfrak{n},\mathfrak{k}}(\boldsymbol{b},\boldsymbol{y})=\sum_{i\in N}n_i(\|b_i\|_i+\|y_i\|_{i,k_i})\quad\text{for}\quad(\boldsymbol{b},\boldsymbol{y})\in\Big(\sum H_i\Big)\times\Big(\sum s_i\Big)\tag{13}$$

where $\|\ \|_i$ denotes the Hilbert norm on the space H_i for $i\in N$ and $\|\ \|_{i,j}$ denotes the pseudonorm on s_i given by the equality

$$\|y\|_{i,j}=\sum_{k=1}^{j}|r_k|\quad\text{for}\quad y=(r_1,r_2,\ldots)\in s_i$$

for every $i,j\in N$. It is obvious that the system of pseudonorms $\{P_{\mathfrak{n},\mathfrak{k}}\colon \mathfrak{n},\mathfrak{k}\in\mathfrak{N}\}$ induces the topology on X.

In the same manner as in the proof of the previous theorem one can show that, for every $n\in N$, there exist a positive integer $m(n)$, a point $q_n\in Y_n$ and a neighbourhood U_n of the point q_n in Y_n such that

(i) *F_{U_n} is differentiable at the point q_n,*

(ii) *$F(U_n)\subset X_{m(n)}$, where $X_k=(H_1\times H_2\times\ldots\times H_k\times\{0\}\times\{0\}\times\ldots)\times\times(s_1\times s_2\times\ldots\times s_k\times\{0\}\times\{0\}\times\ldots)$ for every $k\in N$.*

Let $F_n=(DF_{|U_n})_{q_n}$ for $n\in N$. It is easy to see that the differential $(DF_{|U_n})_{q_n}$ is a Lipschitz embedding of the space Y_n into $X_{m(n)}$ for every $n\in N$. Moreover, F_n regarded as a mapping from the subset Y_n of the space Y into the space X is a Lipschitz embedding of Y_n into X with the same set of constants as F (with respect to the systems of pseudonorms $\{Q_\varphi\colon\varphi\in\Phi\}$ and $\{P_{\mathfrak{n},\mathfrak{k}}\colon \mathfrak{n},\mathfrak{k}\in\mathfrak{N}\}$).

Now, let $\{p_k\colon k\in N\}$ be a sequence of points in the space Y satisfying the conditions

(iii) *$p_n\in Y_n$ for $n\in N$,*

(iv) *$p_1=0$,*

(v) *for every $n\in N$ the set $\{p_k\colon p_k\in Y_n\}$ is dense in Y_n.*

(vi) *the set $\{p_k\colon k\in N\}$ is a linear space over the field Q of rational numbers.*

Put $F_n(p_k)=(\boldsymbol{a}_{k,n}\ \boldsymbol{x}_{k,n})$ for $n\in N$ and $k\leqslant n$. In the sequel we shall identify the space X_n with the space $\bar{X}_n\times\tilde{X}_n$ where $\bar{X}_n=H_1\times H_2\times\times\ldots\times H_n$ and $\tilde{X}_n=s_1\times s_2\times\ldots\times s_n$, for $n\in N$. In the same manner as in the proof of the previous theorem one can show that for every $k\in N$ there exists an $n(k)$ such that the set $\{(a_{k,n},x_{k,n})\colon n\geqslant k\}$ is contained and bounded in $X_{n(k)}$. But this implies that for every $k\in N$ the set $\{\boldsymbol{a}_{k,n}\colon n\geqslant k\}$ is bounded in $\bar{X}_{n(k)}$ and the set $\{\boldsymbol{x}_{k,n}\colon n\geqslant k\}$ is bounded in $\tilde{X}_{n(k)}$. According to our convention, let $\boldsymbol{a}_{k,n}=(a_{1,k,n},a_{2,k,n},\ldots)$, where $\boldsymbol{a}_{i,k,n}\in H_i$ for $i,k,n\in N$ and $a_{i,k,n}=0$ for $i\geqslant n(k)$. Since for every $k\in N$ the set $\{\boldsymbol{a}_{k,n}\colon n\geqslant k\}$ is bounded in $X_{n(k)}$, we have

$$t_k=\sup\{\|a_{i,k,n}\|_i\colon i,n\in N\}<\infty$$

for every $k\in N$.

For every $i\in N$, let $\{e_{j,i}\colon j\in N\}$ be an arbitrary fixed orthonormal basis in H_i and let E_i^j denote the finite-dimensional subspace of H_i spanned by the vectors $\{e_{1,i},e_{2,i},\ldots,e_{j,i}\}$.

For $i,n\in N$, let $I_{i,n}$ be a linear isometry of the space H_i onto itself such that

$$I_{i,n}(a_{i,n,k})\in E_i^{k-1}\ \text{for}\quad k=1,2,\ldots,n.$$

Next put $\bar{a}_{i,k,n}=I_{i,n}(a_{i,n,k})$ for every $i,n\in N$ and $k\leqslant n$ and $\bar{\boldsymbol{a}}_{k,n}=(\bar{a}_{1,k,n},a_{2,k,n},\ldots)$ for $n\in N$ and $k\leqslant n$.

Observe that for every $k\in N$ the set $\{\bar{\boldsymbol{a}}_{k,n}\colon n\geqslant k\}$ is precompact. Indeed, it follows from the definition of the points $\bar{\boldsymbol{a}}_{k,n}$ that

$$\{\bar{\boldsymbol{a}}_{k,n}\colon n\geqslant k\}\subset\bar{X}_{n(k)}\quad\text{for every } k\in N$$

and

$$a_{i,k,n}\in K_{i,k-1}\quad\text{for } i,k\in N \text{ and } n\geqslant k,$$

where

$$K_{i,k-1}=\{a_i\in H_i\colon \|a_i\|_i\leqslant t_k\}\cap E_i^{k-1},$$

and $K_{i,k-1}$ is compact for every $i,k\in N$. By (13) and by the definition of the points $\{\bar{\boldsymbol{a}}_{k,n}\colon n,k\in N\}$, for every pseudonorm $P\in\{P_{\mathfrak{n},\mathfrak{k}}\colon \mathfrak{n},\mathfrak{k}\in\mathfrak{N}\}$, we have

$$P\big((\boldsymbol{a}_{k_1,n},\boldsymbol{x}_{k_1,n})-(\boldsymbol{a}_{k_2,n},\boldsymbol{x}_{k_2,n})\big)=P\big((\bar{\boldsymbol{a}}_{k_1,n},\boldsymbol{x}_{k_1,n})-(\bar{\boldsymbol{a}}_{k_2,n},\boldsymbol{x}_{k_2,n})\big),\tag{14}$$

for every $n\in N$ and $k_1,k_2\leqslant n$.

For every $n\in N$, let $\bar{F}_n$ be the restriction of the mapping F_n to the set $\{p_1,p_2,\ldots,p_n\}$ and let G_n be the mapping defined by the equality

$$G_n(p_k)=(\bar{a}_{k,n},x_{k,n})\quad\text{for}\quad k=1,2,\ldots,n.$$

Formula (14) implies that for every $n\in N$ the mapping G_n is a Lipschitz embedding of the subset $\{p_1,p_2,\ldots,p_n\}$ of Y into X with the same set of constants as F_n (with respect to the systems of pseudonorms $\{Q_\varphi\colon\varphi\in\Phi\}$ and $\{P_{\mathfrak{n},\mathfrak{k}}\colon \mathfrak{n},\mathfrak{k}\in\mathfrak{N}\}$). Hence, for every $n\in N$ the mapping G_n is a Lipschitz embedding of $\{p_1,p_2,\ldots,p_n\}$ into X with the same set of constants as F.

On the other hand, for every $k \in N$, the set $\{\bar{a}_{k,n}: n \geqslant k\}$ is precompact. Similarly, for every $k \in N$, the set $\{x_{k,n}: n \geqslant k\}$ is bounded in the Montel space $\tilde{X}_{n(k)}$ and hence is precompact. But this implies that, for every $k \in N$, the set $\{G_n(p_k): n \geqslant k\}$ is precompact in X.

Now, in the same manner as in the proof of Theorem 3.10 one can prove that there exists a subsequence $\{n_j: j \in N\}$ of positive integers such that, for every fixed $k \in N$, the sequence $\{G_{n_j}(p_k): j \in N\}$ is convergent in X. Put

$$G(p_k) = \lim_{j\to\infty} G_{n_j}(p_k)$$

for $k = 1, 2, \ldots$ As before, it can easily be proved that the mapping G is a Lipschitz embedding of the subset $\{p_k: k \in N\}$ of Y into X with the same set of constants as F. Since the closure of the set $\{p_k: k \in N\}$ is equal to Y, we infer that the mapping G can be uniquely extended to a Lipschitz embedding $\tilde{G}$ of Y into X. Obviously, $\tilde{G}$ is a Lipschitz embedding with the same set of constants as F.

It remains only to prove that the mapping G is linear. To this end it suffices to observe that, for every $n \in N$, the mapping $\tilde{G}_n$ is a restriction of a linear mapping $\bar{G}_n$ from Y_n into X and to apply the same argument as at the end of the proof of Theorem 3.10.

In fact, for $n \in N$, the mapping $\bar{G}_n$ can be explicitly written by the formula

$$\bar{G}_n = I_n \circ F_n,$$

where

$$I_n(\boldsymbol{a}, \boldsymbol{x}) = \big((I_{1,n}(a_1), I_{2,n}(a_2), \ldots), \boldsymbol{x}\big)$$

for $(\boldsymbol{a}, \boldsymbol{x}) \in (\sum H_i) \times (\sum s_i)$; $\boldsymbol{a} = (a_1, a_2, \ldots)$, $a_i \in H_i$. ■

For every pair of LF-spaces X and Y, let the symbol $X \mapsto Y$ mean that the following statements hold:

(i) X is isomorphically embeddable in Y,

(ii) X is not isomorphic to Y,

(iii) if Z is an LF-space such that Z is isomorphically embeddable in Y and X is isomorphically embeddable in Z, then Z is isomorphic either with X or with Y.

By Theorem 3.13, if X is one of the LF-spaces listed in the assumption of this theorem, the problem of finding the LF-spaces Lipschitz embeddable in X reduces to the problem of finding those isomorphically embeddable in X. The latter is completely answered by the following theorem (see [15]):

THEOREM 3.14. *With the notation of Theorem* 3.13 *the diagram below contains all, up to isomorphism,* LF-*subspaces of the space* $\sum H_i \times s_i$ *arranged according to the relation* $\mapsto$

$$\begin{tikzcd}[column sep=small]
 & & \sum H_i \arrow[dr, mapsto] & & \\
 & H_0 \times \sum R_i \arrow[ur, mapsto] \arrow[dr, mapsto] & & s_0 \times \sum H_i \arrow[dr, mapsto] & \\
\sum R_i \arrow[ur, mapsto] \arrow[dr, mapsto] & & H_0 \times s_0 \times \sum R_i \arrow[ur, mapsto] \arrow[dr, mapsto] & & \sum H_i \times s_i \\
 & s_0 \times \sum R_i \arrow[ur, mapsto] \arrow[dr, mapsto] & & H_0 \times \sum s_i \arrow[ur, mapsto] & \\
 & & \sum s_i \arrow[ur, mapsto] & &
\end{tikzcd}$$

By a space of *Hs type* we shall mean an LF-space which is isomorphic to $\sum X_i$, where for each $i \in N$, X_i is either a Hilbert space (finite-dimensional, infinite-dimensional, or even non-separable) or the space s.

Using the argument of [15] one can show

THEOREM 3.15. *Let* $X = \sum X_i$ *be a space of Hs type. An* LF-*space* Y *is isomorphically embeddable in* X *if and only if, for every* $i \in N$, *there exists a closed subspace* Y_i *of* X_i *such that* $Y = \sum Y_i$. *If this is the case, then* Y *is also of Hs type.*

Combining the argument of the proof of Theorem 3.13 with the argument of [16], one can prove

THEOREM 3.16. *If* Y *is an* LF-*space Lipschitz-embeddable in an* LF-*space of Hs type, then* Y *is also of Hs type.*

Remark. Note that two LF-spaces of *Hs* type have the same linear dimension if and only if they are isomorphic.

4. Uniform classification. In this section we shall study the following problem: Let X and Y be LF-spaces which are uniformly homeomorphic. Are the spaces X and Y then isomorphic? We shall solve this problem affirmatively (Theorem 4.8) for separable spaces of *Hs* type and we shall give some partial results concerning Montel spaces (Theorem 4.5). The idea which will be used in order to prove these facts is based on Enflo's proof [6] that a Banach space uniformly homeomorphic with a Hilbert space is isomorphic with it (cf. also Mankiewicz [12] and [14]). Roughly speaking, the method consists of two steps. First we prove that if an LF-space X is uniformly homeomorphic with an LF-space Y then, under some additional assumption on the space Y, the space X is Lipschitz embeddable in Y, and next we apply the theorems on Lipschitz classification of LF-spaces from the previous section.

In the sequal we shall need the notion of "Lipschitz mapping for large distance" (see [2] and [12]).

DEFINITION 4.1. Let X and Y be locally convex linear spaces. We say that a mapping F from a subset A of X into Y *satisfies the Lipschitz condition for large distance* if and only if for every convex neighbourhood U of the origin in X and for every continuous pseudonorm P on Y there exist a continuous pseudonorm Q on X and a positive constant K such that

$$P\big(F(x)-F(z)\big) \leqslant KQ(x-z)$$

for every $x, z \in A$ with $x-z \notin U$. (In other words, F satisfies the Lipschitz condition for large distance iff, for every convex neighbourhood U of the origin, F satisfies the Lipschitz condition for $x-z \notin U$.)

Similarly, if F is a one-to-one mapping from A into Y then we say that *F is a Lipschitz embedding of A into Y for large distance* iff both F and F^{-1} satisfy the Lipschitz condition for large distance.

The following lemma shows the connections between uniform homeomorphisms of locally convex linear spaces and homeomorphism satisfying the Lipschitz condition for large distance.

LEMMA 4.2. *Let F be a uniform homeomorphism of a locally convex vector space X onto a locally convex vector space Y. Then F is an embedding of X into Y satisfying the Lipschitz condition for large distance.*

Proof. The proof of the lemma is in fact the same as the proof of Lemma 5 in [12], so we omit it. ∎

Let X and Y be locally convex vector spaces. Fix arbitrary systems of pseudonorms $\{Q_\varphi\colon \varphi \in \Phi\}$ and $\{P_\vartheta\colon \vartheta \in \Theta\}$ inducing the topologies on X and Y respectively, and consider arbitrary functions $f_1\colon \Phi \to \Theta$, $f_2\colon \Theta \to \Phi$, $g_1\colon \Phi \to R^+$, $g_2\colon \Theta \to R^+$.

DEFINITION 4.3. A subset A of the space X is said to be *Lipschitz embeddable in Y with the set of constants $\{f_1, f_2, g_1, g_2\}$ with respect to the fixed systems of pseudonorms $\{Q_\varphi\colon \varphi \in \Phi\}$ and $\{P_\vartheta\colon \vartheta \in \Theta\}$* if and only if there exists a one-to-one mapping F from A into Y such that for every $\vartheta \in \Theta$ and $\varphi \in \Phi$ the following inequalities hold:

$$P_\vartheta\big(F(x)-F(z)\big) \leqslant g_2(\vartheta) Q_{f_2(\vartheta)}(x-z)$$

and

$$Q_\varphi(x-z) \leqslant g_1(\varphi) P_{f_1(\varphi)}\big(F(x)-F(z)\big)$$

for every $x, z \in A$. If this is the case then we say that the mapping F is a *Lipschitz embedding of A into Y with the set of constants $\{f_1, f_2, g_1, g_2\}$*.

Now let $X = \operatorname{ind\,lim} X_n$ be a separable LF-space and let Y be a Montel LF-space. Fix arbitrary systems of continuous pseudonorms $\{Q_\varphi\colon \varphi \in \Phi\}$ and $\{P_\vartheta\colon \vartheta \in \Theta\}$ inducing the topologies on X and Y, respectively. In the sequel we shall need the following lemma.

LEMMA 4.4. *The space X is Lipschitz embeddable in the space Y with the set of constants $\{f_1, f_2, g_1, g_2\}$, where $f_1\colon \Phi \to \Theta$, $f_2\colon \Theta \to \Phi$, $g_1\colon \Phi \to R^+$, $g_2\colon \Theta \to R^+$ if and only if there exists a sequence $\{z_n\colon n \in N\}$ of points in X satisfying the conditions*

(i) *$z_n \in X_n$ for every $n \in N$,*

(ii) *for every $n \in N$ the set $\{z_k\colon z_k \in X_n\}$ is dense in X_n,*

(iii) *$z_1 = 0$,*

(iv) *for every $k \in N$ the set $\{z_1, z_2, \ldots, z_k\}$ is Lipschitz embeddable in Y with the set of constants $\{f_1, f_2, g_1, g_2\}$.*

Proof. ($\Rightarrow$) is trivial.

($\Leftarrow$). Let $\{z_n\colon n \in N\}$ be a sequence of points in X satisfying conditions (i)–(iv) of the lemma and for every $n \in N$ let F_n be a Lipschitz embedding of the set $\{z_1, z_2, \ldots, z_k\}$ into Y with the set of constants $\{f_1, f_2, g_1, g_2\}$. Without any loss of generality we may assume that $F_n(z_1) = 0$ for $n \in N$. Put

$$y_{n,k} = F_n(z_k)$$

for $n \in N$ and $k \leqslant n$. It can be proved that, for every $k \in N$, the set $\{y_{n,k}\colon n \geqslant k\}$ is bounded, Indeed, for every $\vartheta_0 \in \Theta$, we have

$$P_{\vartheta_0}(y_{n,k}) = P_{\vartheta_0}(y_{n,k} - y_{n,1}) \leqslant g_2(\vartheta_0) Q_{f_2(\vartheta_0)}(z_k - z_1)$$
$$= g_2(\vartheta_0) Q_{f_2(\vartheta_0)}(z_k)$$

for every $n \in N$ and $k \leqslant n$.

Hence, as in the proof of Theorem 3.10, it can be proved that there exists a subsequence $\{n_m\colon m \in N\}$ of positive integers such that, for every $k \in N$, the sequence $\{y_{n_m,k}\colon m \in N\}$ is convergent in Y. Put

$$G(z_k) = \lim_{m\to\infty} y_{n_m,k} = \lim_{m\to\infty} F_{n_m}(z_k)$$

for every $k \in N$.

It can easily be proved that the mapping G is a Lipschitz embedding of the subset $A = \{z_k\colon k \in N\}$ of X into Y with the set of constants $\{f_1, f_2, g_1, g_2\}$. Hence G can be uniquely extended to a Lipschitz embedding $\tilde{G}$ of the closure of A into Y with the set of constants $\{f_1, f_2, g_1, g_2\}$. Since, according to (ii), A is dense in X, we conclude that $\tilde{G}$ is the required Lipschitz embedding of the space X into Y with the set of constants $\{f_1, f_2, g_1, g_2\}$. ∎

Lemma 4.4 enables us to prove the following theorem, which corresponds to Theorem 7 of [12] concerning Fréchet spaces:

THEOREM 4.5. *Let an LF-space $X = \operatorname{ind\,lim} X_n$ be uniformly homeomorphic with a Montel LF-space Y. Then the spaces X and Y have the same* *linear dimension* (i.e. X *is isomorphically embeddable in* Y *and* Y *is isomorphically embeddable in* X).

Proof. Let F be a uniform homeomorphism between the spaces X and Y. Fix arbitrary systems of pseudonorms $\{Q_\varphi: \varphi\in\Phi\}$ and $\{P_\vartheta: \vartheta\in\Theta\}$ inducing the topologies on X and Y, respectively, and fix an arbitrary convex closed symmetric neighbourhood U of the origin in X satisfying the condition $U\cap X_1 \neq X_1$. It follows from Lemma 4.2 that there exist functions $f_1: \Phi\to\Theta$, $f_2: \Theta\to\Phi$, $g_1: \Phi\to R^+$, $g_2: \Theta\to R^+$ such that the mapping F restricted to a subset A of X is a Lipschitz embedding of A into Y with the set of constants $\{f_1, f_2, g_1, g_2\}$ provided that the set A satisfies the condition: $x, z\in A$ with $x\neq z$ implies $x-z\notin U$.

Let Q be the gauge functional of the set U. Obviously, Q is a continuous pseudonorm on X. Hence the set

$$L = \{x\in X: Q(x) = 0\}$$

is a closed linear subspace of X. Put $L_n = X_n\cap L$ for $n\in N$. Obviously, L_n is a closed linear subspace of X_n for every $n\in N$. Since $X_1\cap U\neq X_1$, we have $L_1\neq X_1$. Hence $L_n\neq X_n$ for every $n\in N$. This implies that, for every $n\in N$, the subspace L_n is of the first Baire category in X_n. Hence, owing to the separability of the space X and thus also of X_n for every $n\in N$, there exists a sequence $\{z_{n,k}: k\in N\}$ of points in X_n satisfying the conditions

(a) $\{z_{n,k}: k\in N\}$ is dense in X_n,

(b) $z_{n,k}-z_{n,k'}\notin L_n$ for every $k, k'\in N$, $k\neq k'$.

Hence we infer that there exists a sequence $\{z_n: n\in N\}$ of points in X satisfying the conditions

(i) $z_n\in X_n$ for every $n\in N$,

(ii) for every $n\in N$ the set $\{z_k: z_k\in X_n\}$ is dense in X_n,

(iii) $z_1 = 0$,

(iv) $z_n-z_k\notin L$ for every $n, k\in N$; $n\neq k$.

Observe that condition (iv) is equivalent to the condition

(v) $Q(z_n-z_k) > 0$ for every $n, k\in N$, $n\neq k$.

We shall show that (v) implies the condition

(vi) for every $k\in N$ the set $\{z_1, z_2, \ldots, z_k\}$ is Lipschitz embeddable in Y with the set of constants $\{f_1, f_2, g_1, g_2\}$.

Indeed, fix $k\in N$ and put

$$\lambda_k = 2^{-1}\inf\{Q(z_i-z_j): i, j\leqslant k \text{ and } i\neq j\}.$$

It follows from (v) that $\lambda_k > 0$. Define

$$F_k(z_i) = \lambda_k F(\lambda_k^{-1}z_i)$$

for $i = 1, 2, \ldots, k$. Then, by the definition of the functions f_2, g_2, for every $\vartheta\in\Theta$ we have

$$P_\vartheta(F_k(z_i)-F_k(z_j)) = \lambda_k P_\vartheta(F(\lambda_k^{-1}z_i)-F(\lambda_k^{-1}z_j)) \leqslant \lambda_k g_2(\vartheta)Q_{f_2(\vartheta)}(\lambda_k^{-1}z_i-\lambda_k^{-1}z_j) = g_2(\vartheta)Q_{f_2(\vartheta)}(z_i-z_j)$$

for every $i, j\leqslant k$. In the same manner one can prove that, for every $\varphi\in\Phi$,

$$Q_\varphi(z_i-z_j)\leqslant g_1(\varphi)P_{f1(\varphi)}(F_k(z_i)-F_k(z_j))$$

for every $i, j\leqslant k$. Hence F_k is a Lipschitz embedding of the subset $\{z_1, z_2, \ldots, z_k\}$ into the space Y with the set of constants $\{f_1, f_2, g_1, g_2\}$.

Thus we have proved that the sequence $\{z_n: n\in N\}$ satisfies the assumptions of the previous lemma. Hence the space X is Lipschitz embeddable in the space Y. Now, by Theorem 3.10, we infer that the space X is isomorphically embeddable in the Montel space Y. This implies that X is a Montel space. Knowing that X is a Montel space, we use the symmetrical arguments to prove that the space Y is isomorphically embeddable in X. ■

The following corollary corresponds to Theorem 5.6 in [13] concerning Fréchet spaces.

Corollary 4.6. *Let an* LF-*space* X *be uniformly homeomorphic with an* LF-*space* Y. *Then*

(i) *if* Y *is a Montel space then* X *is a Montel space*,

(ii) *if* Y *is a Schwartz space then* X *is a Schwartz space*,

(iii) *if* Y *is a nuclear space then* X *is nuclear.*

In the sequel, up to the end of this section, we shall assume that Y is one of the following spaces: $H_0\times\sum R_i$, $s_0\times\sum R_i$, $H_0\times s_0\times\sum R_i$, $H_0\times\sum s_i$, $s_0\times\sum H_i$, $\sum H_i$, $\sum s_i$, $\sum H_i\times s_i$, where, for $i = 0, 1, 2, \ldots$, H_i is a separable infinite-dimensional Hilbert space and s_i is a copy of the Fréchet space of all real sequences. By I_Y we shall denote the canonical embedding of Y into $\sum H_i\times s_i$ described in the diagram of Theorem 3.14. Let

$$\tilde{P}_{\mathfrak{k},\mathfrak{n}}(y) = P_{\mathfrak{k},\mathfrak{n}}(I_Y(y)) \quad \text{for} \quad y\in Y,$$

where $\{P_{\mathfrak{k},\mathfrak{n}}: \mathfrak{k}, \mathfrak{n}\in\mathfrak{N}\}$ is the system of pseudonorms on $\sum H_i\times s_i$ defined by formula (13) on p. 132.

A system $\{P_\vartheta: \vartheta\in\Theta\}$ of pseudonorms on Y consisting of all the pseudonorms $\tilde{P}_{\mathfrak{k},\mathfrak{n}}$ will be called a *canonical system of pseudonorms on* Y. It is easy to see that the canonical system of pseudonorms induces the topology on Y.

Lemma 4.7. *Let* $X = \operatorname{ind\,lim} X_n$ *be a separable* LF-*space. Let* $\{Q_\varphi: \varphi\in\Phi\}$ *be an arbitrary system of pseudonorms inducing the topology on* X *and let* $\{P_\vartheta: \vartheta\in\Theta\}$ *be the canonical system of pseudonorms on* Y. *Let* $f_1: \Phi\to\Theta$,

$f_2\colon \Theta \to \Phi$, $g_1\colon \Phi \to R^+$, $g_2\colon \Theta \to R^+$. *Then the space* X *is Lipschitz embeddable in* Y *with the set of constants* $\{f_1, f_2, g_1, g_2\}$ *with respect to the system of pseudonorms* $\{Q_\varphi\colon \varphi \in \Phi\}$ *and* $\{P_\vartheta\colon \vartheta \in \Theta\}$ *if and only if there exists a sequence* $\{z_n\colon n \in N\}$ *of points in* X *satisfying the conditions:*

(i) $z_n \in X_n$ *for every* $n \in N$,

(ii) *for every* $n \in N$ *the set* $\{z_k\colon z_k \in X_n\}$ *is dense in* X_n,

(iii) $z_1 = 0$,

(iv) *for every* $k \in N$ *the set* $\{z_1, z_2, \ldots, z_k\}$ *is Lipschitz embeddable in* Y *with the set of constants* $\{f_1, f_2, g_1, g_2\}$.

Proof. ($\Rightarrow$) is trivial.

($\Leftarrow$) We shall prove the lemma only in the case where Y is isomorphic to the space $\sum H_i \times s_i = (\sum H_i) \times (\sum s_i)$. (The other cases can be proved in the same manner.) It follows from the definition of the system of pseudonorms $\{P_\vartheta\colon \vartheta \in \Theta\}$ that, without loss of generality, we may assume that $Y = (\sum H_i) \times (\sum s_i)$ and the family of pseudonorms $\{P_\vartheta\colon \vartheta \in \Theta\}$ coincides with the system of pseudonorms $\{P_{\mathfrak{n},\mathfrak{k}}\colon \mathfrak{n}, \mathfrak{k} \in \mathfrak{N}\}$.

Let $\{z_n\colon n \in N\}$ be a sequence in X satisfying conditions (i)–(iv) of the lemma and let, for every $n \in N$, the mapping F_n be a Lipschitz embedding of the subset $\{z_1, z_2, \ldots, z_n\}$ into $(\sum H_i) \times (\sum s_i)$ with the set of constants $\{f_1, f_2, g_1, g_2\}$. Without loss of generality we may assume that $F_n(z_1) = 0$ for $n \in N$. Put

$$(\boldsymbol{a}_{k,n}, \boldsymbol{x}_{k,n}) = F_n(z_k)$$

for $n \in N$ and $k \leqslant n$. Then (in the notation of the proof of Theorem 3.13) in the same way as in the proof of that theorem it can be shown that, for every $k \in N$, there exists an $n(k)$ such that the set $\{\boldsymbol{a}_{k,n}\colon n \geqslant k\}$ is contained and bounded in $\overline{X}_{n(k)}$ and the set $\{\boldsymbol{x}_{k,n}\colon n \geqslant k\}$ is contained and bounded in $\tilde{X}_{n(k)}$. Next, in the same way as in the proof of that theorem one can construct a sequence $\{\tilde{F}_n\colon n \in N\}$ of mappings:

$$\tilde{F}_n\colon \{z_1, z_2, \ldots z_n\} \to (\sum H_i) \times (\sum s_i)$$

for $n \in N$ such that for every $n \in N$ the mapping $\tilde{F}_n$ is a Lipschitz embedding of the set $\{z_1, z_2, \ldots, z_n\}$ into $(\sum H_i) \times (\sum s_i)$ with the set of constants $\{f_1, f_2, g_1, g_2\}$ and for every $k \in N$ the set $\{\tilde{F}_n(z_k)\colon n \geqslant k\}$ is precompact in $(\sum H_i) \times (\sum s_i)$.

Thus, using the "diagonal procedure", one can show that there exists a subsequence $\{n_m\colon m \in N\}$ of positive integers such that, for every $k \in N$, the sequence $\{\tilde{F}_{n_m}(z_k)\colon m \in N\}$ is convergent. Put

$$F(z_k) = \lim_{m \to \infty} \tilde{F}_{n_m}(z_k)$$

for $k = 1, 2, \ldots$ It is easy to see that the mapping F is a Lipschitz embedding of the set $\{z_k\colon k \in N\}$ into $(\sum H_i) \times (\sum s_i)$ with the set of constants $\{f_1, f_2, g_1, g_2\}$. Hence, applying the same argument as before, according to (ii), we infer that X is Lipschitz embeddable in the space $Y = (\sum H_i) \times (\sum s_i)$ with the set of constants $\{f_1, f_2, g_1, g_2\}$. ■

In the sequel we shall need the following lemma:

Lemma 4.8. *If an* LF-*space* X *is uniformly homeomorphic with the space* Y, *then the space* X *is Lipschitz embeddable in* Y.

Proof. The proof of this lemma is based on the construction of a sequence $\{z_n\colon n \in N\}$ of points in X satisfying the conditions (i)–(iv) of the previous lemma and is, in fact, a "word for word" repetition of the first part of the proof of Theorem 4.5, and so we omit it. ■

Now we are ready to prove

Theorem 4.9. *Let an* LF-*space* X *be uniformly homeomorphic with the space* Y, *where* Y *is a space isomorphic to one of the spaces*: $H_0 \times \sum R_i$, $s_0 \times \sum R_i$, $H_0 \times s_0 \times \sum R_i$, $H_0 \times \sum s_i$, $s_0 \times \sum H_i$, $\sum H_i$, $\sum s_i$, $\sum H_i \times s_i$. *Then the space* X *is isomorphic with* Y.

Proof. It follows from the previous lemma that the space X is Lipschitz-embeddable in Y. Hence by Theorem 3.13 the space X is isomorphically embeddable in Y. Hence by Theorem 3.14, X is isomorphic to one of the spaces: $H_0 \times \sum R_i$, $\sum H_i$, $s_0 \times \sum R_i$, $\sum s_i$, $H_0 \times s_0 \sum R_i$, $H_0 \times \sum s_i$, $s_0 \times \sum H_i$, $\sum H_i \times s_i$. Now, in the same manner as before, we deduce that Y is isomorphically embeddable in X. In other words, we have proved that the spaces X and Y have the same linear dimension. To complete the proof it is enough to observe that, owing to Theorem 3.14, this implies that X is isomorphic to Y. ■

Note that, combining the argument of the proof of the previous theorem with the method used in [16], one can also obtain

Theorem 4.10. *If an* LF-*space* X *is uniformly homeomorphic with an* LF-*space of Hs type then* X *is also of that type.*

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Received May 25, 1973 (683)

STUDIA MATHEMATICA, T. LII. (1974)

On linear properties of separable conjugate spaces of C^*-algebras

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Abstract. It is proved that the conjugate space of a separable C^*-algebra in which any hermitian element has a countable spectrum is isometric to $(\sum_{n=1}^{\infty} N(H_n))_1$ where $N(H_n)$ are nuclear operators on separable Hilbert spaces H_n. This implies that a C^*-algebra with a separable conjugate space has a Schauder basis.

The present paper is a study of some linear properties of a class of C^*-algebras which can be considered as a generalization of spaces of continuous functions on countable compact spaces. We prove an isometric representation of a conjugate space of such an algebra. This result can be considered as a generalization to the C^*-algebra setting of a theorem of Rudin [5]. The method of the proof was influenced by [7]. From our representation theorem we deduce some corollaries on the linear structure of such algebras.

Our terminology on C^*-algebras agrees with that of [6] and our terminology on Banach spaces is that usually adopted in Banach space theory (cf. [3]).

DEFINITION. A C^*-algebra is called *countably scattered* if it is separable and each abelian $*$-subalgebra has a scattered spectrum.

LEMMA 1. *The class of countably scattered C^*-algebras is closed under taking $*$-subalgebras, $*$-homomorphic images and sums in the sense of c_0.*

Proof. Obvious from the definition and the following

SUBLEMMA. *A C^*-algebra X is countably scattered iff X is separable and every hermitian element in X has a countable spectrum.*

Recall that a W^*-algebra is a C^*-algebra X isometric to a conjugate space of some Banach space X_*. This space X_* is unique (cf. [6], 1.13.3).

LEMMA 2. *Let (Ω, μ) be a measure space which is a disjoint sum of sets of finite measure* (call such a space a *localizable measure space*) *and let W be a factor, i.e., a W^*-algebra such that an element commuting with any other is a multiple of identity. If every central projection in $L_\infty(\Omega, W)$ contains a minimal projection, then (Ω, μ) is purely atomic.*