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TWO CRITERIA THRUSTING SIMPLE CONNECTEDNESS ON MANIFOLDS

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We give in the sequel two conditions that force a closed n-dimensional manifold to be simply connected. One of these conditions is the entirely topological one that the manifold have a residual set of codimension at least two whereas the other requires the existence of a retraction of the entire manifold onto its residual set. Though the conditions seem not to be related at first glance, it will be seen that the proofs have a remarkable similarity.

In the original decomposition theorem [4], it was shown that each topological closed manifold (connected) may be written as the union of two disjoint sets, one of which is a copy of euclidean n-space E^n while the other is a nowhere dense continuum R, the residual set. From the construction of R there exists in each neighborhood of R an n-manifold N with a sphere boundary. Such an N contains R in its interior. It was then shown in [2] that R may be selected so that it is the continuous image of the boundary of a closed n-cell under a mapping of the closed n-cell onto the manifold that is 1-1 on the interior of the n-cell. Finally, it is shown in both [6] and [3] that R may be chosen to be a strong deformation retraction of N under a pseudo-isotopy of N; thus R can be selected a rather nice ANR for which N acts as a topological analogue of a regular neighborhood [8].

Homology and cohomology is over the integers unless indicated.

LEMMA 0. *An n-manifold X that is compact and has a null-homotopic sphere boundary is simply connected.*

Proof. $n = 1$ or 2 is trivial. Otherwise close X with an attached n-cell to get a closed manifold Y. $Y-p$ (a point) has the homotopy type of X. Sets Y and X have the same fundamental group. Let K be the universal covering space of Y. If Y is not simply connected, the counter image of the point p under projection contains at least two points. K less this counter image is not trivial in dimension $(n-1)$-homotopy. But this group is isomorphic to that of X; worse $\mathrm{Bd}\,X$ is not trivial (homologically) in X.

COROLLARY 1. *Let K be a closed manifold of dimension at least three. If K less a point has no $(n-1)$-homotopy, then each factor of K is simply connected.*

COROLLARY 2. *Let M be a closed n-manifold with a residual set R while N is a compact neighborhood of R with a bicollared sphere boundary. If R is a retract of M, then M is simply connected and so orientable $(n \geqslant 2)$.*

LEMMA 1. *Let M be a closed n-manifold with an ANR residual set R. If N is a manifold neighborhood of R with a bicollared sphere boundary, then M retracts to R is equivalent to M retracts to N.*

Proof. Certainly, if M retracts to N, then M retracts to R. Conversely, if M retracts to R, one has that BdN is null-homotopic in N. As R is an ANR, there is an epsilon such that all maps of a sphere into N within epsilon of R are homotopic [1].

NON-TRIVIAL EXISTENCE. *There exist non-sphere manifolds that are closed and have residual retracts with non-vanishing $(n-1)$-homotopy.*

Proof. Let M be a closed PL 4-manifold with a 2-sphere R. The 2-dimensional homology of M over rationals has R as a non-trivial 2-cycle. Therefore R is a retract of M.

By the above equivalence N is also a retract of M. Thus BdN is null-homotopic in N. Further the three homotopy of N is non-trivial.

Call an n-manifold that is closed *R-retractile* if it retracts to its residual set.

THEOREM 1. *An R-retractile n-manifold is simply connected $(n \geqslant 2)$.*

Proof. One may repeat the argument of Lemma 0 after disposing of the dimension 2.

SEWING THEOREM. *The set of R-retractile n-manifolds forms a semi-group under sums.*

Proof. One takes the sum of two such manifolds, one joins the old residual sets in disjoint manifolds with sphere boundaries; join the residual sets by a flat arc meeting each residual set at one endpoint and each boundary sphere in a point, shrink the arc to a point and retract. It is true that if a manifold is a sum of two others, there is a nice map of the manifold onto the one point union of the two factors.

LOW DIMENSIONAL TRIVIALITY. *If the Poincaré conjecture is true, all R-retractile manifolds in dimensions three or less are spheres.*

Indeed, they are all simply connected except for the 1-sphere.

THEOREM 2. *Let M be a closed manifold and let R be an arbitrary residual set (perhaps indecomposable). If dimension R is $n-2$ or less, then M is simply connected.*

Proof. We take the universal covering space of M and note that the counter image of R cannot separate it and conclude that the projection is 1-1; this follows immediately by intersecting R with a finite set of closed evenly covered sets and applying Theorem III.2 of [5].

COROLLARY 1. *There is no non-simply connected finite k-complex in a PL n-manifold (k less than $n-1$) whose regular neighborhood has a sphere boundary.*

COROLLARY 2. *A closed n-manifold that is not simply connected has only $(n-1)$-dimensional residual sets.*

DISCUSSION, APPLICATION, AND QUESTIONS

We note that the only homology value among the 4-dimensional R-retractile spaces is the second Betti number.

ABSOLUTENESS PROPERTY. *Let M^n be R-retractile and N a nice neighborhood of R, an ANR again. Then N is a retract of every n-manifold in which it embeds.*

Proof. Put N in K, say. Attach a cone over boundary of N. Map the closure of $K-N$ into the cone with $\mathrm{Bd}\,N$ fixed. Now N plus the cone is M and the retraction follows.

COROLLARY TO PROOF. *If N is embedded in any k-manifold, there is a map of the manifold into N that is fixed on boundary of N.*

Note that an N derived from an R-retractile space has many cell--like properties, though the fixed point property is not one of them by the non-trivial existence above.

Manifolds fall into three classes according to properties of *their residual sets*:

Simply connected with necessarily $(n-1)$-dimensional R.

Not simply connected with necessarily $(n-1)$-dimensional R.

Rest.

Since if M^n is R-retractile, $\pi_1(M)=1$, then $H_1(M)=0$ and $H_{n-1}(M)$ has no torsion and $H^{n-1}(M)=0$.

Because an R-retractile M^4 has no 1 or 3 non-zero Betti numbers mod 2, we observe that an M^4 that is R-retractile necessarily has Euler characteristic $\chi(M^4)=B_2+2$ and $\chi(R)=B_2+1$. Thus $\chi(M^4)\geqslant 2$ and if $B_2=0$, M^4 is a homotopy sphere since $\pi_1(M^4)=0$. In the case of all others we may obtain an M^4 with $B_2\geqslant 0$, by summing.

However, these conditions are not even sufficient since $S^2\times S^2$ has not $S^2\wedge S^2$ as a retract. In this case $\mathrm{Bd}\,N(S^2\wedge S^2)\to N$ is essential. Also a retraction $r\colon M^4\to R$ is impossible because if ε_1, ξ_2 generate the H^2 of $S^2\wedge S^2$, then $r_*(\xi_1\cup\varepsilon_2)=0$. Hence in dimension 4 it is required that M^4 have the property that the cohomology ring be such that if ξ, δ are not the identity nor of dimensions 0 or 4, then $\xi\delta=0$, since $\dim\xi+\dim\delta=4$.

In light of the above observations it is interesting to inquire to what extent the cohomology ring can be used to obtain results on retractile manifolds. In particular, can they be characterized using this ring and some simple condition? (**P 908**) It is rather easy to get sufficiency results that a product space need not be retractile.

The following question appears as P 819 in [7]. If dimension of R is less than $n-1$ in a closed n-manifold M while K is a compact connected $(n-1)$-manifold in M that is closed, does K separate M? It appears that Theorem 2 above along with the argument in [7] yields an affirmative answer to this question.

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