La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange	Address of the Editorial Board and of the exchange	Die Adresse der Schriftleitung und des Austausches	Адрес редакции и книгообмена

ACTA ARITHMETICA

ul. Śniadeckich 8, 00-950 Warszawa

Les auteurs sont priés d'envoyer leurs manuscrits en deux exemplaires
The authors are requested to submit papers in two copies
Die Autoren sind gebeten um Zusendung von 2 Exemplaren jeder Arbeit
Рукописи статей редакция просит предлагать в двух экземплярах

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

ACTA ARITHMETICA
XXVI (1975)

Boundary conditions for expanding domains

by

G. L. Cohen (Broadway, N. S. W., Australia)

1. Introduction. I. V. Chulanovskii [1] obtained a law for the distribution of Gaussian primes in expanding domains, using only elementary methods. (See also Gelfond and Linnik [2]; see Hardy and Wright [3] for an introductory discussion of Gaussian primes.) His result was derived for domains which expand in such a way that

(i) the boundary consists of a bounded number of simple closed curves,

(ii) the ratio of the number of unit lattice-squares through which the boundary passes to the radius of the domain is bounded.

We consider the effect of replacing (ii) with the more natural, and weaker, condition:

(ii′) the boundary is rectifiable and the ratio of its length to the radius of the domain is bounded.

It will appear that only in extremely pathological cases is Chulanovskii's result not obtainable, while at other times it may be strengthened.

2. Notation. By the words "integers" and "primes" we mean Gaussian integers and primes. Lower case Greek letters (except π and ξ) denote integers, with ϱ reserved for primes; ξ is any complex number. For any domain D in the complex plane, we denote by ∂D and $\sim D$, its boundary and complement, respectively. (D need not be connected or open.) The radius R of D is

$$R = \sup_{\xi\in D} |\xi|.$$

By $[\xi]$ we mean the integer satisfying

$$\mathrm{Re}[\xi]-\tfrac{1}{2} < \mathrm{Re}\,\xi \leqslant \mathrm{Re}[\xi]+\tfrac{1}{2}, \quad \mathrm{Im}[\xi]-\tfrac{1}{2} < \mathrm{Im}\,\xi \leqslant \mathrm{Im}[\xi]+\tfrac{1}{2}.$$

Put

$$S_\lambda = \{\xi\colon [\xi] = \lambda\}$$

and for a given domain D, put

$$M_1 = \{\lambda\colon \lambda = [\xi] \text{ for some } \xi \in \partial D\},$$

$$M_n = \{\mu\colon |\mu-\lambda| \leqslant \sqrt{2},\ \lambda \in M_{n-1}\} \quad (n = 2, 3, \ldots);$$

$$B_n = \bigcup_{\lambda \in M_n} S_\lambda \quad (n = 1, 2, \ldots),$$

and

$$Z = \sum_{\lambda \in M_1} 1.$$

We note that $M_{n-1} \subset M_n$, and so $B_{n-1} \subset B_n$ $(n \geqslant 2)$.

3. The construction. Given any domain D, whose boundary is not necessarily rectifiable, we give a method for constructing sequences $\{D_{(n)}\}$ and $\{D^{(n)}\}$ $(n = 1, 2, \ldots)$ of domains with rectifiable boundaries whose areas do not differ significantly from that of D. This is in the following rough sense: D will be allowed to expand so that always $Z = O(R)$; then D has area $O(R^2)$ but the difference in areas of D and $D_{(n)}$ and of D and $D^{(n)}$ (for any n) will be $O(R)$. As D expands, the effect of $D_{(n)}$ and $D^{(n)}$ is, for sufficiently large n, to remove long "arms" or "recesses", respectively, in D, providing these have only a bounded width.

Our theorem is:

Let D be any domain of radius R. For $n = 1, 2, \ldots$, define

$$D_{(n)} = D \cap \sim B_n, \qquad D^{(n)} = D \cup B_n.$$

Then each of $D_{(n)}$ and $D^{(n)}$ has a boundary which is a union of simple closed curves, and

$$\begin{cases} D \supset D_{(1)} \supset D_{(2)} \supset \ldots, \\ R \geqslant R_{(1)} \geqslant R_{(2)} \geqslant \ldots\ (R_{(n)} \textit{ is the radius of } D_{(n)}), \\ \textit{length of } \partial D_{(n)} \leqslant 4(2n-1)Z, \\ \Big|\sum\limits_{\nu\in D} 1 - \sum\limits_{\nu \in D_{(n)}} 1\Big| \leqslant (2n-1)^2 Z, \end{cases}$$

and

$$\begin{cases} D \subset D^{(1)} \subset D^{(2)} \subset \ldots, \\ R \leqslant R^{(1)} \leqslant R^{(2)} \leqslant \ldots\ (R^{(n)} \textit{ is the radius of } D^{(n)}), \\ R^{(n)} \leqslant R + n\sqrt{2}, \\ \textit{length of } \partial D^{(n)} \leqslant 4(2n-1)Z, \\ \Big|\sum\limits_{\nu\in D} 1 - \sum\limits_{\nu \in D^{(n)}} 1\Big| \leqslant (2n-1)^2 Z. \end{cases}$$

Proof. We shall verify the properties listed for the sequence $\{D_{(n)}\}$, those for $\{D^{(n)}\}$ being verified in a similar manner. Obviously, $D_{(1)} \subset D$. Let $\xi \in D_{(n)}$ $(n \geqslant 2)$. Then $\xi \notin B_n$, so $\xi \notin B_{n-1}$; that is, $\xi \in \sim B_{n-1}$ and since also $\xi \in D$, we have $\xi \in D_{(n-1)}$. Then obviously $R \geqslant R_{(1)} \geqslant R_{(2)} \geqslant \ldots$ Notice that B_n is in fact the union of squares with vertices $[\xi] \pm (n+\frac{1}{2}) \pm (n+\frac{1}{2})i$ for each $\xi \in \partial D$ $(i = \sqrt{-1})$; these squares each have perimeter of length $4(2n-1)$ and contain $(2n-1)^2$ lattice points. Since $D - D_{(n)} = D \cap B_n \subset B_n$, we have

$$\Big|\sum_{\nu\in D} 1 - \sum_{\nu\in D_{(n)}} 1\Big| = \sum_{\nu \in D - D_{(n)}} 1 \leqslant \sum_{\nu\in B_n} 1 \leqslant (2n-1)^2 Z.$$

Finally, by definition of M_1 and B_1, we have $\partial D \subset B_1$, so

$$\partial D_{(n)} = \partial(D \cap \sim B_n) \subset \partial D \cup \partial(\sim B_n) \subset B_1 \cup \partial B_n \subset B_n \cup \partial B_n.$$

But since $D_{(n)} \subset \sim B_n$, we must have $\partial D_{(n)} \subset \partial B_n$. Since B_n is the union of squares described above, its boundary may be specified as a union of simple closed polygonal curves of total length $\leqslant 4(2n-1)Z$, and so $\partial D_{(n)}$ also has this property. This completes the proof.

These constructions are related to an idea in, for example, Schechter [4], Chapter VI, Section 7.

4. Application. Chulanovskii proved that provided D expands in such a way that (i) ∂D consists of a bounded number of simple closed curves, and (ii) $Z = O(R)$, then

$$\sum_{\varrho\in D} 1 = \frac{2}{\pi \log R} \sum_{\nu\in D} 1 + O\big(f(R)\big), \quad \text{as} \quad R \to \infty,$$

where

$$f(R) = \frac{R^2}{\log R \sqrt[3]{\log\log R}}.$$

The same result may be proved, with slightly less difficulty, when we replace (ii) by the condition:

(ii′) ∂D is rectifiable and its length l satisfies $l = O(R)$.

In the following, we take a domain D satisfying (ii), but only require Chulanovskii's result under (ii′) to obtain similar results.

Let D be a domain in the complex plane which expands in such a way that $Z = O(R)$. For suitable finite m and n, let $D_{(m)}$ and $D^{(n)}$ be domains associated with D as described above. Then

(1) *if $\partial D_{(m)}$ consists of a bounded number of connected pieces and its length l satisfies $l = O(R_{(m)})$, we have*

$$\sum_{\varrho\in D}1 = \frac{2}{\pi\log R_{(m)}}\sum_{\nu\in D_{(m)}}1+O\big(f(R_{(m)})\big)+O(R);$$

(2) *if $\partial D^{(n)}$ consists of a bounded number of connected pieces, we have*

$$\sum_{\varrho\in D}1 = \frac{2}{\pi\log R^{(n)}}\sum_{\nu\in D^{(n)}}1+O\big(f(R)\big).$$

Proof. (1) We have

$$\sum_{\varrho\in D_{(m)}}1 = \frac{2}{\pi\log R_{(m)}}\sum_{\nu\in D_{(m)}}1+O\big(f(R_{(m)})\big)$$

and

$$\Big|\sum_{\varrho\in D}1-\sum_{\varrho\in D_{(m)}}1\Big| = \sum_{\varrho\in D-D_{(m)}}1 \leqslant \sum_{\nu\in D-D_{(m)}}1 \leqslant (2m-1)^2Z,$$

so $\sum_{\varrho\in D}1 = \sum_{\varrho\in D_{(m)}}1+O(R)$, since $Z = O(R)$.

(2) The length l of $\partial D^{(n)}$ satisfies

$$l \leqslant 4(2n-1)Z = O(R) = O(R^{(n)}),$$

since $R \leqslant R^{(n)}$. Hence

$$\sum_{\varrho\in D^{(n)}}1 = \frac{2}{\pi\log R^{(n)}}\sum_{\nu\in D^{(n)}}1+O\big(f(R^{(n)})\big) = \frac{2}{\pi\log R^{(n)}}\sum_{\nu\in D^{(n)}}1+O\big(f(R)\big),$$

since $R^{(n)} \leqslant R+n\sqrt{2}$, and again $\sum_{\varrho\in D}1 = \sum_{\varrho\in D^{(n)}}1+O(R)$.

References

[1] I. V. Chulanovskii, *An elementary proof of the law of distribution of primes of a Gaussian field* (in Russian), Vestnik LGU 13 (1956), pp. 43–62.

[2] (i) A. O. Gelfond and Yu. V. Linnik, *Elementary Methods in Analytic Number Theory*, Allen and Unwin 1965.
(ii) A. O. Gelfond and Yu. V. Linnik, *Elementary Methods in the Analytic Theory of Numbers*, Pergamon Press, Oxford–London–New York–Paris 1966.

[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fourth edition, Oxford 1962.

[4] Martin Schechter, *Principles of Functional Analysis*, Academic Press, New York–London 1971.

N. S. W. INSTITUTE OF TECHNOLOGY
Broadway, N. S. W., Australia

Received on 21. 4. 1973 (399)

ACTA ARITHMETICA
XXVI (1975)

Metrische Ergebnisse über den Kotangensalgorithmus

von

Fritz Schweiger (Salzburg)

1. Einleitung. D. H. Lehmer hat in seiner Arbeit [2] die Theorie des Kotangensalgorithmus begründet. Das wichtigste Resultat lautet: Ist $x > 0$ irrational, so läßt sich x eindeutig als Reihe der Form

$$x = \cot\sum_{k=0}^{\infty}(-1)^k \operatorname{arccot} n_k$$

schreiben, wobei $n_0 \geqslant 0$ und $n_{k+1} \geqslant h(n_k) = n_k^2+n_k+1$ erfüllt sind. Umgekehrt ist jede Reihe dieser Art konvergent. Ist

$$\operatorname{arccot} x = \sum_{k=0}^{g-1}(-1)^k\operatorname{arccot} n_k+(-1)^g\operatorname{arccot} x_g$$

so ist $n_g = [x_g]$ und

$$\operatorname{arccot} x_{g+1} = \operatorname{arccot} n_g - \operatorname{arccot} x_g.$$

Wir setzen $h(n_k) = n_k^2+n_k+1$, und weiters sei

$$y_g = \frac{h(n_g)}{x_{g+1}}.$$

Das Hauptergebnis dieser Arbeit is enthalten in

Satz 1. *Es ist*

$$P(y_{g+p} < t\,|\,n_0,\ldots,n_g) = t+O(t^2 2^{-g-p}).$$

Abschnitt 2 ist dem Beweis von Satz 1 gewidmet, Abschnitt 3 enthält einige weitere Sätze, die aus Satz 1 folgen. Herrn Prof. J. R. Kinney danke ich für die Anregung zu dieser Arbeit. Es sei noch erwähnt, daß dieser Algorithmus Berührpunkte mit den von Galambos [1] betrachteten Algorithmen aufweist.