References

[1] Y. Dupain et J. Lesca, *Répartition des sous-suites d'une suite donnée*, Ac Arith. 23 (1973), p. 307–314.

[2] M. Mendès France, *Suites et sous-suites équiréparties modulo* 1, Journées Arit métiques Françaises, Université de Provence, 1971.

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On some estimates in the theory of $\zeta(s,\chi)$-functions

by

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1. Let K be an algebraic number field, ν and Δ the degree and the discriminant of the field K respectively (see [2]).

Denote by $\mathfrak{N}\mathfrak{a}$ the norm of an ideal $\mathfrak{a}$ of K, by $\mathfrak{f}$ a given ideal of K and by $\mathfrak{p}$ a prime ideal of K (see [2]) .

Denote further by $\mathscr{H}$ (mod $\mathfrak{f}$) an ideal-class mod $\mathfrak{f}$ ([3], Def. VIII), by $\mathscr{H}_0$ (mod $\mathfrak{f}$) the principal class mod $\mathfrak{f}$ and by $h(\mathfrak{f})$ the class-number.

Let $\chi(\mathscr{H})$ be a character of the abelian group of ideal-classes $\mathscr{H}$ (mod $\mathfrak{f}$), $\chi(\mathfrak{a})$ the extension of $\chi(\mathscr{H})$ ([3], Def. X) and χ_0 — the principal character mod $\mathfrak{f}$.

Denote by $\zeta_K(s)$ the Dedekind Zeta-function and by $\zeta(s,\chi)$ the Hecke–Landau Zeta-functions ([3], Def. XVII).

C_i, $i=1,2,3,\ldots$, are positive constants independent of K.

Denote further

$$\Delta(x,\mathscr{H}) = \sum_{n\leqslant x}\gamma(n) - \frac{x}{h(\mathfrak{f})}, \tag{1.1}$$

$$\Delta(x,\mathscr{H}_0) = \sum_{n\leqslant x} g(n) - \frac{x}{h(\mathfrak{f})}, \tag{1.2}$$

where

$$\gamma(n) = \sum_{\substack{(\mathfrak{N}\mathfrak{p})^m = n\\ \mathfrak{p}^m\in\mathscr{H}(\mathrm{mod}\,\mathfrak{f})}} \log\mathfrak{N}\mathfrak{p}, \tag{1.3}$$

$$g(n) = \sum_{\substack{(\mathfrak{N}\mathfrak{p})^m = n\\ \mathfrak{p}^m\in\mathscr{H}_0(\mathrm{mod}\,\mathfrak{f})}} \log\mathfrak{N}\mathfrak{p} \tag{1.4}$$

and

$$E_0 = E_0(\chi) = \begin{cases} 1 & \text{if } \chi=\chi_0,\\ 0 & \text{if } \chi\neq\chi_0.\end{cases} \tag{1.5}$$

The aim of this note is to determine the equivalence between the domain in which $\zeta(s,\chi)\neq 0$ and the upper estimate of $|\Delta(x,\mathscr{H}_0)|$. In the case of the Riemann Zeta-function such equivalence was discovered and proved by P. Turán (see [8], p. 150). The case of the Dedekind Zeta-functions was the subject of our previous note (see [7]).

2. We will prove the following theorems.

THEOREM 1. *Suppose that $\prod_{\chi}\zeta(s,\chi)\neq 0$ in the domain*

$$\sigma > 1-C_0\eta(|t|), \qquad C_0\leqslant 1, \tag{2.1}$$

where C_0 is a constant depending on the ideal $\mathfrak{f}$ and the field K, $\eta(t)$ is for $t\geqslant 0$ a decreasing function, having a continuous derivative $\eta'(t)$ and satisfying the conditions

(a) $0<\eta(t)\leqslant\frac{1}{2}$,

(b) $\eta'(t)\to 0$ *as* $t\to\infty$,

(c) $\dfrac{1}{\eta(t)} = O(\log t)$ *as* $t\to\infty$.

Let a be a fixed number satisfying $0<a<1$.

Then for every $\mathscr{H}\ (\operatorname{mod}\mathfrak{f})$

$$|\Delta(x,\mathscr{H})| < C_1C_0^{-1}\nu\log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\cdot x\exp\left(-\frac{aC_0}{2}\,\omega(x)\right), \tag{2.2}$$

where $\omega(x)=\min_{t\geqslant 1}\{\eta(t)\log x+\log t\}$ and C_1 depends only on a and on the function $\eta(t)$.

THEOREM 2. *Let $0<a<1$ and $\eta_1(t)$ be a function satisfying except* (a), (b), (c) *also the additional condition*

(d) $\eta_1(t)\leqslant C_2$ *for* $t>C_3$,

where C_2 is a sufficiently small positive number and let

$$\omega_1(x) = \min_{t\geqslant 1}\{\eta_1(t)\log x+\log t\}.$$

Suppose further the estimate

$$|\Delta(x,\mathscr{H}_0)| < C_4C_0^{-1}\nu\log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\cdot x\exp\left(-\frac{aC_0}{2}\,\omega_1(x)\right), \tag{2.3}$$

where $\mathscr{H}_0(\operatorname{mod}\mathfrak{f})$ is the principal class $\operatorname{mod}\mathfrak{f}$ and the constant C_4 depends only on a and $\eta_1(t)$.

Then $\prod_{\chi}\zeta(s,\chi)\neq 0$ in the domain

$$\sigma > 1-\frac{\log t}{400\log\omega_1^{-1}(\log t^{4(aC_0)^{-1}})}, \tag{2.4}$$

$$t>\max\Big\{C_5,\ \big(C_0^{-1}h^2(\mathfrak{f})\nu^2\log^2(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\big)^5,\ \eta_1^{-1}\big(\exp(-\nu^2h^2(\mathfrak{f}))\big),(|\Delta|+2)\mathfrak{N}\mathfrak{f}\Big\},$$

where $\omega_1^{-1}(x)$, $\eta_1^{-1}(x)$ denote functions inverse to $\omega_1(x)$ and $\eta_1(x)$ respectively.

THEOREM 3. *Under the conditions of Theorem 2, we have $\prod_{\chi}\zeta(s,\chi)\neq 0$ in the region*

$$\sigma > 1-\frac{aC_0}{(40)^2}\,\eta_1(t^{4(aC_0)^{-1}}), \tag{2.5}$$

$$t>\max\Big\{C_5,\ \big(C_0^{-1}h^2(\mathfrak{f})\nu^2\log^2(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\big)^5,\ \eta_1^{-1}\big(\exp(-\nu^2h^2(\mathfrak{f}))\big),\ (|\Delta|+2)\mathfrak{N}\mathfrak{f}\Big\}.$$

Choosing $\eta_1(t)=\eta(t)=\dfrac{1}{\log^{\gamma}(t+2)}$, $0<\gamma\leqslant 1$, we obtain from Theorems 1 and 2 the following

THEOREM 4. *If γ_1 is the supremum of the numbers γ for which*

$$\Delta(x,\mathscr{H}_0) = O\{x\exp(-C_6\log^{\gamma}x)\} \tag{2.6}$$

and γ_2 is the infimum of the numbers γ' for which $\prod_{\chi}\zeta(s,\chi)\neq 0$ in the region

$$\sigma > 1-\frac{C_7}{\log^{\gamma'}|t|}, \qquad |t|\geqslant C_8, \tag{2.7}$$

then

$$\gamma_1 = 1/(1+\gamma_2),$$

and the constants depend on γ, $\mathfrak{f}$ and the field K.

3. The proofs of Theorems 1 and 2 will rest on the following lemmas.

LEMMA 1. *Let $z_1, z_2, \ldots, z_h$ be complex numbers such that*

$$|z_1|\geqslant|z_2|\geqslant\ldots\geqslant|z_h|, \qquad |z_1|\geqslant 1$$

and let $b_1, b_2, \ldots, b_h$ be any complex numbers. Then, if m is positive and $N\geqslant h$, there exists an integer ν such that $m\leqslant\nu\leqslant m+N$,

$$|b_1z_1^{\nu}+b_2z_2^{\nu}+\ldots+b_hz_h^{\nu}| \geqslant \left(\frac{N}{48e^2(2N+m)}\right)^N \min_{1\leqslant j\leqslant h}|b_1+b_2+\ldots+b_j|. \tag{3.1}$$

This lemma is Turán's second main theorem (see [8], p. 52).

LEMMA 2. *For $\sigma=2$, $-\infty<t<+\infty$,*

$$|\zeta(s,\chi)| > K_1, \tag{3.2}$$

where $K_1=(6/\pi^2)^{\nu}$.

LEMMA 3. *In the region $-1/100\leqslant\sigma\leqslant 4$, $-\infty<t<+\infty$*

$$|(s-1)\zeta(s,\chi)| \leqslant K_2(|t|+1)^{K_3}, \tag{3.3}$$

where

$$K_2 = C_9^{\nu}|\Delta|^{51/100}(\mathfrak{N}\mathfrak{f})^{152/100}, \qquad K_3=\tfrac{51}{100}\nu+1$$

and C_9 is a numerical constant.

LEMMA 4. *For $\sigma > 1$ it follows*

$$-\frac{\zeta'}{\zeta}(s,\chi) = \sum_{n} \frac{G(n,\chi)}{n^s},$$

where

$$G(n,\chi) = \sum_{(\mathfrak{N}\mathfrak{p})^m = n} \chi(\mathfrak{p}^m)\log \mathfrak{N}\mathfrak{p}$$

and

$$|G(n,\chi)| \leqslant K_4 \log^2 n \tag{3.4}$$

where $K_4 = \nu/\log 2$.

For the proofs of Lemmas 2–4 see [5].

LEMMA 5. *If $s_0 = 1+\mu+it$, $0<\mu\leqslant 1/40$, $t\geqslant 10$ and N_1 stands for the number of roots of $\zeta(s,\chi)$ in the circle $|s-s_0|\leqslant 8\mu$, then*

$$N_1 < C_{10}\nu \log(|\Delta|\mathfrak{N}\mathfrak{f}t)\cdot \log^{-1}(8\mu)^{-1}. \tag{3.5}$$

This lemma follows from (3.3) by the use of Jensen's inequality (compare [8], p. 187).

LEMMA 6. *Denote by $V(T)$ the number of zeros of $\zeta(s,\chi)$ in the rectangle $\sqrt{\delta}\leqslant\sigma\leqslant 1$, $T\leqslant t\leqslant T+1$ where $0<\delta\leqslant(3/16)^2$. Then for $-\infty<T<+\infty$*

$$V(T) < \frac{8}{3}\delta^{-5/6}\log\left(\frac{K_2}{K_1}(|T|+4)^{K_3}\right). \tag{3.6}$$

LEMMA 7. *There exists a broken line L in the vertical strip*

$$\tfrac{4}{3}\sqrt{\delta}\leqslant\sigma\leqslant\tfrac{8}{3}\sqrt{\delta},\qquad 0<\delta\leqslant(3/16)^2$$

consisting of horizontal and vertical segments alternately having the following property: if we denote by T_m the ordinates of horizontal segments, so for each integer m there exists only one T_m such that $m<T_m<m+1$ and

$$\left|\sum_{\chi}\frac{\zeta'}{\zeta}(s,\chi)\right| < 17\delta^{-11/6}h^2(\mathfrak{f})\log^2\left(\frac{K_2}{K_1}(|t|+5)^{K_3}\right) \tag{3.7}$$

holds for $s\in L$.

If $\frac{4}{3}\sqrt{\delta}\leqslant\sigma\leqslant 3$, $t=T_m$, $|m|\geqslant 2$, then

$$\left|\sum_{\chi}\frac{\zeta'}{\zeta}(s,\chi)\right| < 15\delta^{-4/3}h^2(\mathfrak{f})\log^2\left(\frac{K_2}{K_1}(|T_m|+5)^{K_3}\right). \tag{3.8}$$

For the proofs of Lemmas 6 and 7 see [4] and [5].

LEMMA 8. *If $0<\delta\leqslant(3/16)^2$, $1<\sigma\leqslant 3/2$, $\xi>1$ and $l\geqslant 2$ is a positive integer, then*

$$\left|(-1)^l\sum_{n\geqslant\xi}\frac{g(n)}{n^s}\,\frac{\log^{l+1}(n/\xi)}{(l+1)!}+\frac{1}{h(\mathfrak{f})}\left(\frac{\xi^{1-s}}{(1-s)^{l+2}}+\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{l+2}}\right)\right| < 171\delta^{-11/6}\frac{\xi^{\sigma_0-\sigma}h(\mathfrak{f})\log^2\left(\frac{K_2}{K_1}(|t|+6)^{K_3}\right)}{\min\left(1,(\sigma-\sigma_0)^{l+2}\right)}, \tag{3.9}$$

where $\sigma_0=\frac{8}{3}\sqrt{\delta}$ and the sum is taken over all zeros of $\prod_{\chi}\zeta(s,\chi)$ lying to the right of the line L.

This lemma can be proved following *mutatis mutandis* the Appendix V of [8].

4. We pass over to the proof of Theorem 1. Similarly as in [1], pp. 60–62, we can prove that

$$\frac{\zeta'}{\zeta}(s,\chi)+\frac{E_0}{s-1} = O\{C_0^{-1}\nu\log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2))\cdot\log(|t|+2)\} \tag{4.1}$$

in the region

$$\begin{gathered}1-aC_0\eta(|t|)\leqslant\sigma\leqslant 1+aC_0\eta(|t|),\qquad |t|\geqslant T_0,\\ 1-aC_0\eta(T_0)\leqslant\sigma\leqslant 1+aC_0\eta(|t|),\qquad |t|\leqslant T_0,\end{gathered} \tag{4.2}$$

where T_0 depends on the function $\eta(t)$, and the constant implied by the O notation depends on $\eta(t)$ and a only.

Next we deduce the simple formula

$$\sum_{n\leqslant x}(x-n)\gamma(n) = \frac{x^2}{2\pi h(\mathfrak{f})i}\sum_{\chi}\frac{1}{\chi(\mathscr{H})}\int_{C-\infty i}^{C+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s,\chi)\right)ds \tag{4.3}$$

where $C>1$.

From Lemma 4 and (4.1)–(4.3), by contour integration we get

$$\sum_{n\leqslant x}(x-n)\gamma(n) = \frac{x^2}{2h(\mathfrak{f})}+O\{C_0^{-1}\nu x^2\log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\cdot\exp(-aC_0\omega(x))\}. \tag{4.4}$$

The constant in (4.3) depends on a and $\eta(t)$ only.

Using the relation between $\sum_{n\leqslant x}(x-n)\gamma(n)$ and $\sum_{n\leqslant x}\gamma(n)$ we get similarly as in [1], p. 64, the estimate (2.2).

5. Proof of Theorem 2 (compare [6]). Put $t \geqslant 2$. By (1.2) we have for $n > 1$

$$g(n) = \Delta(n, \mathscr{H}_0) - \Delta(n-1, \mathscr{H}_0) + \frac{1}{h(\mathfrak{f})}.$$

Hence

$$\Big|\sum_{N_1 \leqslant n \leqslant N_2} g(n)\exp(-it\log n)\Big| \leqslant \frac{1}{h(\mathfrak{f})}\Big|\sum_{N_1 \leqslant n \leqslant N_2} \exp(-it\log n)\Big| + \tag{5.1}$$
$$+\Big|\sum_{N_1 \leqslant n \leqslant N_2} \{\Delta(n, \mathscr{H}_0) - \Delta(n-1, \mathscr{H}_0)\}\cdot\exp(-it\log n)\Big| = \frac{1}{h(\mathfrak{f})} I_1 + I_2.$$

We choose N_1, N_2 so large, that

$$\omega_1^{-1}(\log t^{4(aC_0)^{-1}}) \leqslant N/2 < N_1 < N_2 \leqslant N. \tag{5.2}$$

Then by

$$\omega_1(1+t^2) < \log(1+t^2) < \log t^{4(aC_0)^{-1}} \tag{5.3}$$

and the estimate (2.3), we get

$$I_2 < C_{11}\, MNt^{-1}, \tag{5.4}$$

where $M = C_0^{-1}\nu\log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)$ and C_{11} depends on a and $\eta_1(t)$ only (see [6]).

From the estimate of I_1 (see [8], p. 153) and from (5.4) we get

$$\Big|\sum_{N_1 \leqslant n \leqslant N_2} g(n)\exp(-it\log n)\Big| \leqslant C_{12}\, MNt^{-1}. \tag{5.5}$$

Suppose

$$1 < \sigma \leqslant 3/2. \tag{5.6}$$

By partial summation and (5.5) we have

$$\Big|\sum_{N_1 \leqslant n \leqslant N_2} g(n)n^{-s}\Big| \leqslant C_{13}\, MNt^{-1}. \tag{5.7}$$

We choose

$$\eta \geqslant \omega_1^{-1}(\log t^{4(aC_0)^{-1}}) \tag{5.8}$$

and apply the inequality (5.7) for

$$N_1^j = \eta\cdot 2^j, \quad N_2^j = \eta\cdot 2^{j+1}, \quad j = 0, 1, 2, \ldots$$

Hence

$$\Big|\sum_{n \geqslant \eta} \frac{g(n)}{n^s}\Big| \leqslant C_{14} M \frac{\eta^{1-\sigma}}{t(\sigma-1)}. \tag{5.9}$$

We choose further

$$\xi \geqslant \omega_1^{-1}(\log t^{4(aC_0)^{-1}}). \tag{5.10}$$

Denoting by l a positive integer and following [8], p. 154, we get, by (5.9),

$$\Big|\sum_{n \geqslant \xi} \frac{g(n)}{n^s}\log^{l+1}\Big(\frac{n}{\xi}\Big)\Big| < C_{15} M \frac{(l+1)!\,\xi^{1-\sigma}}{t(\sigma-1)^{l+2}}. \tag{5.11}$$

Hence by Lemma 8 with $\delta = (3/16)^2$ and by (5.3) we get the estimate

$$\Big|\sum_{\varrho} \frac{\xi^{\varrho-s}}{(\varrho-s)^{l+2}}\Big| < C_{16}\Big\{\xi^{1/2-\sigma}h^2(\mathfrak{f})\frac{\nu^2\log^2(|\Delta|\mathfrak{N}\mathfrak{f}t)}{(\sigma-\frac{1}{2})^{l+2}} + M\frac{\xi^{1-\sigma}}{t(\sigma-1)^{l+2}}\Big\} \tag{5.12}$$
$$< C_{17} M_1 \frac{\xi^{1-\sigma}}{t(\sigma-1)^{l+2}}\log^2 t,$$

where $M_1 = C_0 h^2(\mathfrak{f}) M^2$.

Let us suppose now that our theorem is not true. Hence there exist such zeros

$$\varrho^* = \sigma^* + it^*, \quad t^* \to \infty$$

that

$$\sigma^* > 1 - \frac{\log t^*}{400\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}, \tag{5.13}$$

$$t^* > \max\Big\{e^{28}, \big(C_0^{-1}h^2(\mathfrak{f})\nu^2\log^2(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\big)^5, \tag{5.14}$$
$$\eta_1^{-1}\big(\exp(-\nu^2 h^2(\mathfrak{f}))\big), (|\Delta|+2)\mathfrak{N}\mathfrak{f}\Big\}.$$

Putting in the estimate (5.12)

$$S = s_1 = 1 + \frac{1}{10}\frac{\log t^*}{\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})} + it^* = \sigma_1 + it^*, \tag{5.15}$$

$$\xi = \exp\big((l+2)\lambda\big), \tag{5.16}$$

where

$$\log t^* \leqslant l+2 \leqslant \tfrac{5}{4}\log t^*, \tag{5.17}$$

$$\lambda = \frac{\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}{\log t^*}, \tag{5.18}$$

we may without difficulty verify that then (5.6) and (5.10) are satisfied. Multiplying both sides of (5.12) by

$$|\xi^{s_1-\varrho^*}(s_1-\varrho^*)^{l+2}| = \xi^{\sigma_1-\sigma^*}(\sigma_1-\sigma^*)^{l+2}$$

we have

$$\Big|\sum_{\varrho} \xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < C_{18} M_1 \frac{\xi^{1-\sigma^*}}{t^*}\Big(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\Big)^{l+2}\log^2 t^*. \tag{5.19}$$

In virtue of (5.13), (5.14) and (5.17) it follows

$$\left(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\right)^{l+2} = \left(1+\frac{1-\sigma^*}{\sigma_1-1}\right)^{l+2} \leqslant t^{*1/32}.$$

Hence from (5.19) we get

$$\left|\sum_{\varrho} \xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{l+2}\right| < t^{*-2/3}\xi^{1-\sigma^*} \tag{5.20}$$

if (5.14) is satisfied and $t^* > C_{19}(C_4)$.

By virtue of Lemma 6 we get similarly to [8], p. 156, the estimates

$$\left|\sum_{\substack{\varrho \\ t_\varrho \geqslant t^*+6(\sigma_1-\sigma^*)}} \xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{l+2}\right| < C_{20}t^{*-1}\xi^{1-\sigma^*},$$

$$\left|\sum_{\substack{\varrho \\ t_\varrho \leqslant t^*-6(\sigma_1-\sigma^*)}} \xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{l+2}\right| < C_{21}t^{*-1}\xi^{1-\sigma^*},$$

$$\left|\sum_{\substack{\varrho \\ |t_\varrho-t^*|<6(\sigma_1-\sigma^*) \\ \sigma_\varrho<1-3(\sigma_1-\sigma_0)}} \xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{l+2}\right| < C_{22}t^{*-1}\xi^{1-\sigma^*}.$$

From the above estimates and (5.20) it follows, for $t^* > C_{23}(C_4)$

$$V = \left|\sum_{\substack{\varrho \\ |t_\varrho-t^*|\leqslant 6(\sigma_1-\sigma^*) \\ \sigma_\varrho\geqslant 1-3(\sigma_1-\sigma^*)}} \left(e^{\lambda(\varrho-\varrho^*)}\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{l+2}\right| < \xi^{1-\sigma^*}t^{*-2/3}. \tag{5.21}$$

We estimate the sum V from below by the use of Lemma 1. We choose

$$z_j = \frac{s_1-\varrho^*}{s_1-\varrho}\exp\big(\lambda(\varrho-\varrho^*)\big)$$

and

$$m = \log t^*. \tag{5.22}$$

We have now to determine the number N. The region

$$1-3(\sigma_1-\sigma^*) \leqslant \sigma < 1, \quad |t-t^*| \leqslant 6(\sigma_1-\sigma^*)$$

is contained in the circle $|s-s_1| \leqslant 8(\sigma_1-1)$. Hence denoting by N_1 the number of roots of $\prod_{\chi}\zeta(s,\chi)$ in the circle, and using Lemma 5 with $C_2 = \exp\big(-(28C_{10})^2\big)$ we have for

$$s_0 = s_1 = \sigma_1+it^*, \quad \mu = \frac{\log t^*}{10\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}$$

and under (5.14) and $t^* > C_{24}$ the estimate

$$N_1 < \log t^*/14.$$

In virtue of Lemma 1 there exists an exponent $l+2$ such that then

$$V > t^{*-0.66}. \tag{5.23}$$

From (5.21) and (5.23) it follows

$$1-\sigma^* > \frac{1}{400}\cdot\frac{\log t^*}{\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}$$

and this is a contradiction to (5.13).

This completes the proof of Theorem 2.

References

[1] A. E. Ingham, *The distribution of prime numbers*, Cambridge 1932.

[2] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Leipzig und Berlin 1927.

[3] — *Über Ideale und Primideale in Idealklassen*, Math. Zeitschr. 2. Band, Berlin 1918, pp. 52–154.

[4] W. Staś, *Über eine Anwendung der Methode von Turán, auf die Theorie des Restgliedes im Primidealsatz*, Acta Arith. 5 (1959), pp. 179–195.

[5] — *Über einige Abschätzungen in Idealklassen*, Acta Arith. 6 (1960), pp. 1–10.

[6] — *Über die Umkehrung eines Satzes von Ingham*, Acta Arith. 6 (1961), pp. 435–446.

[7] — and K. Wiertelak, *Some estimates in the theory of Dedekind Zeta-functions*, Acta Arith. 23 (1973), pp. 127–135.

[8] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953.

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