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Katowice

Reçu par la Rédaction le 16. 2. 1973

Ideals in subalgebras of $C(X)$

by

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Abstract. For a k-space X let $C(X)$ denote the continuous real or complex-valued functions on X; consider a uniformly closed subalgebra A of $C(X)$. If $E = \{x \in X: |f(x)| = \sup_{t\in X}|f(t)|\}$ for some $f \in A$, and if $p \in E$ is isolated in the boundary of E or if X is first countable, then $C(N_\infty)$ ($N_\infty =$ the one point compactification of the natural numbers) is a homomorphic image of A so that 1) the maximal ideal $I_p = \{f \in A: f(p) = 0\}$ contains 2^c mutually disjoint infinite chains of prime ideals of A; 2) I_p is not countably generated; 3) if A has a peak point nonisolated in X, then A has a finitely generated ideal which is not principal and krull $\dim A = \infty$.

Under further restrictions on X and A, countably generated ideals and chains of ideals of A are discussed. Applications to generalizations of the disc algebra are considered.

Let $C(X)$ denote the algebra of complex-valued continuous functions on a space X. The relationship between X and $C(X)$ has been studied for a long time; it is known that if X is nontrivial in almost any sense, $C(X)$ has an intricate ideal structure with an abundance of prime ideals in particular. For fixed X what aspects of this ideal structure do various subalgebras A of $C(X)$ share? We discover, roughly speaking, that when A is uniformly closed and X has a modicum of compact parts, the algebra of all continuous functions on the one point compactification N_∞ of the natural numbers is a homomorphic image of A and from known properties of $C(N_\infty)$, we deduce that A and $C(X)$ share many qualitative aspects (2.1 ff). For example both will contain chains of prime ideals of arbitrary length.

Even more can be said about a class of subalgebras (§ 1) which generalize to noncompact spaces the familiar notion of uniform algebra [15]. Here a Silov boundary can be introduced and as with $C(X)$ a closed countably generated ideal has a hull which meets this boundary in an open-closed set (3.6). Under further restrictions on X, chains of arbitrary ideals are also discussed (§ 4). Our results generalize theorems in [10], [11] and [13], and bear on the problem of characterizing $C(X)$ among its subalgebras discussed in [5] and [18].

The concept of peak point (§ 1) serves as our motif. It is such points, together with uniform closure, which breed the intricacies which distinguish A from such subalgebras as polynomials in one or several variables where complications, while plentiful, are more laconic and controlled.

1. Preliminaries. The topology of uniform convergence on the compact parts K of X is a locally m-convex topology on $C(X)$ given by the seminorms $\|f\|_K = \sup\{|f(x)|: x \in K\}$. A *function algebra* A on X is a subalgebra of $C(X)$ with identity which separates the points of X and is closed in this topology. Much of the theory of uniform algebras [15] generalizes to this setting in a more or less straightforward way [9]. Here we sketch only the concept of boundary which will be useful in what follows.

The dual of $C(X)$ is $M_c(X)$, the complex-valued regular Borel measures on X of compact support. A probability measure $\mu \in M_c(X)$ is a positive measure of total mass 1. Given a function algebra A on X, we call a probability measure μ a *representing measure for* A at $x \in X$ if $f(x) = \int_X f\,d\mu$ for all $f \in A$. We define the *Choquet boundary* of $A[\partial A]$ as the set of those $x \in X$ which admit only unit mass at x for representing measures; for example, if X is completely regular, $\partial C(X) = X$ [8, p. 203]. $p \in X$ is a *peak point* for A if for some $f \in A$, $f(p) = 1$ and $|f(x)| < 1$ when $x \neq p$. Evidently every peak point p is a boundary point; indeed if μ is a representing measure at p, $f^n \to \chi_{\{p\}}$ boundedly so that $1 = f(p)^n = \int_X f^n d\mu \to \mu(\{p\})$.

Standard arguments [9] reveal that the points of ∂A, considered as elements of the dual A' under evaluation, consist of the extreme points of the weakly closed, convex set of those continuous functionals whose action on A is described by integration against some probability measure. If X is compact the Krein–Milman theorem guarantees that ∂A is nonvoid, but this will not be true in general (cf. 1.3). Nevertheless we define the *Silov boundary* of $A[\Gamma_A]$ as the closure of ∂A in X. Evidently this need not be a boundary in the sense that, say, restriction $A \to A|\Gamma_A$ is a topological algebra equivalence, but as we shall see, it does make geometric sense in certain cases, it agrees with the notion in the compact case and it reveals, as we wish to emphasize, where the algebraic subtilties of A coagulate.

EXAMPLES OF FUNCTION ALGEBRAS. 1.1. If A is a locally convex topological algebra with identity, the Gelfand transforms of A form a subalgebra $\hat{A}$ of $C(\mathcal{M}_A)$ which contains the constants and separates the points of the completely regular space $\mathcal{M}_A$. The closure of $\hat{A}$ is therefore a function algebra on $\mathcal{M}_A$.

1.2. For any subset X of complex n-space $\boldsymbol{C}^n$, let $P(X)$ and $R(X)$ denote the closure in $C(X)$ of the polynomials and the rational functions with poles off X respectively. Let $A(X)$ denote those continuous functions on X which are holomorphic on the interior of X. Evidently $P(X) \subset R(X) \subset A(X)$ are function algebras on X. It follows from Cauchy's integral formula that the Silov boundary of each is contained in the topological boundary of X in $\boldsymbol{C}^n[\partial X]$; for compact planar X, $\Gamma_{R(X)} = \Gamma_{A(X)} = \partial X$ [15, II. 1.3, p. 27], but even here $\Gamma_{P(X)}$ is typically proper in ∂X.

Call X *polynomial* (*rational*) *convex* if X coincides with $\{\lambda \in \boldsymbol{C}^n$: for some compact $K \subset X$, $|f(\lambda)| \leqslant \|f\|_K$ for all polynomials f (rational functions f with poles off X)$\}$. In the plane every X is rational convex, and every convex set is polynomial convex [9, 3.2.17, 3.2.21]; in general, X is polynomial (rational) convex if and only if evaluation $X \to \mathcal{M}_{P(X)}(\mathcal{M}_{R(X)})$ is a homeomorphism.

1.3. Let D be the open unit disc and for any $F \subset T = \partial D$, set $A_F = A(F \cup D)$. The Choquet and Silov boundary of A_F is F, since $f(z) = \frac{1}{2}(\bar{a}z+1)$ is an A_F function which peaks at $a \in F$. The boundary of the algebra of holomorphic functions on D is therefore empty.

Notice also $A_F = P(F \cup D)$. For given $f \in A_F$ and $\delta > 0$, $f_\delta(z) = f\left(\frac{z}{1+\delta}\right)$ lies in $P(F \cup D)$, since it is uniformly approximable even on $\bar{D}$ by the partial sums of its Taylor series about 0. For compact $K \subset F \cup D$ and $\varepsilon > 0$, f is uniformly continuous on $K' = \{rz: r \in [0,1], z \in K\}$; this implies $\|f_\delta - f\|_K < \varepsilon$ for all sufficiently small δ. Since $F \cup D$ is convex evaluation is evidently a homeomorphism $F \cup D \equiv \mathcal{M}_{A_F}$; from this it follows that A_F is a Fréchet algebra if and only if F is open in T.

1.4. Example 1.3 can be generalized following [3]. Suppose G is an ordered locally compact Abelian group with non-negative elements P. Let Δ denote the space of all nonzero continuous multiplicative maps $\sigma: P \to D$ $\big(\sigma(x\dot{+}y) = \sigma(x)\sigma(y)\big)$ with the compact-open topology. If Γ is the dual group of G then for each $F \subset \Gamma|P \subset \Delta$, let $\underline{A}_F$ denote the closed linear span of the functions $\{\omega_x: x \in P\}$ in $C(F \cup \mathring{\Delta})$ where $\omega_x(\sigma) = \sigma(x)$ and $\mathring{\Delta} = \Delta - \Gamma|P$. Because P is a semigroup, $\underline{A}_F$ is a function algebra. As before $\partial \underline{A}_F = F = \Gamma_{\underline{A}_F}$; notice for $G = Z$, $\underline{A}_F \cong A_F$.

1.5. Given a family of function algebras A_a on spaces X_a, the closed linear span in $C(X)$ of products $f_{a_1} f_{a_2} \cdots f_{a_n}$, interpreted as functions on $X = \Pi_a X_a$, is a function algebra on X whose Choquet and Silov boundaries are $\Pi_a \partial A_a$ and $\Pi_a \Gamma_{A_a}$ respectively. The polydisc algebra [26] arises from n copies of the disc algebra is just this way.

OTHER EXAMPLES. We are also interested in subalgebras A of $C(X)$ which are *uniformly closed*: if $f_n \in A$ and $f_n \to f$ uniformly on X, then $f \in A$. Of course every function algebra is uniformly closed; there are other obvious examples: $C_0(X)$ and $C^*(X)$ (the continuous functions vanishing at ∞ and bounded respectively). The left (equiv. right) almost periodic functions on a topological group provide another well-known example [23, Section 41, p. 165].

1.6. If I is a proper ideal of $C(X)$, its uniform closure I^u is a proper subalgebra of $C(X)$; in fact I^u consists of those $f \in C(X)$ whose Stone–Čech extension f^β vanishes on a fixed nonvoid $\Delta \subset \beta X$ [24, 2.3]. $I^u + \boldsymbol{C}$ is also a uniformly closed subalgebra of $C(X)$ which is proper if Δ is not a point of X.

1.7. If X, τ completely regular, it has the structure of a uniform space [22, 6.17, p. 188] and $C^*_\mu(X)$, the bounded functions on X which are uniformly continuous with respect to a fixed uniformity μ inducing τ, forms a uniformly closed subalgebra of $C(X)$. Evidently $C_0(X) \subset C^*_\mu(X) \subset C^*(X)$; if X is metric but neither compact nor discrete, each inclusion is proper.

1.8. If G is a family of bounded functions on X for which $G \subset G^2$ and A is a uniformly closed subalgebra of $C(X)$, then so is $A_G = \{f \in A$: for all $g \in G$ and $\varepsilon > 0$ there is a compact $K \subset X$ so that $|fg| < \varepsilon$ off $K\}$. For example if $\boldsymbol{R}_n^+$ $[\boldsymbol{R}_n^-]$ denotes those n-tuples whose entries are all positive (negative) and $\frac{1}{2}E \subset E \subset \boldsymbol{R}_n^-$, set $G = \{e^{a\cdot x}: a \in E\} \subset C(\boldsymbol{R}_n^+)$. Then $C^*(\boldsymbol{R}_n^+) \subset C(\boldsymbol{R}_n^+)_G \subset C(\boldsymbol{R}_n)$ (strictly).

1.9. If B is any algebra of functions on X containing the constants and A is a uniformly closed subalgebra of $C(X)$, then so is $A_B = \{f \in A: |f| \leqslant g$ for some $g \in B\}$. For instance if B is the polynomial, $C^*(\boldsymbol{R}) \cup B^u \subset C(\boldsymbol{R})_B \subset C(\boldsymbol{R})$ (strictly). More arcane examples of uniformly closed subalgebras of still other genres can be found in [21, 1.21 and 1.22, pp. 107–109]. See also [19, Section 3].

1.10. Of course, an intersection of uniformly closed subalgebras in $C(X)$ is also one; tensor products can be defined as in 1.5.

2. Prime ideals. Call a Hausdorff space X a *k-space* if its topology is weakly induced by its compact subspaces or equivalently, if X is the quotient of some locally compact Hausdorff space [14, XI. 9.4, p. 248]. Examples include first countable spaces, locally compact spaces and finite products of such.

For a subset E of X and a subalgebra A of $C(X)$, let I_E denote the ideal of functions in A which vanish on E; for $B \subset A$ let $Z(B)$ stand for the set of zeros common to the functions in B. Our central insight into the prime ideal structure of A is the following.

THEOREM 2.1. *Suppose X is a k-space, A is a uniformly closed subalgebra of $C(X)$, J is an ideal of A and $E = \{x \in Z(J): |f(x)| = \sup_{t\in X} |f(t)|\}$ for some $f \in A$. Then if p is isolated in the boundary ∂E of E in $Z(J)$, the maximal ideal I_p will contain 2^c mutually disjoint infinite chains of prime ideals of A with each prime containing J.*

Proof. The closure of $E^c = Z(J) - E$ in $Z(J)$ is a closed subset G of X. It follows that G is a k-space. Indeed, X is the quotient of a locally compact space Y under some identification p; because G is closed in X, a restriction of p makes G a quotient of the locally compact space $p^{-1}(G)$ [14, VI. 2.1 (1)].

If $U = E^c \cup \{p\}$ is open in G, then applying [14, VI. 2.1] we see as above that U is a k-space. If in addition p is open in U, it is open in G so some neighborhood of p in $Z(J)$ misses E^c: $p \notin \partial E^c = \partial E$, a contradiction. So p cannot be open in the k-space U: p is not isolated in some compact C in U. We conclude that every neighborhood of p in X meets $C \cap E^c$.

In case U is not open in the k-space G, there is a compact set C in G so that $C \cap U$ is not open in C. $p \in C$ (otherwise $C \cap U = E^c \cap C$ is open in C); in fact p is not isolated in C (otherwise $C \cap U = E^c \cap C \cup \{p\}$ is open in C). So if a neighborhood W of p in X misses $E^c \cap C$, $W \cap C \subset \partial E$; because p is isolated in ∂E, we can choose a neighborhood V of p in X so that $V \cap \partial E = p$. Thus $p \in W \cap V \cap C \subset V \cap \partial E = p$; that is, p is isolated in C — a contradiction.

It must be then, that in either case, there is a compact set C containing p so that $C \cap E^c$ meets every X-neighborhood of p.

Since E is proper and nonempty, we may assume $\sup_{t\in X}|f(t)| = 1$. Since C is regular, there is a compact neighborhood U of p in C so that $U \subset V \cap C$. Inductively select $p_k \in U \cap E^c$ so that

$$|f(p)-f(p_k)| < \min\{1/k+1, |f(p)-f(p_{k-1})|\}.$$

p is the only possible accumulation point q of the set $\{p_k\}$. For $f(q)$ is an accumulation point of the distinct points $\{f(p_k)\}$; since $|f(p_k)| \to |f(p)|$, $q \in E$. In fact then $q \in U \cap \partial E \subset V \cap \partial E = p$.

Because $\{p_k\} \subset C$ must have an accumulation point, we conclude that $K = \{p_k\} \cup \{p\}$ is compact, f is a homeomorphism of K onto $f(K)$ and finally that $p_k \to p$.

$S = \{f(p_k)\} \cup \{0\}$ is a subset of the open unit disc D and passing to a subsequence, we may assume S is a Carleson–Newman interpolating sequence for the bounded analytic functions H^∞ on Δ: $H^\infty|S = l^\infty$ [20, p. 204]. From this we conclude $A|K = C(K)$. For if $g \in C(K)$, $g \circ f^{-1}$ has a continuous extension s to the closed unit disc so that $s(0) = 0$. According to [20, Ex. 7, p. 208], there is some $h \in H^\infty$ matching s on S which has continuous boundary values. h can be approximated uniformly on the closed unit disc $\bar{D}$ by polynomials p_n which have no constant term. Thus $p_n(f) \in A$ and for $x \in X$, $|p_n(f(x)) - h(f(x))| \leqslant \|p_n - h\|_{\bar{D}} \to 0$ so that actually $h(f) \in A$. But $h(f)|K = g \circ f^{-1}|f(K) = g$.

K is homeomorphic to N_∞, the one-point compactification of the positive integers, and composing the induced isomorphism $C(K) \cong C(N_\infty)$

with restriction $A \to A|K$, we obtain an algebra homomorphism Φ of A onto $C(N_\infty)$ so that $\Phi(I_p) = I_\infty$ and $J \subset \ker\Phi$. Exactly as in [13, Thm. 1] and [12, 1.2] we can now exploit properties of $C(N_\infty)$ [16, 14G, p. 213] to find the required primes.

Remark. If X is first countable we obtain such primes in I_p for every $p \in \partial E$, since in that case we can easily choose $p_k \in Z(J) - E$ converging to p. But even in the compact case, the isolation of p in ∂E is necessary in 2.1. Consider for instance the compact, connected F-space $\beta R^+ - R^+$ [16, 6.10]. If $f \in A = C(\beta R^+ - R^+)$ has nonconstant modulus and $J = 0$, then $\partial E \neq \emptyset$ but since the primes of $C(\beta R^+ - R^+)$ contained in any maximal ideal form a chain [16, 14.25; 12, 1.1], 2.1 fails for this f, A and J.

Remark. If A is a real subalgebra of $C_r(X)$ (the real valued continuous functions on X) the proof of 2.1 takes a very simple form: $f(K)$ lies in $[-1, 1]$ and the classical Weierstrass approximation theorem guarantees that $A|K = C_r(K)$; everything else goes as before. Thus *mutatis mutandis*, 2.1 and its applications below can be phrased for uniformly closed real subalgebras of $C_r(X)$.

The following improves [13, Theorem 1] even when A is a uniform algebra.

COROLLARY 2.2. *Suppose A is a function algebra on a k-space and J is an ideal of A. If p is a peak point for A which is not isolated in $Z(J)$, there are 2^c pairwise disjoint, infinite chains of prime ideals of A with each prime containing J and contained in I_p. Further there is an infinite ascending sequence of primes containing J which are densely contained in I_p. In particular, the krull dimension of A/J is infinite.*

Proof. The first statement is clear from 2.1. If $f \in A$ peaks at p, $I = (1-f)I_p + J$ is dense in I_p: for $g \in I_p$ and $K \subset X$ compact, $(1-f^n)g = (1-f)(1+f+f^2+ \ldots +f^{n-1})g \in I$ and $\|(1-f^n)g - g\|_K = \|f^n g\|_K \to 0$. [10, 2.1, p. 65] yields a prime Q of $C(N_\infty)$ strictly between $\Phi(I) = \Phi(1-f)I_\infty$ and I_∞ [Φ as in 2.1], and successive application of [10, 3.2] provides an infinite ascending sequence $\{Q_n\}$ of primes between Q and I_∞: $\{\Phi^{-1}(Q_n)\}$ provides the required dense chain of primes in A.

If X is separable A has cardinality c so 2.2 with $J = 0$ implies that the maximal ideal at each non-isolated peak point will contain *exactly* 2^c primes.

In the function algebra A_F (1.3) every $p \in F$ is a nonisolated peak point: krull $\dim A_F = \infty$ whenever F is nonvoid. Actually the krull dimension of $A_\emptyset = H(D)$ is also infinite [1], but the conclusion of 2.2 fails: the primes of $H(D)$ contained in a given maximal ideal form a chain [1].

None of the primes constructed in 2.1 can be finitely generated (cf. [12, 1.2 ff]). Nonetheless we observe

COROLLARY 2.3. (Compare [16, 14.25].) *A function algebra on a k-space which has a nonisolated peak point has a finitely generated ideal which is not principal.*

Proof. Otherwise the primes contained in a fixed maximal ideal form a chain [16, 14L] in violation of 2.2.

In the first countable case 2.1 yields

COROLLARY 2.4. *If X is first countable and a uniformly closed subalgebra A of $C(X)$ contains a function which attains its maximum modulus at points which do not form an open set in X, then A has infinite krull dimension and not every finitely generated ideal is principal.*

For example every uniformly closed subalgebra of $C(R_n)$ (or $C_r(R_n)$) which admits a nonconstant function attaining its maximum modulus will have infinite krull dimension and exactly 2^c nonmaximal prime ideals.

Suppose now A, p and X satisfy the conditions of 2.1 for some $f \in A$ and $J \subsetneq A$. Let F_p denote the ideal of functions in A which vanish on some neighborhood of p. Using the homomorphism $A \to C(N_\infty)$, the following two results can be verified exactly as in [12, 1.4, 1.5].

PROPOSITION 2.5. *If I is an ideal of A strictly between F_p and I_p, there are ideals $\underline{I}$ and $\bar{I}$ of A so that $F_p \subset \underline{I} \subset I \subset \bar{I} \subset I_p$ (strictly).*

By repeated use of 2.5 we see that I is a member of infinite ascending and descending chains of ideals of A lying between F_p and I_p. Although in general $\bar{J}$ will not be prime, it can be chosen so if J is countably generated.

PROPOSITION 2.6. *If $I \subset I_p$ is a countably generated ideal of A, there is a prime ideal of A strictly between I and I_p. In particular, the maximal ideal I_p is not countably generated.*

There is a way to construct primes in I_p which does not depend on interpolation (see also Theorem 3.5).

PROPOSITION 2.7. *Suppose E is a peak set for a function algebra A on any space X. If I is a closed ideal of A properly contained in I_E, there is a prime ideal P of A so that $I \subset P \cap I_E \subsetneq I_E$.*

Proof. Choose $f \in A$ with $f|E = 1$ and $|f| < 1$ off E. The multiplicatively closed set $S = \{(1-f)^n \colon n = 1, 2, \ldots\}$ does not meet I since $(1-f)^k I_E$ is dense in I_E for each k. Indeed $(1-f)^{k+1} I_E$ will be dense in I_E if both $(1-f)^k I_E$ and $(1-f) I_E$ are, so it suffices to show $(1-f) I_E$ is. But for $g \in I_E$ and $K \subset X$ compact, $(1-f^n)g = (1-f)(1+f+\ldots+f^{n-1})g \in (1-f) I_E$ and $\|(1-f^n)g - g\|_K = \|f^n g\|_K \to 0$. Applying Zorn's lemma, we can find a prime P containing I and disjoint from S: plainly $I \subset P \cap I_E \subsetneq I_E$.

If $Z(I)$ is a peak point p and X is compact, then $Z(P) = p$ and we conclude that there is a prime ideal between I_p and every nonmaximal closed primary ideal at p. In the disc algebra A_T the nonmaximal closed primary ideals at any $p \in T$ form a chain [20, p. 88] and applying the above argument to the union I of this chain, we actually find a prime of A_T properly in I_p which contains every other closed primary ideal at p. Such a prime is necessarily dense in I_p; it is not known if every proper non-zero prime in A_T is dense in some maximal ideal.

3. Countably generated ideals. Here we extend [10, 2.1, p. 65] and [11, 3.9, p. 72] to function algebras on certain k-spaces in two different ways (3.3, 3.5).

LEMMA 3.1. *In a Banach algebra every maximal ideal which has a bounded approximate identity lies in the Silov boundary.*

Proof. Adjoin an identity to the algebra A if necessary. The Gelfand transforms of a bounded approximate identity for a maximal ideal M in A are an approximate identity for I_M in the uniform algebra $\mathrm{cl}\hat{A} \subset C(\mathcal{M}_A)$ with the same bound. Thus [7, 4.1, p. 178] $M \in \partial \mathrm{cl}\hat{A} \subset \Gamma_A$.

THEOREM 3.2. *If A is a Banach algebra and J is a closed, finitely generated ideal of A with bounded approximate identity, the hull of J is open in the maximal ideal space of A.*

Proof. We may assume A has 1 (cf. [11, 2.4]). $J+\boldsymbol{C}$ is a Banach algebra and restriction $r\colon \mathcal{M}_A \to \mathcal{M}_{J+C}$ is an identification which collapses the hull of J to a point $q \in \mathcal{M}_{J+C}$ [15, p. 12]. Actually J is the maximal ideal of $J+\boldsymbol{C}$ which corresponds to q and is finitely generated over $J+\boldsymbol{C}$. (cf. [11, 2.2]). Since $J \in \Gamma_{J+C}$ (3.1), Gleason's theorem [17, 2.1] makes q isolated in $\mathcal{M}_{J+C}$, whence $r^{-1}(q)$ is open in $\mathcal{M}_A$.

For example I_E is not finitely generated if E is a nonopen peak set for a uniform algebra. Actually we have

THEOREM 3.3. *If X is locally compact or first countable and E is a nonopen peak set for a uniformly closed subalgebra A of $C(X)$, then I_E is not a countably generated ideal of A.*

Proof. If X is first countable the proof of 2.1 can be modified to produce an algebra homomorphism of A onto $C(N_\infty)$ which takes I_E onto I_∞; if I_E is countably generated, I_∞ is also: a violation of [10, 2.1].

If X is locally compact and I_E has generators $\{b_n\}$, choose $q \in \partial E$ and take some compact neighborhood C of q. Since $\{b_n/2^n\|b_n\|_C\}$ generate I_E, we may assume $|b_n| \leqslant 1/2^n$ on C. $\emptyset = \sum_{i=1}^{\infty} |b_i|$ defines a continuous function on C; since $\bigcap Z(b_n) = Z(I_E) \subset Z(1-f) = E$, $Z(\emptyset) = E \cap C$. Select $p_n \in C$ so that $0 < \emptyset(p_n) < 1/2^n$ and $0 < |1-f(p_n)| < 1/n$. Passing to a subsequence we may assume $\{f(p_n)\} \cup \{0\} \subset D$ is an interpolating sequence for H^∞: there is some $h \in A_T$ with $h(f(p_n)) = n\emptyset(p_n)$ and $h(0) = 0$ [20, p. 208]. $h(1) = \lim_{n\to\infty} n\emptyset(p_n) = 0$, so there are polynomials p_n which approximate h uniformly on $\bar{D}$ and which all vanish at 0 and 1. Since $p_n(f) \in I_E$ and $\|p_n(f)-h(f)\|_X \leqslant \|p_n - h\|_{\bar{D}} \to 0$, $h(f)$ lies in I_E: $h(f) = \sum_{i=1}^{k} (s_i+a_i)b_i$, $s_i \in A$, $a_i \in \boldsymbol{C}$. We reach the absurd conclusion:

$$n = \frac{h(f(p_n))}{\emptyset(p_n)} \leqslant \sum_{i=1}^{k} |s_i(p_n)+a_i| \frac{|b_i(p_n)|}{\emptyset(p_n)} \leqslant \sum_{i=1}^{k} \|s_i\|_C + |a_i| \,.$$

Other restrictions on X and A produce 3.3 for any closed ideal of A whose zero set meets Γ_A with boundary in Γ_A. Call a Hausdorff space X *hemicompact* when it is the union of an ascending sequence of compact parts $\{K_n\}$, and every compact $K \subset X$ lies in some K_n. Every function algebra on X is a Fréchet algebra if and only if X is a hemicompact k-space [27, Thm. 2, p. 267]. Given a function algebra A on X and subset $B \subset A$, let BA denote the ideal of finite sums $\sum f_i g_i$, $f_i \in B$, $g_i \in A$.

THEOREM 3.5. *Let A be a function algebra on a hemicompact space X which is either locally compact or first countable. Suppose J is a closed ideal and $B \subset A$ is countable. If $E = Z(B) \cap \Gamma_A$ has a nonempty boundary ∂E in Γ_A and if E contains $Z(J) \cap \Gamma_A$, then for some $q \in \partial E$, there are infinitely many prime ideals P of A so that $BA \subset P \subset I_q$ but $J \not\subset P$.*

Proof. If $B = \{b_n\}$ and we set $B' = \{b_n/2^n\|b_n\|_{K_n}\}$ (K_n as above), then $BA = B'A$: we may assume $\|b_n\|K_n \leqslant 1/2^n$. Thus the series $\sum_{i=1}^{\infty} |b_i|$ converges uniformly on the compact parts of X; since X is a k-space, it represents some continuous function $\emptyset$ on Γ_A whose zero set is E.

We claim that for some $q \in \partial E$ there is a sequence of distinct points $\{p_k\}$ on $\partial A - E$, contained in a fixed compact set C, so that q is a cluster point of $\{p_k\}$, the p_k's can be surrounded by pairwise disjoint neighborhoods $\{V_k\}$ in X and $0 < \emptyset(p_k) < 1/k!$ for each k. If X is first countable, take any $q \in \partial E$ and select from $\partial A - E$ some sequence of distinct points $\{q_n\}$ converging to $q \in Z(\emptyset)$. ($\partial A - E$ is dense in the open subset $\Gamma_A - E$ of Γ_A.) $0 < \emptyset(q_n) \to 0$: pick a subsequence $\{p_k\}$ so that $0 < \emptyset(p_k) < 1/k!$ and set $C = \{p_k\} \cup \{q\}$. Plainly the distinct elements of this convergent sequence can be surrounded by pairwise disjoint neighborhoods $\{V_k\}$ is the Hausdorff space X.

If X is locally compact, fix a neighborhood V of some $x \in \partial E$ whose closure C is compact. For each $\delta > 0$, $U_\delta = \{x \in V \cap \Gamma_A\colon 0 < \emptyset(x) < \delta\}$ is an nonvoid open set in Γ_A. Select $p_k \in \partial A \cap U_{k!}$ inductively so that $\emptyset(p_1) > \emptyset(p_2) > \ldots > \emptyset(p_n) > \ldots$ by taking $p_{k+1} \in \partial A \cap U_{\min\{\emptyset(p_k), (k+1)!\}}$. Select pairwise disjoint open intervals $\{I_k\}$ around the $\emptyset(p_k)$ and, considering $\emptyset$ as a continuous map on all of X for the moment, set $V_k = \emptyset^{-1}(I_k)$. Since C is compact, $\{p_k\}$ has some cluster point q; evidently $q \in \partial E$.

Since $Z(J) \cap \Gamma_A \subset E$, there is some $t_k \in J$ with $t_k(p_k) = 1$. Because A separates points it is easy to see that for each finite set $F \subset X$ and each $x \notin F$, there is some $g \in A$ which vanishes on F but is 1 at x. Select for each k, then, some $s_k \in A$ so that $s_k(p_k) = 1$ and $s_k(p_i) = 0$ if $1 \leqslant i < k$. Set $g_n = s_n t_n \in J$, $W_n = V_n \cap |g_n|^{-1}$ $(-2, 2)$ and $C_n = K_n \cup C$.

Observe that for each n there is some $f_n \in A$ with $f_n(p_n) = 1$, $\|f_n\|_{C_n} \leqslant 2$ and $|f_n| < 1/2\|g_n\|_{C_n}$ on $C_n - W_n$. Indeed for $K = C_n$, $p_n \in K \cap \partial A \subset \partial A_K$ is a strong boundary point for the algebra A_K [4, 2.3.4]: select $s \in A_K$ with $s(p_n) = 1 = \|s\|_K$ and $|s(x)| < 1$ on the compact set $K - W_n$. By raising s to a high enough power we can assume $|s|$ is less than the positive number $\varepsilon = 1/2\|g_n\|_{C_n}$ on $K - W_n$. There is a sequence $\{s_r\}$ on A converging to s uniformly on K; since $s_r(p_n) \to s(p_n) = 1$, we may assume $s_r(p_n) = 1$ for each r. By taking r large enough we have $\|s_r\|_{K-W_n} \leqslant \|s_r - s\|_K + \|s\|_{K-W_n} < \varepsilon$; since $\|s_r\|_K \to \|s\|_K = 1$, taking r perhaps even larger we also have $\|s_r\|_K \leqslant 2$: set $f_n = s_r$.

Finally let $h_n = f_n g_n \in J$. Observe that $h_k(p_i) = 0$ if $1 \leqslant i < k$, $h_k(p_k) = 1$ and for $i > k$, $|h_k(p_i)| \leqslant |f_k(p_i)|\|g_k\|_C < 1/2$. Further $\|h_n\|_{C_n} \leqslant 4$ since if $x \in W_n \cap C_n$, $|h_n(x)| \leqslant 2|f_n(x)| \leqslant 2\|f_n\|_{C_n} \leqslant 4$; if $x \in C_n - W_n$, $|h_n(x)| \leqslant \|g_n\|_{C_n} \dfrac{1}{2\|g_n\|_{C_n}} < 4$. Because every compact part of X lies in some C_n, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} h_n$ converges in A to some element f in the closed ideal J. Notice

$$|f(p_k)| \geqslant 1/2^k - \Big|\sum_{n>k} 1/2^n h_n(p_k)\Big| \geqslant 1/2^k - \sum_{n>k} 1/2^{n+1} = 1/2^{k+1}.$$

Consider the multiplicatively closed set $S = \{f^m g\colon m$ is a nonnegative integer, $g(q) \neq 0\}$. We observe that S does not meet BA. For if instead $f^m g = \sum\limits_{i=1}^{p} b_i a_i$, $a_i \in A$, find a neighborhood V of q and a $\delta > 0$ so that $|g(x)| \geqslant \delta$ for $x \in V$. Since q is a cluster point of $\{p_k\}$, p_k lies in V infinitely often and for such k,

$$\frac{\delta}{2^m} \cdot \frac{k!}{(2^m)^k} = \frac{\delta(1/2^{k+1})^m}{1/k!} \leqslant \frac{|f^m(p_k) g(p_k)|}{\emptyset(p_k)} \leqslant \sum_{i=1}^{p} \frac{|b_i(p_k)|}{\emptyset(p_k)} |a_i(p_k)| \leqslant \sum_{i=1}^{p} \|a_i\|_C.$$

But this is quite impossible since $k!/a^k \to +\infty$ for all $a > 0$.

It follows that $BA \cap S = \emptyset$ and applying Zorn's lemma, we find a prime ideal P of A containing BA and disjoint from S. Clearly $BA \subset P \subset I_q$ and $J \not\subset P$. To obtain infinitely many such primes repeat the process with B successively enlarged by elements from $J \setminus P$. That is for the $f \in J$ constructed above, set $B_1 = B \cup \{f\}$ and $\emptyset_1 = \emptyset + |f|$. Since $|f(p_k)| \leqslant 1/2^{k-1}$ we may choose a subsequence $\{p_{n_k}\}$ so that $0 < \emptyset(p_{n_k}) < 1/k!$ in such a way that q remains a cluster point of $\{p_{n_k}\}$. Now repeat the above argument to obtain a prime P_1 and some $f_1 \in J \setminus P_1$ with $B_1 A \subset P_1 \subset I_q$. $P \neq P_1$ since $f \in P_1 \setminus P$. Continuing in this way ($B_2 = B_1 \cup \{f_1\}$, $\emptyset_2 = \emptyset_1 + |f_1|$), we obtain infinitely many primes P_n so that $BA \subset P_n \subset I_q$ but $J \not\subset P_n$.

The following easy corollary generalizes [11, 3.9].

COROLLARY 3.6. *Let A be a function algebra on a hemicompact space which is first countable or locally compact. If J is a closed, countably generated ideal of A, then $Z(J) \cap \Gamma_A$ is open-closed in Γ_A. In particular the maximal ideal I_p requires uncountably many generators when p is nonisolated in Γ_A.*

The argument in 3.5 applies to the example A_F (1.3) for arbitrary $F \subset T$ at least when B is finite. For in this case we can take $p_n \to q \in \partial E$; since A_T interpolates $C(K)$ on the peak set $K = \{p_n\} \cup \{q\}$ with preservation of norm [15, 12.6, p. 58], we can choose the $h_n \in J$ in 3.5 to be globally bounded, so that $f = \sum\limits_{n=1}^{\infty} 1/2^n h_n$ still defines an element of A_F. Following 3.6 we have in particular

COROLLARY 3.7. *If J is a closed, finitely generated ideal of A_F, then $Z(J) \cap F$ is open-closed in F.*

A close reading of 3.5 further reveals that I_p will contain infinitely many primes whenever p is nonisolated and lies in the discrete boundary in Γ_A of some countable intersection of zero sets from A; in particular I_p captures infinitely many primes if p is the zero set of some $f \in A$.

4. Chains of ideals. Using Cohen's factorization theorem [6] we obtained the following in [11, 1.10].

THEOREM 4.1. *Let A be a uniform algebra on a compact space. If $I \subsetneq J$ are ideals of A with $Z(I) = Z(J) \subset \partial A$, there is an ideal strictly between them whenever one of them is closed.*

In the present setting we have the following version of this.

THEOREM 4.2. *Let A be a function algebra on a hemicompact k-space, and suppose E is a set of peak points or itself a peak set for A. If $I \subsetneq J$ are ideals of A with $Z(I) = Z(J) = E$, if either I or J is closed and if I contains finitely many functions $f_1, \ldots, f_n$ whose Gelfand transforms have a compact set K of common zeros in $\mathcal{M}_A$, then there is an ideal of A strictly between I and J.*

Proof. If not, we can find a maximal ideal M containing I so that $MJ \subset I$ [11, 1.1]. $\mathcal{F} = \{Z(g) \cap K\colon g \in M\}$ has the finite intersection property: if $\emptyset = \bigcap\limits_{j=1}^{l} Z(g_j) \cap K$, a theorem of Arens [2, 1.32, p. 178] produces $h_i, k_j \in A$ so that $1 = \sum\limits_i h_i f_i + \sum\limits_j k_j g_j \in M$, a contradiction. Since K is compact, $Z(M) = \bigcap \mathcal{F}$ contains some $p \in \mathcal{M}_A$; thus $M = I_p$. Since $p \in Z(I) = E$, we have $I_E J \subset I_p J \subset I$.

If $f|E = 1$ and $|f| < 1$ off E, $\{1-f^n\}$ is an approximate identity for I_E and $\|1-f^n\|_K \leqslant 2$ for every compact $K \subset X$. Thus if I is closed, $J \subset \mathrm{cl}\, I_E J \subset I$; if J is closed, viewing J as a Fréchet module over I_E, we may apply a generalization [25] of Cohen's factorization theorem to also obtain $J = I_E J \subset I$, a contradiction. If p is a peak point for A, the argument is the same.

Since for $F \subset T$ open, A_F is a Fréchet algebra with maximal ideal space $F \cup D$ for which every $p \in F$ is a peak point, 4.2 becomes

COROLLARY 4.3. *Suppose $F \subset T$ is open and $I \subsetneq J$ are ideals of A_F with $Z(I) = Z(J) \subset F$. When either I or J is closed and I contains functions $f_1, \ldots, f_n$ whose set of common zeros is compact, there is an ideal of A_F strictly between I and J.*

Remark. 4.2 improves 4.1 since there are uniform algebras which have peak sets not contained in the Choquet boundary — for example each fiber $\{\emptyset \in \mathcal{M}_{H^\infty}: \emptyset(z) = a\}$ is a connected peak set for H^∞ [20, Chap. 10]. Also we only need assume that I and J share a hull contained in E when J is closed. If both I and J are closed, successive application of 4.2 produces infinite ascending and descending chains of ideals of A between I and J. Using 3.6 and an idea from 2.7, one can also produce infinite ascending and descending chains of ideals of A which are all *densely* contained in I_E if E is a nonopen peak set and X is hemicompact and first countable or locally compact. The unenlightening details are left for the interested reader.

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Reçu par la Rédaction le 2. 4. 1973