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Accepté par la Rédaction le 28. 10. 1973

λ-complete near-rings *

by

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Abstract. Let N be a near-ring. For each cardinal λ, a radical like ideal $C_\lambda(N)$ is introduced and used to describe the structure of N in terms of λ-complete near-rings of transformations. The radical $J_2(N)$ of Betsch is extended to near-rings in which $0n = 0$ is not assumed and it is shown that $J_2(N) \subseteq C_1(N) \cap N_c$. Finally, the result of Berman and Silverman on simplicity of near-rings of transformations is extended for infinite groups and several illustrative examples are given.

1. Introduction. The most natural example of a near-ring is given by the collection of all transformations of a group. Several authors (for example [3, 4, 7, 8]) have studied the structure of near-rings by extending well known radical concepts of rings to near-rings. In this paper a radical like ideal $C_\lambda(N)$ is introduced and used to describe the structure of near-rings in terms of λ-complete near-rings of transformations. The radical $J_2(N)$ of Betsch is extended to near-rings in which $0n = 0$ is not assumed and it is shown that $J_2(N) \subseteq C_1(N) \cap N_c$. Finally, the principal result of [2] is extended for infinite groups and several illustrative examples are given.

2. Definitions. A *near-ring* N is a system (containing at least two elements) with two binary operations $+$ and $\cdot$ satisfying

(i) $(N, +)$ is a group.
(ii) $(N, \cdot)$ is a semigroup.
(iii) $a(b+c) = ab+ac$ for all $a, b, c \in N$.

If N is a near-ring then an additive group Γ ($\neq \{0\}$) is an N-*group* if and only if for all $\gamma \in \Gamma$ and $n \in N$, γn belongs to Γ and

(i) $\gamma(m+n) = \gamma m + \gamma n$ for all $\gamma \in \Gamma$ and $m, n \in N$.
(ii) $\gamma(mn) = (\gamma m)n$ for all $\gamma \in \Gamma$ and $m, n \in N$.

A subgroup Δ of an N-group Γ is an N-*subgroup* if and only if $\Delta N \subseteq \Delta$. Observe that any N-subgroup of Γ must contain $\Gamma_0 = 0N$ (where 0 is the identity element of Γ). If Γ and Γ' are N-groups and

* Research performed at the U.S.A.F. Aerospace Research Laboratories while in the capacity of an Ohio State University Research Foundation Visiting Research Associate under Contract F33615-67-C-1758.

θ: $\Gamma \to \Gamma'$ is a group homomorphism, then θ is an *N-homomorphism* if and only if $(\gamma\theta)n = (\gamma n)\theta$ for all $\gamma \in \Gamma$ and $n \in N$.

A subset K of a near-ring N is an *ideal* in N if and only if K is the kernel of a near-ring homomorphism. Thus K is an ideal in N if and only if

(i) K is an additive normal subgroup of N.

(ii) $NK \subseteq K$.

(iii) $(n+k)m - nm \in K$ for all $k \in K$ and m, $n \in N$ [5].

If K satisfies (i) and (iii), then K is called a *right ideal* in N. If Γ is an N-group and $\gamma \in \Gamma$ $(B \subseteq \Gamma)$ then $A(\gamma)(A(B))$ denotes the annihilating right ideal of $\gamma(B)$. Notice that $A(\Gamma)$ is an ideal in N for any N-group Γ.

Berman and Silverman [1] have shown that every near-ring N can be expressed as the supplementary sum of its maximal sub-C-ring N_c and its maximal sub-Z-ring N_z where $N_c = \{n \in N|\ 0n = 0\}$ and $N_z = \{n \in N|\ 0n = n\}$.

3. The radical $C_\lambda(N)$. Let Γ be an N-group and let λ $(\geqslant 1)$ be a cardinal number. Γ is *of class* λ if and only if for any Ω $(\subseteq \Gamma)$ of cardinality not exceeding λ and for any function f: $\Gamma \to \Gamma$ there exists $n \in N$ such that $\omega n = \omega f$ for all $\omega \in \Omega$. Thus if Γ is an N-group of class λ then Γ is of class μ for all $\mu \leqslant \lambda$. Also, if Γ is of class $\lambda = \operatorname{ord}\Gamma$ then Γ is of class μ for any cardinal number μ.

The near-ring N is *λ-complete* if and only if N possesses an N-group Γ of class λ and $A(\Gamma) = (0)$. The ideal K in N is *λ-complete* if and only if N/K is a λ-complete near-ring.

THEOREM 1. *Let K be an ideal in N. K is λ-complete if and only if there exists an N-group Γ of class λ such that $A(\Gamma) = K$.*

Proof. The proof is quite analogous to the proof of Lemma 2.6 [4].

The *radical* $C_\lambda(N)$ of the near-ring N is the intersection of all annihilating ideals $A(\Gamma)$ of N-groups Γ of class λ. If no such N-groups exist then $C_\lambda(N) = N$. Thus if $\lambda \leqslant \mu$, then $C_\lambda \subseteq C_\mu$.

COROLLARY 1. *$C_\lambda(N)$ is the intersection of all λ-complete ideals of N.*

COROLLARY 2. *$C_\lambda(N) = (0)$ if and only if N is isomorphic to a subdirect sum of λ-complete near-rings.*

THEOREM 2. *If Γ is an N-group of class λ and $A(\Gamma) = (0)$ then Γ is N-isomorphic to N_z.*

Proof. The mapping θ: $N \to \Gamma$ defined by $n\theta = 0n$ is an N-homomorphism onto Γ. Evidently $N_c \subseteq \operatorname{Ker}\theta$. If $n \in \operatorname{Ker}\theta$ and $\gamma \in \Gamma$, then $\gamma(0n) = (\gamma 0)n = 0n = 0$. Hence $0n \in A(\Gamma) = (0)$. Thus Γ is N-isomorphic to $N^+ - N_c$. Similarly, N_z is N-isomorphic to $N^+ - N_c$.

COROLLARY 3. *The near-ring N is 1-complete if and only if $A(N_z) = (0)$.*

The right ideal M is *modular* if and only if there exists $e \in N$ such that $n - en \in M$ for all $n \in N$ [3, 4]. M is *C_λ-modular* if and only if M is modular and $N^+ - M$ is an N-group of class λ.

THEOREM 3. *$C_\lambda(N)$ is the intersection of all C_λ-modular right ideals of N.*

Proof. The proof proceeds in a manner similar to the proof of Theorem 3.2 [4]. The main difference is that for any N-group Γ of class λ, $\gamma N = \Gamma$ for all $\gamma \in \Gamma$.

If B, Δ are subsets of the N-group Γ then $(B: \Delta)$ denotes $\{n \in N|\ \Delta n \subseteq B\}$.

THEOREM 4. *The ideal K is λ-complete if and only if $K = (M: N)$ where M is a C_λ-modular right ideal in N.*

Proof. Let K be a λ-complete ideal. By Theorem 1, there exists an N-group Γ of class λ such that $K = (0: \Gamma)$. By Theorem 2, $(0: \Gamma) = (0: N^+ - N_c) = (N_c: N^+)$. Conversely, suppose $K = (M: N)$ where M is a C_λ-modular right ideal. Then $K = (0: N^+ - M)$ and by Theorem 1, K is λ-complete.

COROLLARY 4. *$C_\lambda(N) = \bigcap_M (M: N^+)$ where M varies over all. C_λ-modular right ideals in N.*

THEOREM 5. *$C_\lambda(N/C_\lambda(N)) = (0)$ for any near-ring N.*

Proof. The proof is essentially the same as the proof of Theorem 2.4 [4].

The N-group Γ is minimal if and only if the only N-subgroups of Γ are Γ_0 and Γ. Γ is essentially minimal if and only if Γ is minimal and $\Gamma N \neq \{0\}$. The *radical* $J_2(N)$ of N is the intersection of all annihilating ideals $A(\Gamma)$ of esentially minimal N-groups Γ. If no such N-groups Γ exist, then $J_2(N) = N$. Notice that if $N_z = \{0\}$, then the above definitions coincide with those of Betsch [3, 4].

THEOREM 6. *$J_2(N) \subseteq C_1(N) \cap N_c$ for any near-ring N.*

Proof. If Γ is an N-group of class 1, then $\gamma N = \Gamma$ for all $\gamma \in \Gamma$. Thus Γ is a minimal N-group. Hence $J_2(N) \subseteq C_1(N)$. If $N_z = \{0\}$, then obviously $J_2(N) \subseteq N_c$. Otherwise N_z is an essentially minimal N-group. Thus $n \in A(N_z)$ implies $0n = 0$. In either case $J_2(N) \subseteq N_c$.

4. Examples.

(1) Let Γ be any additive group ($\operatorname{ord}\Gamma \geqslant 2$) and let N_1 be the near-ring of all constant functions on Γ into Γ. Then $C_1(N_1) = (0)$ and $C_2(N_1) = N_1$.

(2) Let Γ be the additive group of any field F and let N_2 be the near-ring of all polynomials of degree 1 or less. Then N_2 is 2-complete whence $C_2(N_2) = (0)$, while N_2 is not 3-complete. Furthermore, the only nontrivial ideal in N_2 is $(N_2)_z$. It follows that $C_3(N_2) = N_2$.

(3) Let Γ be the additive group of any field F of characteristic 2. Let N_3 be the near-ring of all polynomials $xp = a_n x^{\alpha_{n-1}} + \ldots + a_1 x^{\alpha_0} + a_0$ over F where $\alpha_i = 2^i$ $(i = 0, 1, \ldots, n-1)$. Since N_3 contains all quadratic polynomials, N_3 is 3-complete. Thus $C_3(N_3) = (0)$. However, due to the characteristic of the field, for any $a \in F$, $(a+1)p = ap + 1p + 0p$. Thus if $0, 1, a$ are distinct elements of F and $f\colon F \to F$ has the property that

$$0f = 1, \quad 1f = 1, \quad af = 1, \quad (a+1)f = 0$$

then there exists no polynomial $p \in N_3$ such that $xf = xp$ $(x = 0, 1, a, a+1)$ since $0p = 1, 1p = 1, ap = 1$ implies $(a+1)p = 1+1+1 \neq 0 = (a+1)f$. Thus $(F, +)$ is an N_3-group which is not of class 4. Now, if N_3 has an N_3-group Γ of class 4 such that $A(\Gamma) = (0)$, by Theorem 2 $(\Gamma, +)$ is N_3-isomorphic to $(N_{3z}, +)$. Obviously, $(N_{3z}, +)$ is N_3-isomorphic to $(F, +)$ which implies that $(F, +)$ is an N_3-group of class 4. This contradiction shows that N_3 is not 4-complete.

(4) Let Γ be the additive group of integers and let N_4 be the near-ring of all constant functions on Γ. Let N_5 be the near-ring of all functions f on Γ such that f is constant outside a bounded set Ω_f. Let N_6 be the near-ring of all transformations of Γ. It is known that N_6 is simple [2].

THEOREM 7. *N_5 is simple.*

Proof. Let K be a non-zero ideal in N_5. Then K must contain a non-zero element k. Thus there exists an integer α_0 such that $\alpha_0 k = \gamma_0 \neq 0$. If $\bar{\gamma}$ denotes the constant function whose range is $\{\gamma\}$, then $\bar{\alpha}_0 k = \bar{\gamma}_0 \in K$. Let $m \in N_5$ be defined by:

$$\gamma m = \begin{cases} 0 & \text{if} \quad \gamma \neq \gamma_0, \\ \gamma_1 & \text{if} \quad \gamma = \gamma_0 \end{cases}$$

where $\gamma_1 \in \Gamma$. Then $\bar{0}m = \bar{0}$ whence $(\bar{0}+\bar{\gamma}_0)m - \bar{0}m = \bar{\gamma}_0 m \in K$. But $\bar{\gamma}_0 m = \bar{\gamma}_1$. Hence K contains all constant functions. Furthermore, for any $\beta, \delta \in \Gamma$, let $m_\beta, n_\delta \in N_5$ be defined by

$$\gamma m_\beta = \begin{cases} 0 & \text{if} \quad \gamma \neq 3, \\ \beta & \text{if} \quad \gamma = 3, \end{cases}$$

$$\gamma n_\delta = \begin{cases} -1 & \text{if} \quad \gamma \neq \delta, \\ 1 & \text{if} \quad \gamma = \delta. \end{cases}$$

Then $(n_\delta + \bar{2})m_\beta - n_\delta m_\beta \in K$. But

$$\gamma[(n_\delta+\bar{2})m_\beta - n_\delta m_\beta] = \begin{cases} 0 & \text{if} \quad \gamma \neq \delta, \\ \beta & \text{if} \quad \gamma = \delta. \end{cases}$$

Thus if $f_{\beta,\delta} \in N_5$ denotes the function defined by

$$\gamma f_{\beta,\delta} = \begin{cases} 0 & \text{if} \quad \gamma \neq \delta, \\ \beta & \text{if} \quad \gamma = \delta, \end{cases}$$

then $f_{\beta,\delta} \in K$. Now, let $f \in N_5$. Then there exist $\beta_1, \beta_2, \ldots, \beta_n, \delta_1, \delta_2, \ldots, \delta_n, \gamma \in K$ such that

$$f = f_{\beta_1,\delta_1} + f_{\beta_2,\delta_2} + \ldots + f_{\beta_n,\delta_n} + \bar{\gamma}.$$

Hence $f \in K$. Thus N_5 is simple.

Let N be the direct sum $N_4 + N_5 + N_6$ and let λ be any integer greater than 1. Also let μ be any infinite cardinal. It follows then that

$$(0) = C_1(N) \subsetneqq C_\lambda(N) \subsetneqq C_\mu(N).$$

In fact $C_\lambda(N)$ is equal to N_4 and $C_\mu(N)$ is equal to the direct sum $N_4 + N_5$.

(5) Let Γ be an additive group of infinite order λ. For every infinite cardinal μ not exceeding λ let N_μ denote the near-ring of all transformations f of Γ such that ran f contains at most μ elements. Set N equal to the direct sum $\sum_\mu N_\mu$ where μ ranges overall infinite cardinals less than or equal to λ.

THEOREM 8. *N_μ is simple.*

Proof. Let K be a non-zero ideal in N_μ. As in the proof of Theorem 7, K contains all constant functions. Let $f \in N_\mu$. Define an equivalence relation $\overset{f}{\simeq}$ on Γ by:

If $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 \overset{f}{\simeq} \gamma_2$ if and only if $\gamma_1 f = \gamma_2 f$. Let the $\overset{f}{\simeq}$ equivalence classes be denoted A_i, $i \in \mathcal{A}$ where $\mathcal{A}$ is an indexing set. From each equivalence class A_i select an element γ_i. Let $B = \{\gamma_i - \gamma_j \mid i, j \in \mathcal{A}\}$. Evidently the cardinality of B cannot exceed μ. Since the theorem is true for $\mu = \lambda$ [2] we may assume $\mu < \lambda$. Then there exists $\delta \in \Gamma \setminus B$. Define functions $g, h\colon \Gamma \to \Gamma$ as follows: For any $\gamma \in \Gamma$, $\gamma g = \gamma_i$ where $\gamma \in A_i$ and

$$\gamma h = \begin{cases} 0 & \text{if} \quad \gamma \notin \{\gamma_i + \delta \mid i \in \mathcal{A}\}, \\ \gamma_i f & \text{if} \quad \gamma = \gamma_i + \delta. \end{cases}$$

Then $g, h \in N_\mu$ and $\bar{\delta} \in K$. Hence $(g+\bar{\delta})h - gh \in K$. However, if $\gamma \in \Gamma$ then $\gamma(g+\bar{\delta})h - \gamma gh = (\gamma_i + \bar{\delta})h - \gamma_i h = \gamma_i f = \gamma f$. Thus $f \in K$; i.e., $K = N_\mu$.

It follows therefore if $\aleph_0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \lambda$, then $C_{\nu_1}(N) \subsetneqq C_{\nu_2}(N)$. In fact, $C_{\nu_i}(N)$ is isomorphic to the direct sum $\sum_\mu N_\mu$ where μ ranges over all infinite cardinals less than ν_i $(i = 1, 2)$.

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Accepté par la Rédaction le 5. 11. 1973