A Lefschetz-type fixed point theorem

by

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Abstract. In the present note, we consider a new class of multi-valued maps called by us admissible. The class of admissible maps contains acyclic maps and it is essentially larger.

This note gives the following two generalizations on the Lefschetz fixed point theorem:

(i) for admissible maps of compact, metric, approximative ANR-spaces,

(ii) for admissible maps of metric ANR-spaces.

Moreover, in this paper we give a modern proof of Eilenberg-Montgomery coincidence theorem which depend on methods given by A. Dold in [4].

S. Eilenberg and D. Montgomery have proved [5] the following theorem:

Let X be a compact metric ANR, and $p, q\colon Y \to X$ a pair of continuous maps, where Y is a compact metric space and p a Vietoris map. If the Lefschetz number $\lambda(q_* p_*^{-1})$ does not equal 0, then there exists a point $y \in Y$ such that $p(y) = q(y)$.

This note gives a modern proof of this theorem. Our proof may be viewed as an application and generalization of methods given by A. Dold in [4].

Moreover, we generalize the above Eilenberg–Montgomery theorem onto the following two cases:

(i) X is a compact metric AANR of a finite type ([3]),

(ii) X is a metric ANR and q is a compact map.

Applications of these results to multi-valued maps are given.

1. Homology preliminaries. All spaces are assumed to be Hausdorff. By H we denote the Čech homology functor with compact carriers ([6, 10]) from the category of topological pairs to the category of graded vector spaces (the coefficient group is the field Q of rationals).

A non-empty space X is called *acyclic* provided (i) $H_n(X) = 0$ for all $n > 0$ and (ii) $H_0(X) = Q$.

We recall that a *Vietoris map* is a proper map $p\colon (Y, B) \to (X, A)$ with $p(Y) = X$, $p^{-1}(A) = B$ and $p^{-1}(x)$ is an acyclic space for each $x \in X$.

The Vietoris–Begle theorem (see [1]) and the five lemma gives:

(1.1) THEOREM. *If* $p\colon (Y, B)\to(X, A)$ *is a Vietoris map, then* $p_*\colon H(Y, B)\xrightarrow{\sim} H(X, A)$ *is an isomorphism.*

Consider the category $\mathcal{C}$ of all pairs (U, V) such that U and V are open subsets in the Euclidean space R^n, for some $n\geqslant 0$, or U is a finite polyhedron and V is an open subset of U and continuous maps of such pairs.

Since the family of pairs of finite polyhedra $\{(K, L)\}$ is confinal in the family of compact pairs $\{(X, A)\}$ contained in (U, V), we obtain the following:

(1.2) *On the category* $\mathcal{C}$ *the functors* H *and* $\bar{H}$ *are naturally isomorphic* ($\bar{H}$ *denote the singular homology functor with coefficients in* Q).

From the excision axiom for singular homology and (1.2) we deduce (comp. [11]):

(1.3) *If* $A\subset U\subset R^n\subset S^n=R^n\cup\{\infty\}$, *where* A *is a compact and* U *an open subset in* R^n, *then the inclusion map* $j\colon (U, U\setminus A)\to(S^n, S^n\setminus A)$, *induces an isomorphism* $j_*\colon H(U, U\setminus A)\to H(S^n, S^n\setminus A)$.

Let $K\subset U\subset R^n$, where K is a finite polyhedron and U an open subset of R^n. Consider the diagram:

$$U\xleftarrow{p} Y\xrightarrow{q} K$$

in which p is Vietoris map and q is a continuous map. With the above diagram we associate the following:

$$(U, U\setminus K)\xleftarrow{\bar{p}}\big(Y, Y\setminus p^{-1}(K)\big)\xrightarrow{\bar{q}}(R^n, R^n\setminus\{0\}),$$

where $\bar{p}(y)=y$ and $\bar{q}(y)=p(y)-q(y)$ for each $y\in Y$. We observe that $\bar{p}$ is a Vietoris map.

Let $\Delta\colon (U, U\setminus K)\to(U, U\setminus K)\times U$ be a map given by $\Delta(x)=(x, x)$ for each $x\in U$ and $d\colon (U, U\setminus K)\times K\to(R^n, R^n\setminus\{0\})$ given by $d(x, x')=x-x'$, for each $x\in U$ and $x'\in K$.

(1.4) LEMMA. *The diagram*

$$\begin{tikzcd}
H(U, U\setminus K) \arrow[r, "\Delta_*"] \arrow[dr, "\bar{q}_*\bar{p}_*^{-1}"'] & H(U, U\setminus K)\otimes H(U) \arrow[r, "\mathrm{Id}_*\otimes q_*p_*^{-1}"] & H(U, U\setminus K)\otimes H(K) \arrow[dl, "d_*"] \\
 & H(R^n, R^n\setminus\{0\}) &
\end{tikzcd}$$

commutes.

Proof. We consider the commutative diagram:

$$\begin{tikzcd}
(U, U\setminus K)\times U & (U, U\setminus K)\times Y \arrow[l, "\mathrm{Id}\times p"'] \arrow[r, "\mathrm{Id}\times q"] & (U, U\setminus K)\times K \arrow[d, "d"] \\
(U, U\setminus K) \arrow[u, "\Delta"] & \big(Y, Y\setminus p^{-1}(K)\big) \arrow[l, "\bar{p}"] \arrow[r, "\bar{q}"'] \arrow[u, "f"] & (R^n, R^n\setminus\{0\})
\end{tikzcd}$$

in which the map f is given by $f(y)=\big(p(y), y\big)$ for each $y\in Y$. It is easy to see that, for spaces in the above diagram, the Künneth theorem is true. Hence from the commutativity of the above diagram, in view of the Künneth theorem for H, we obtain (1.4).

2. The Lefschetz number. Let $M=\{M_i\}_{i\in Z}$ and $N=\{N_i\}_{i\in Z}$ be graded vector spaces over the field of rationals.

Consider the following vector spaces:

(a) $M^*=\{M_i^*\}$, where $M_i^*=\mathrm{Hom}(M_{-i}, Q)$,

(b) $M^*\otimes N=\{M_i^*\otimes N_j\}$,

(c) $\mathrm{Hom}(M, N)=\{\mathrm{Hom}(M_{-i}, N_j)\}$.

Let us define a map: $\theta\colon M^*\otimes N\to\mathrm{Hom}(M, N)$, where $\theta_{i,j}\colon M_i^*\otimes N_j\to\mathrm{Hom}(M_{-i}, N_j)$ is given by the formula: $\theta_{i,j}(f\otimes n)(m)=(-1)^{ij}f(m)\cdot n$.

We note that of M is a graded space of finite type, i.e., $\dim M_i<+\infty$ for all i and $M_i=0$ for almost all i, then θ is an isomorphism (comp. [4]).

The homomorphisms of $\mathrm{Im}\,\theta$ said to be of *finite rank*.

(2.1) DEFINITION (comp. [4]). Let M be a graded vector space of finite type. The *Lefschetz number* $\lambda(f)$ of a homomorphism $f\in\mathrm{Hom}(M, M)$ of finite rank is given by the equality

$$\lambda(f)=e\big(\theta^{-1}(f)\big),$$

where $e\colon M^*\otimes M\to Q$ is given by

$$e_{ij}(f_i\otimes m_j)=\begin{cases}0 & \text{for}\quad i\neq -j,\\ f_i(m_j) & \text{for}\quad i=-j.\end{cases}$$

Remark. It is well known that the $\lambda(f)$ just defined coincides with the Lefschetz number defined as the alternating sum of traces.

Let $f\colon M\to M$ be an endomorphism of M; then by $\Lambda(f)$ we denote the *generalized Leftschetz number* of f (see [8]); call f a Leray endomorphism, if $\Lambda(f)$ is defined.

(2.2) *If in the category of graded vector spaces the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "g"] & M' \arrow[dl, leftarrow] \\
M \arrow[u, "f"] \arrow[r, "g"'] & M' \arrow[u, "f'"']
\end{tikzcd},$$

then f is a Leray endomorphism if and only if f' is a Leray endomorphism and in this case $\Lambda(f) = \Lambda(f')$.

For the proof of (2.2) see [8].

3. The coincidence index. Let us fix for each $n \geqslant 0$ an orientation $1 \in H_n(S^n) = Q$ of the nth sphere $S^n = R^n \cup \{\infty\}$.

Consider the diagram

$$S^n \xrightarrow{i} (S^n, S^n \setminus A) \xleftarrow{j} (U, U \setminus A),$$

in which A is a compact subset of U, U is open in R^n, and i, j are inclusions.

From (1.3) we conclude that j_* is isomorphic map.

(3.1) Definition (cf. [4]). The *fundamental class* O_A of the pair (U, A) is defined by the equality

$$O_A = j_*^{-1} i_*(1).$$

Let $A \subset A_1 \subset V \subset U \subset R^n$, where U, V are open in R^n and A, A_1 are compact. Let $k\colon (V, V \setminus A_1) \to (U, U \setminus A)$ be the inclusion; then we have

(3.2) $k_*(O_{A_1}) = O_A$.

A coincidence of a pair of maps $p, q\colon Y \to X$ is a point $y \in Y$ with $p(y) = q(y)$.

Consider the diagram

$$U \xleftarrow{p} Y \xrightarrow{q} U,$$

in which U is an open subset in R^n, p is a Vietoris map and q is a compact map, i.e., $q(Y)$ is contained in some compact subset A of U.

Then the set $\varkappa_{p,q} = \{x \in U;\ x \in qp^{-1}(x)\}$ is compact and we have the diagram

$$(1)\qquad (U, U \setminus \varkappa_{p,q}) \xleftarrow{\bar{p}} (Y, Y \setminus p^{-1}(\varkappa_{p,q})) \xrightarrow{\bar{q}} (R^n, R^n \setminus \{0\}),$$

where $\bar{p}(y) = p(y)$ and $\bar{q}(y) = p(y) - q(y)$ for each $y \in Y$. Since p is a Vietoris map, we may define the *index of coincidence* $I(p, q)$ of the pair (p, q) by putting

$$I(p, q) = \bar{q}_* \bar{p}_*^{-1}(O_{\varkappa_{p,q}}) \in H_n(R^n, R^n \setminus \{0\}) \approx Q.$$

We note the following simple facts:

(3.3) *If $I(p, q) \neq 0$, then the pair (p, q) has a coincidence.*

(3.4) *If A is a compact set such that $\varkappa_{p,q} \subset A \subset U$, then $I(p, q) = \tilde{p}_* \tilde{q}_*^{-1}(O_A)$, where $\tilde{p}, \tilde{q}$ are given by the same formula as $\bar{p}, \bar{q}$ in (1).*

(3.5) Lemma. *Let K be a finite polyhedron such that $q(Y) \subset K$, then there exists an element $a \in (H(K))^* \otimes H(K)$ such that $I(p, q) = e(a)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
H(U, U \setminus K) \arrow[r, "\Delta_*"] \arrow[dr, "\tilde{q}_* \tilde{p}_*^{-1}"'] & H(U, U \setminus K) \otimes H(U) \arrow[r, "\mathrm{Id}_* \otimes q_{1*} p_*^{-1}"] \arrow[d, phantom, "\mathrm{I}"] & H(U, U \setminus K) \otimes H(K) \arrow[dl, "d_*"'] \arrow[d, "\hat{d} \otimes \mathrm{Id}_*"] \arrow[dl, phantom, "\mathrm{II}"] \\
 & Q \approx H(R^n, R^n \setminus \{0\}) & (H(K))^* \otimes H(K) \arrow[l, "e"]
\end{tikzcd}$$

where $q_1\colon Y \to K$ is a contraction of q to the pair (Y, K) and $\hat{d}$ is given by

$$\hat{d}(u)(v) = d_*(u \otimes v) \quad \text{for} \quad u \in H(U, U \setminus K),\ v \in H(K).$$

The subdiagram (I) commutes (comp. (1.4)). The commutativity of (II) follows by easy computation.

We define $a = (\hat{d} \otimes \mathrm{Id})(\mathrm{Id} \otimes q_{1*} p_*^{-1}) \Delta_*(O_K)$. The commutativity of above diagram and (3.4) imply that $I(p, q) = e(a)$ and the proof is completed.

Let $K \subset U \subset R^n$ be as in (3.5), $t\colon U \times K \to K \times U$, $t(x, x') = (x', x)$ and $O_K^*\colon H(K) \to H(U, U \setminus K) \otimes H(K)$, $O_K^*(u) = O_K \otimes u$.

Applying (1.2) to the Dold's lemma ((4.2) in [4]), we obtain the following:

(3.6) *The composite*

$$\begin{aligned}
L = L(K, U)\colon H(K) &\xrightarrow{O_K^*} H(U, U \setminus K) \otimes H(K) \\
&\xrightarrow{\Delta_* \otimes \mathrm{Id}_*} H(U, U \setminus K) \otimes H(U) \otimes H(K) \xrightarrow{\mathrm{Id}_* \otimes t_*} H(U, U \setminus K) \otimes H(K) \otimes H(U) \\
&\xrightarrow{d_* \otimes \mathrm{Id}_*} Q \otimes H(U) \xrightarrow{(O_{\{0\}}^*)^{-1}} H(U),
\end{aligned}$$

coincides with the inclusion homomorphism $i_*\colon H(K) \to H(U)$.

4. The coincidence theorem for open subsets of the Euclidean space. In this section we prove the following:

(4.1) Theorem. *Let $p, q\colon Y \to U$ be a pair of maps from a space Y to an open subset U in R^n. If p is a Vietoris map and q is a compact map, then $q_* p_*^{-1}$ is a Leray endomorphism and if $\Lambda(q_* p_*^{-1}) \neq 0$ then the pair (p, q) has a coincidence.*

Proof. Since q is a compact map, there exists a finite polyhedron K such that $q(Y) \subset K \subset U$. We have the following commutative diagram:

$$\begin{tikzcd}
H(K) \arrow[rr, "i_*"] & & H(U) \\
H(p^{-1}(K)) \arrow[u, "q'_*"] \arrow[r, "j_*"] & H(Y) \arrow[ul, "q_{1*}"'] \arrow[r, "\mathrm{Id}_*"] & H(Y) \arrow[u, "q_*"'] \\
H(K) \arrow[u, "(p'_*)^{-1}"] \arrow[rr, "i_*"'] & & H(U) \arrow[u, "p_*^{-1}"'] \arrow[ul, "p_*^{-1}"]
\end{tikzcd}$$

From (2.2) we deduce that $\Lambda(q_*p_*^{-1}) = \lambda\big(q'_*(p'_*)^{-1}\big)$ and hence $q_*p_*^{-1}$ is a Leray endomorphism.

Assume that $\Lambda(q_*p_*^{-1}) \neq 0$. For the proof it suffice shows that:

(1) $$\lambda\big(q'_*(p'_*)^{-1}\big) = I(p, q) \quad \text{(comp. (3.3))}.$$

Consider the diagram

$$\begin{tikzcd}
H(U, U\setminus K) \otimes H(U) \otimes H(K) \arrow[r, "\mathrm{Id}*\otimes i_*"] \arrow[d, "\hat{d}\otimes q_{1*}p_*^{-1}\otimes \mathrm{Id}_*"] & H(U, U\setminus K) \otimes H(K) \otimes H(U) \arrow[r, "d_*\otimes \mathrm{Id}_*"] \arrow[d, "\hat{d}\otimes \mathrm{Id}_*\otimes q_{1*}p_*^{-1}"] & Q \otimes H(U) \approx H(U) \arrow[d, "q_{1*}p_*^{-1}"] \\
(H(K))^* \otimes H(K) \otimes H(K) \arrow[r, "\mathrm{Id}_*\otimes i_*"] & (H(K))^* \otimes H(K) \otimes H(K) \arrow[r, "e\otimes \mathrm{Id}_*"] & Q \otimes H(K) \approx H(K),
\end{tikzcd}$$

where q_1 is the induced homomorphism by the contraction of q to the pair (Y, K). The commutativity of the above diagram follows by simple calculation.

Let $a = (\hat{d} \otimes \mathrm{Id}_*)(\mathrm{Id}_* \otimes q_{1*}p_*^{-1})\Delta_*(O_K) \in \big(H(K)\big)^* \otimes H(K)$. Since $e(a) = I(p, q)$, (see (3.5)), for the proof of (1) it is suffice to show that

(2) $$\theta(a) = q'_*(p'_*)^{-1} \quad \text{(comp. also (2.1))}.$$

If we follow $\Delta_*(O_K) \otimes k \in H(U, U\setminus K) \otimes H(U) \otimes H(K)$ along $\downarrow \to \to$, we get $\big(\theta(a)\big)(k)$. If we follow it along $\to \to$ we get $i_*(k)$, (see (3.6)). Therefore, for the proof of (2) it is suffice to show that

(3) $$q_{1*}p_*^{-1}i_* = q'_*(p'_*)^{-1}.$$

Let j: $p^{-1}(K) \to Y$ be the inclusion map. Consider the following commutative diagram:

$$\begin{tikzcd}
U & Y \arrow[l, "p"'] \arrow[r, "q_1"] & K \\
K \arrow[u, "i"] & p^{-1}(K) \arrow[l, "p'"] \arrow[u, "j"'] \arrow[ur, "q'"'] &
\end{tikzcd}$$

Applying to the above diagram the functor H we obtain (3) and the proof of (4.1) is completed.

5. Multi-valued maps. In what follows, the symbols φ, ψ will be reserved for multi-valued maps; the single-valued maps will be denoted by $f, g, \ldots, p, q, r$.

Let φ: $X \to X$ be a multi-valued map. A point $x \in X$ is called a *fixed point* for φ provided $x \in \varphi(x)$.

If φ: $X \to Y$ and ψ: $Y \to Z$ are multi-valued maps, then the *composition* of φ and ψ is denoted by $\psi\varphi$: $X \to Z$ and is defined $\psi\varphi(x) = \bigcup_{y \in \varphi(x)} \psi(y)$.

We associate with φ: $X \to Y$ the following diagram of continuous maps

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y$$

in which

$$\Gamma_\varphi = \{(x, y) \in X \times Y;\ y \in \varphi(x)\}$$

is the *graph* of φ and the *natural projections* p_φ, q_φ are given by $p_\varphi(x, y) = x$ and $q_\varphi(x, y) = y$.

A multi-valued map φ: $X \to Y$ is said to be *continuous* provided the graph Γ_φ of φ is closed in $X \times Y$. The *image* of a subset A of X under φ is $\varphi(A) = \bigcup_{x \in A} \varphi(x)$. A continuous multi-valued map φ: $X \to Y$ is called *compact* provided the image $\varphi(X)$ of X under φ is contained in a compact subset of Y.

A compact multi-valued map φ: $X \to Y$ is said to be *acyclic* provided the set $\varphi(x)$ is acyclic for every point $x \in X$. A continuous multi-valued map φ: $X \to Y$ is said to be an $*$*-map* provided the natural projection p_φ is a Vietoris map.

We note the following:

(5.1) *Every acyclic map is an* $*$*-map.*

Let φ: $X \to Y$ be a $*$-map. Using (1.1) we define the linear map

$$\varphi_*\colon H(X) \to H(Y)$$

by putting

$$\varphi_* = (q_\varphi)_*(p_\varphi)_*^{-1},$$

φ_* is said to be induced by the multi-valued map φ.

Let p: $Y \to X$ be a Vietoris map. We associate with p the multi-valued map φ_p: $X \to Y$ given by $\varphi_p(x) = p^{-1}(x)$.

(5.2) ([7]). *If* p: $Y \to X$ *is a Vietoris map, then the map* φ_p: $X \to Y$ *is an* $*$*-map and* $(\varphi_p)_* = p_*^{-1}$.

(5.3) Definition ([7]). A multi-valued map φ: $X \to X_1$ is called *admissible* provided there exists a space Y and a pair (p, q) of continuous maps of the form $X \xleftarrow{p} Y \xrightarrow{q} X_1$ which satisfy the following conditions:

(i) p is a Vietoris map,

(ii) q is a compact map,

(iii) $q(p^{-1}(x)) \subset \varphi(x)$ for each $x \in X$.

In this case the pair (p, q) is called a *selected pair* of φ. If $\varphi\colon X \to X_1$ is an acyclic map then, for example, the pair (p_φ, q_φ) is the selected pair of φ.

(5.4) PROPOSITION. *Let $\varphi\colon X \to X_1$ and $\psi\colon X_1 \to X_2$ be two admissible maps. Then the composition $\psi\varphi\colon X \to X_2$ is an admissible map and for every selected pair (p_1, q_1) of φ and (p_2, q_2) of ψ there exists a selected pair (p, q) of $\psi\varphi$ such that: $q_{2*}(p_2)_*^{-1} q_{1*}(p_{1*})^{-1} = q_* p_*^{-1}$.*

Proof. Let (p_1, q_1) and (p_2, q_2) be selected pairs of φ and ψ respectively.

Consider the following commutative diagram

$$\begin{tikzcd}
X & Y_1 \arrow[l, "p_1"'] \arrow[r, "q_1"] & X_1 & Y_2 \arrow[l, "p_2"'] \arrow[r, "q_2"] & X_2 \\
 & & Y \arrow[ull, "p"] \arrow[ul, "f_1"] \arrow[u, "q"] \arrow[ur, "f_2"] \arrow[urr, "q"'] & &
\end{tikzcd}$$

where $Y = \{(y_1, y_2) \in Y_1 \times Y_2,\ q_1(y_1) = p_2(y_2)\}$, $p(y_1, y_2) = p_1(y_1)$, $q(y_1, y_2) = q_2(y_2)$, $f_1(y_1, y_2) = y_1$, $f_2(y_1, y_2) = y_2$, $g(y_1, y_2) = q_1(y_1)$. Since $f_1^{-1}(y_1)$ is homeomorphic to $p_2^{-1}(q_1(y_1))$ and p_2 is a Vietoris map, we infer that f_1 is a Vietoris map. Hence p, as the composite pf_1, is a Vietoris map. We observe that q, as the composite $q_2 f_2$, is a compact map. Moreover, we have $q(p^{-1}(x)) \subset \psi\varphi(x)$ for each $x \in X$. Applying to above diagram the functor H, we obtain: $q_{2*}(p_{2*})^{-1} q_{1*}(p_{1*})^{-1} = q_* p_*^{-1}$ and the proof of (5.4) is completed.

Remark. We observe that (5.4) remain true if we assume only that one of the maps q_1, q_2 is compact.

From (4.1) we infer:

(5.5) THEOREM. *Let U be an open subset in R^n and $\varphi\colon U \to U$ an admissible map. Then for every selected pair (p, q) of φ the endomorphism $q_* p_*^{-1}$ is a Leray endomorphism and if $\Lambda(q_* p_*^{-1}) \neq 0$ for some selected pair (p, q) of φ, then φ has a fixed point.*

6. Consequences of (4.1). First we prove the following:

(6.1) THEOREM. *Let X be a retract of an open subset $U \subset R^n$ and let $p, q\colon Y \to X$ be a pair of maps such that p is a Vietoris map, q is a compact map. Then the endomorphism $q_* p_*^{-1}$ is a Leray endomorphism and if $\Lambda(q_* p_*^{-1}) \neq 0$ then there exists a point $y \in Y$ such that $p(y) = q(y)$.*

Proof. Let $r\colon U \to X$ be a retraction map and $\varphi\colon U \to U$ the multi-valued map given by $\varphi = iq\varphi_p r$, where $i\colon X \to U$ is the inclusion map.

Then from (5.4) we conclude that φ is an admissible map and hence we may choose a selected pair (p, q) of φ such that

$$(1) \qquad q_* p_*^{-1} = i_* q_* p_*^{-1} r_* \quad \text{(comp. also (4.3))}.$$

Consider the following diagram:

$$\begin{tikzcd}
H(X) \arrow[r, "i_*"] & H(U) \\
H(X) \arrow[u, "q_* p_*^{-1}"] \arrow[r, "i_*"'] & H(U) \arrow[u, "i_* q_* p_*^{-1} r_*"'] \arrow[ul, "q_* p_*^{-1} r_*"']
\end{tikzcd}$$

Since $r_* i_* = \mathrm{Id}_*$, this diagram commutes. From (4.1), (1) and (2.2) we obtain that $q_* p_*^{-1}$ is a Leray endomorphism. Assume that $\Lambda(q_* p_*^{-1}) \neq 0$. Then (5.5) implies that φ has a fixed point. From the fact that φ has a fixed point we infer that there exists a point $y \in Y$ such that $p(y) = q(y)$ and the proof of (6.1) is completed.

Remark. We note that every Euclidean neighbourhood retract (see [4] or [2]) or, in particular, every finite polyhedron is a retract of an open subset in R^n.

A space X is called of *finite type* if the graded vector space $H(X)$ is of finite type.

(6.2) ([2]). *Every compact metric* ANR *is a space of finite type.*

From (6.1) we infer the following:

(6.3) EILENBERG-MONTGOMERY THEOREM ([5]). *Let X be a compact metric* ANR *and $p\colon Y \to X$ a Vietoris map. If $q\colon Y \to X$ is a continuous map such that $\lambda(q_* p_*^{-1}) \neq 0$, then the pair (p, q) has a coincidence.*

(6.4) Remark. In [5] Y is a compact metric space; in (6.3) Y is an arbitrary compact Hausdorff space.

(6.5) COROLLARY. *Let X be a compact metric* ANR *and $\varphi\colon X \to X$ an admissible map. If for some selected pair (p, q) of φ the Lefschetz number $\lambda(q_* p_*^{-1}) \neq 0$, then φ has a fixed point.*

Now we generalize the Eilenberg–Montgomery theorem to the case where X is a compact metric AANR of finite type.

Let Y be a metric space, X a compact subset of Y and ε a positive real number. A continuous map $r_\varepsilon\colon Y \to X$ is an ε-retraction, if $d(r_\varepsilon(x), x) < \varepsilon$ for each $x \in X$.

A compact subspace $X \subset Y$ is called a *neighbourhood approximative retract* of Y provided for every $\varepsilon > 0$ there exists an open neighbourhood V_ε of X in Y and an ε-retraction $r_\varepsilon\colon V_\varepsilon \to X$.

(6.6) DEFINITION (comp. [3]). A compact metric space X is said to be AANR provided for each embedding $h\colon X\to Y$, Y being a metric space, the space $h(X)$ is a neighbourhood approximative retract of Y.

(6.7) LEMMA ([7]). *If V is an open subset of a Banach space E and $X\subset V$ is compact, then there exists a compact (metric)* ANR *C such that: $X\subset C\subset V$.*

Using the Kuratowski–Wojdysławski embedding theorem ([2], p. 79) and (6.7) we infer:

(6.8) LEMMA. *If X is a neighbourhood approximative retract of a Banach space E, then for every $\varepsilon>0$ there exist a compact metric* ANR *C_ε and an ε-retraction $r_\varepsilon\colon C\to X$.*

(6.9) LEMMA ([9]). *Let X be an* AANR *of finite type. There exists an $\varepsilon_0=\varepsilon(X)>0$ such that if $f,g\colon X\to X$ are two continuous single-valued maps and $d\big(f(x),g(x)\big)<\varepsilon_0$ for each $x\in X$, then $f_*=g_*$.*

(6.10) THEOREM. *Let X be an* AANR *of finite type and $p\colon Y\to X$ a Vietoris map. If for a map $q\colon Y\to X$ the Lefschetz number $\lambda(q_*p_*^{-1})$ does not equal* 0, *then there exists a point $y\in Y$ such that $p(y)=q(y)$.*

Proof. For each $n=1,2,\dots$ let $r_n\colon C_n\to X$ be the corresponding $(1/n)$-retraction of some compact metric ANR ((6.8)). For each n, we define the map $\varphi_n\colon C_n\to C_n$ by putting $\varphi_n=i_nq\varphi_pr_n$, where $i_n\colon X\to C_n$ is the inclusion map. Then the map φ_n as the composite of admissible maps, is admissible. Applying (5.4) to the map φ_n, we choose a selected pair (p_n,q_n) of φ_n such that:

$$q_{n*}p_{n*}^{-1}=i_{n*}q_*p_*^{-1}r_{n*}\quad\text{for all } n>0. \tag{1}$$

Consider now, for each n, the diagram:

$$\begin{tikzcd}
H(X) \arrow[r, "i_{n*}"] & H(C_n) \\
H(X) \arrow[u, "q_*p_*^{-1}"] \arrow[r, "i_{n*}"'] & H(C_n) \arrow[u, "i_{n*}q_*p_*^{-1}r_{n*}"'] \arrow[ul, "q_*p_*^{-1}r_{n*}"']
\end{tikzcd}$$

The identity map $\mathrm{Id}\colon X\to X$ is a uniform limit of the sequence $\{r_ni_n\}$. Applying Lemma (6.9) to the space X, we conclude that there exists an integer n_0 such that:

$$\mathrm{Id}_*=(r_ni_n)_*=r_{n*}i_{n*}\quad\text{for all } n\geqslant n_0\,.$$

This implies that for all $n\geqslant n_0$ the above diagram commutes. From (2.2), (1) and (6.5) we deduce that φ_n has a fixed point for each $n\geqslant n_0$.

This implies that there exists a point $y\in Y$ such that $p(y)=q(y)$ (comp. the proof of Theorem 1 in [7]). The proof of this Theorem is completed.

(6.11) COROLLARY. *Let X be an* AANR *of finite type and $\varphi\colon X\to X$ an admissible map. If for some selected pair (p,q) of φ the Lefschetz number $\lambda(q_*p_*^{-1})$ does not equal* 0, *then φ has a fixed point.*

We note that in the case where $\varphi=f$ is a continuous single-valued map (6.11) was given in [3].

7. The coincidence theorem for arbitrary metric ANR-s. We shall make use of the following:

(7.1) APPROXIMATION THEOREM ([8]). *Let U be an open subset of a normed space E and let $f\colon Y\to U$ be a compact map. Then for every $\varepsilon>0$ there exists a finite polyhedron $K_\varepsilon\subset U$ and a map $f_\varepsilon\colon Y\to U$, called an ε-approximation of f, such that:*

(i) *$\|f(y)-f_\varepsilon(y)\|<\varepsilon$ for all $y\in Y$,*

(ii) *$f_\varepsilon(Y)\subset K_\varepsilon$,*

(iii) *f_ε is homotopic to f.*

First we prove the following:

(7.2) THEOREM. *Let U be an open subset of a normed space E, and $p,q\colon Y\to U$ a pair of maps such that:*

(i) *p is a Vietoris map,*

(ii) *q is a compact map.*

Then

(a) *$q_*p_*^{-1}$ is a Leray endomorphism.*

(b) *$\Lambda(q_*p_*^{-1})\neq 0$ implies that there exists a point $y\in Y$ such that $p(y)=q(y)$.*

Proof. By applying to q the Approximation theorem (7.1) we get a sequence $\{K_n\}$ of finite polyhedra $K_n\subset U$ and a sequence of maps $q_n\colon Y\to U$ such that

(i) $\|q_n(y)-q(y)\|<1/n$ for each $y\in Y$ and for every n,

(ii) $q_n(Y)\subset K_n$ for every n,

(iii) q_n is homotopic to q for every n.

Suppose that $q_n'\colon Y_n=p^{-1}(K_n)\to K_n$, $\bar q_n\colon Y\to K_n$, $p_n\colon Y_n\to K_n$ are contractions of q_n and p respectively and $i_n\colon K_n\to U$, $j_n\colon Y_n\to Y$ are inclusions.

Now, for every n, we have the commutative diagram

$$\begin{tikzcd}
K_n \arrow[rr, "i_n"] & & U \\
Y_n \arrow[u, "q'_n"] \arrow[r, "j_n"] \arrow[d, "p_n"'] & Y \arrow[ul, "\bar{q}_n"'] \arrow[r, "\mathrm{Id}"] \arrow[dr, "p"] & Y \arrow[u, "q"'] \arrow[d, "p"] \\
K_n \arrow[rr, "i_n"'] & & U
\end{tikzcd}$$

Since every K_n is of a finite type, $(q'_n)_*(p_{n*})^{-1}$ is a Leray endomorphism. Consequently, by (2.2), $q_{n*}p_*^{-1}$ is also a Leray endomorphism and $\lambda(q'_{n*}p_*^{-1}) = \Lambda(q_{n*}p_*^{-1})$.

Now (iii) implies that $q_*p_*^{-1}$ is a Leray endomorphism and moreover we have:

(iv) $\lambda(q'_{n*}p_{n*}^{-1}) = \Lambda(q_{n*}p_*^{-1}) = \Lambda(q_*p_*^{-1})$.

To prove (b) assume that $\Lambda(q_*p_*^{-1}) \neq 0$. Then, in view of (iv), we have $\lambda(q'_{n*}p_{n*}^{-1}) \neq 0$ for every n.

Now we apply the Eilenberg–Montgomery theorem to $p_n, q'_n\colon Y_n \to K_n$ for each n and obtain a sequence $\{y_n\}$ of points $y_n \in Y$ such that $p_n(y_n) = q'_n(y_n) = p(y_n) = q_n(y_n)$.

We put: $p(y_n) = q_n(y_n) = x_n \in U$ and $q(y_n) = \bar{x}_n \in U$. Since q is compact map we may assume that there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ such that $\lim_k \bar{x}_{n_k} = x$. Then from (i) we deduce that $\lim_k x_{n_k} = x$ and hence we have

(v) $\bar{x}_{n_k} \in qp^{-1}(x_{n_k})$, $\{\bar{x}_{n_k}\} \to x$ and $\{x_{n_k}\} \to x$.

Since qp^{-1} is a continuous multi-valued map (comp. (5.2) and (5.4)) from (v), we infer that $x \in qp^{-1}(x)$.

This implies that there exists a point $y \in p^{-1}(x)$ such that $p(y) = q(y)$ and the proof of (7.2) is completed.

(7.3) THEOREM. *Let X be a retract of an open subset U of a normed space E. Assume that $p, q\colon Y \to X$ is a pair of maps such that:*

(i) *p is a Vietoris map,*

(ii) *q is a compact map.*

Then

(a) *$q_*p_*^{-1}$ is a Leray endomorphism,*

(b) *$\Lambda(q_*p_*^{-1}) \neq 0$ implies that there exists a point $y \in Y$ such that $p(y) = q(y)$.*

The proof of (7.3) is analogous to the proof of (6.1).

Remarks. 1. Every metric ANR is a retract of an open subset of a normed space (see [2] or [8]).

2. We underline that in (7.3) Y is an arbitrary Hausdorff space.

(7.4) COROLLARY. *Let X be as in (7.3). Assume that $\varphi\colon X \to X$ is an admissible map. Then for every selected pair (p, q) of φ, $q_*p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1}) \neq 0$ implies that φ has a fixed point.*

(7.5) COROLLARY. *If X is a metric AR, then any admissible map $\varphi\colon X \to X$ has a fixed point.*

References

[1] E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. of Math. 51 (1950), pp. 534–543.

[2] K. Borsuk, *Theory of Retracts*, Warszawa 1967.

[3] M. H. Clapp, *On a generalization of absolute neighborhood retracts*, Fund. Math. 70 (1971), pp. 117–130.

[4] A. Dold, *Fixed point index and fixed points for* ENR-*s*, Topology 4 (1965), pp. 1–8.

[5] S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math. (1946), pp. 214–222.

[6] — and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[7] L. Górniewicz, *On a coincidence theorem*, Bull. Acad. Polon. Sci. 2 (1974), pp. 147–152.

[8] A. Granas, *Topics in the fixed point theory*, Séminaire Jean Leray, Paris 1969/70.

[9] G. Skordev, *Fixed points of maps of* AANR-*s*, Bull. Acad. Polon. Sci. 21 (1973), pp. 173–180.

[10] E. Spanier, *Algebraic Topology*, New York 1966.

[11] C. Teleman, *Elemente de Topologie si varietati diferentiable*, Bucaresti 1964.

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