Locally weakly confluent mappings on hereditarily locally connected continua

by

T. Maćkowiak (Wrocław)

Abstract. A continuous mapping f of a topological space X onto a topological space Y is said to be *locally weakly confluent* provided for each point y of Y there exists a closed neighbourhood V of y in Y such that the partial mapping $f|f^{-1}(V)$ is a weakly confluent mapping of $f^{-1}(V)$ onto V, i.e., for each continuum Q in V there is a component C of the set $f^{-1}(Q)$ such that $f(C) = Q$.

We study some properties of this class of mappings. Moreover, it is proved in the paper that the property of being a hereditarily locally connected continuum or a graph is an invariant under a locally weakly confluent mapping. Further, we characterize graphs which are images under locally weakly confluent mappings of an arbitrary graph. This is a partial answer to a problem asked in [2] for weakly confluent mappings.

1. Introduction. In this paper we localize a well-known class of weakly confluent mappings (see [5]) in the same way as the class of confluent mappings is localized in [3], p. 239. We obtain in this manner a new kind of continuous mappings, called locally weakly confluent. This natural class of locally weakly confluent mappings comprises weakly confluent mappings. Some theorems on weakly confluent mappings will be generalized to locally weakly confluent ones. Moreover, we will characterize graphs which are images under locally weakly confluent mappings of an arbitrary graph. This is a partial answer to a problem asked in [2] for weakly confluent mappings.

The author is very much indebted to Dr. J. J. Charatonik for his valuable suggestions and help during the preparation of this paper.

2. Preliminaries. The topological spaces under consideration are assumed to be metric, and the mappings — to be continuous and surjective.

Recall that a mapping f of a topological space X onto a topological space Y is said to be

(i) *confluent* if for every subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [1], p. 213);

(ii) *weakly confluent* if for every subcontinuum Q of Y there exists a component C of the inverse image $f^{-1}(Q)$ such that $f(C) = Q$ (see [5] and [7]; compare also [6], Sections 4 and 5).

In [3], p. 239, a class of locally confluent mappings is introduced as follows. We say that a mapping f of X onto Y is *locally confluent* provided for each point y of Y there exists a closed neighbourhood V of y in Y such that the partial mapping $f|f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V.

We define a class of locally weakly confluent mappings in a similar way, namely: a mapping f of X onto Y is called *locally weakly confluent* provided for each point y of Y there exists a closed neighbourhood V of y in Y such that the partial mapping $f/f^{-1}(V)$ is weakly confluent.

We have the following immediate consequences of the definitions mentioned above.

(2.1) PROPOSITION. *Any weakly confluent mapping is locally weakly confluent.*

(2.2) PROPOSITION. *Any locally confluent mapping is locally weakly confluent.*

The class of locally weakly confluent mappings is essentially larger than the class of weakly confluent mappings, even for hereditarily locally connected continua unlike the class of locally confluent mappings, which coincides with the class of confluent mappings onto hereditarily arcwise connected continua (see [6], 5.3) or onto locally connected continua (see [6], Corollary 5.2).

(2.3) EXAMPLE. There exists a locally weakly confluent mapping f of a simple triod T onto T such that the mapping f is not weakly confluent.

The triod T will be considered as a subset of the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy. The triod T consists of the straight line interval joining points $(-1, 0)$ and $(1, 0)$ and of the straight line interval joining points $(0, 0)$ and $(0, 1)$. We define a mapping f of T onto itself as follows:

$$f\big((x, y)\big) = \begin{cases} (x, y) & \text{if} \quad x \leqslant 0 \text{ and } y = 0, \\ (3x, y) & \text{if} \quad 0 \leqslant x \leqslant 1/3 \text{ and } y = 0, \\ (-3x+2, y) & \text{if} \quad 1/3 \leqslant x \leqslant 2/3 \text{ and } y = 0, \\ (y, 3x-2) & \text{if} \quad 2/3 \leqslant x \leqslant 1 \text{ and } y = 0, \\ (x, y/2) & \text{if} \quad x = 0 \text{ and } y \neq 0. \end{cases}$$

Take $V_1 = \{(x, y) \in T\colon y \leqslant 1/2\}$ and $V_2 = \{(x, y) \in T\colon 1/3 \leqslant y\}$. Observe that for $i = 1, 2$ the partial mapping $f|f^{-1}(V_i)$ is weakly confluent, and for each point $y \in T$, we have either $y \in \operatorname{Int} V_1$ or $y \in \operatorname{Int} V_2$; therefore f is locally weakly confluent.

Let $T' = \{(x, y) \in T\colon x \leqslant 2/3 \text{ and } y \leqslant 2/3\}$. The set T' is a continuum and for each component C of the set $f^{-1}(T')$ we have $f(C) \neq T'$. Thus we conclude that f is not weakly confluent.

(2.4) EXAMPLE. There exist locally weakly confluent mappings f and g of a simple triod T onto itself such that the composed mapping gf is not locally weakly confluent.

Adopt the notation of Example (2.3). Define a mapping g of T onto itself as follows:

$$g\big((x, y)\big) = \begin{cases} (x, 0) & \text{if} \quad y \leqslant 1/2, \\ (x, 2y-1) & \text{if} \quad y \geqslant 1/2. \end{cases}$$

Observe that the mapping g is monotone, and thus also locally weakly confluent. Put $h\big((x, y)\big) = g\Big(f\big((x, y)\big)\Big)$ for $(x, y) \in T$. There exists no neighbourhood V of the point $(0, 0)$ in T such that the partial mapping $h|h^{-1}(V)$ is weakly confluent. Therefore the mapping h is not locally weakly confluent.

(2.5) THEOREM. *If a mapping $f\colon X \to Y$ is weakly confluent and a mapping $g\colon Y \to Z$ is locally weakly confluent, then gf is locally weakly confluent.*

Proof. Let z be an arbitrary point of Z and let V be a closed neighbourhood of z in Z such that $g|g^{-1}(V)$ is weakly confluent. Let $K \subset V$ be a continuum and let C be a component of $g^{-1}(K)$ such that $g(C) = K$. Since f is weakly confluent, we have a component R of $f^{-1}(C)$ such that $f(R) = C$. Therefore $gf(R) = g(C) = K$, we conclude that $gf|(gf)^{-1}(V)$ is weakly confluent. Thus gf is a locally weakly confluent mapping.

(2.6) THEOREM. *If a mapping f of X onto Y is locally weakly confluent and if a subset B of Y is closed, then the partial mapping $f|f^{-1}(B)$ is locally weakly confluent.*

Proof. Let $y \in B$ and let V be a closed neighbourhood of y in Y such that the mapping $g = f|f^{-1}(V)$ is weakly confluent. Put $V_0 = B \cap V$ and $g_0 = g|g^{-1}(V_0)$. It follows from 4.7 in [6] that the mapping g_0 is weakly confluent. Since $g_0 = g|g^{-1}(V_0) = f|f^{-1}(V_0)$, we conclude that the partial mapping $f|f^{-1}(B)$ is locally weakly confluent.

(2.7) THEOREM. *If $h = gf$ is locally weakly confluent, then g is locally weakly confluent.*

Proof. Let f map X onto Y, and let g map Y onto Z. Suppose that V is a closed neighbourhood of z in Z such that the partial mapping $h|h^{-1}(V)$ is weakly confluent and let K be an arbitrary continuum contained in V. Then there is a component C of $h^{-1}(K)$ such that $h(C) = K$. Moreover, $f(C)$ is a subcontinuum of Y which is contained in some component R of the set $g^{-1}(K)$. Therefore $K = h(C) = g\big(f(C)\big) \subset g(R)$, and thus $g(R) = K$.

This implies that the partial mapping $g|g^{-1}(V)$ is weakly confluent, i.e., the mapping g is locally weakly confluent.

Recall that a mapping f from X onto Y is called *light* provided $\dim f^{-1}(y) = 0$ for each $y \in Y$ (see [10], p. 130). We have (see [10], (4.41), p. 131) the following

(2.8) LEMMA. *In order that a continuous mapping f from X onto Y be light it is necessary and sufficient that for any $\varepsilon > 0$ there should exist a $\delta > 0$ such that if B is any continuum in Y of diameter less than δ, any component of $f^{-1}(B)$ is of diameter less than ε.*

As a direct consequence of Whyburn's factorization theorem (see [9], (2.3), p. 297 and [10], (4.1), p. 141) and of Theorem (2.7) we obtain the following

(2.9) THEOREM. *If a mapping h of X onto Y is locally weakly confluent, then there exists a unique factorization of h into two locally weakly confluent mappings*

$$h(x) = g(f(x)) \quad \textit{for each } x \in X\,,$$

where f is monotone and g is light.

It is proved (see [4], § 41, VI, Corollary 4d, p. 24) that

(2.10) LEMMA. *Let $\mathcal{C}$ be an open covering of a compact metric space X. Then there is a number $\varepsilon > 0$ such that each subset of X of diameter less than ε is contained in some element of $\mathcal{C}$.*

We have the following characterization of locally weakly confluent mappings:

(2.11) THEOREM. *A mapping f of X onto Y is locally weakly confluent if and only if there is a number $\varepsilon > 0$ such that for each continuum Q of diameter less than ε in Y there exists a component K of $f^{-1}(Q)$ such that $f(K) = Q$.*

Proof. Suppose that f is locally weakly confluent. Then there are sets $F_1, F_2, \ldots, F_n$ such that $X = \operatorname{Int}F_1 \cup \operatorname{Int}F_2 \cup \ldots \cup \operatorname{Int}F_n$ and $f|f^{-1}(F_i)$ is a weakly confluent mapping for $i = 1, \ldots, n$. The family

$$\mathcal{C} = \{\operatorname{Int}F_1, \operatorname{Int}F_2, \ldots, \operatorname{Int}F_n\}$$

is an open covering of Y. Therefore, by Lemma (2.10), there is a number $\varepsilon > 0$ such that each continuum Q of diameter less than ε in Y is contained in some F_{i_0}. Since $f|f^{-1}(F_{i_0})$ is weakly confluent, there is a component K of $f^{-1}(Q) = (f|f^{-1}(F_{i_0}))^{-1}(Q)$ such that $f(K) = (f|f^{-1}(F_{i_0}))(K) = Q$.

Conversely, suppose that there is a number $\varepsilon > 0$ such that for each continuum Q of diameter less than ε in Y there exists a component K of $f^{-1}(Q)$ such that $f(K) = Q$. Let y be an arbitrary point of Y and let V be the closed ball with diameter equal to $\varepsilon/2$ and centre at y. Obviously $f|f^{-1}(V)$ is a weakly confluent mapping, because each continuum Q contained in V has diameter less than ε. The proof of (2.11) is complete.

A similar theorem holds also for locally confluent mappings; namely

(2.12) THEOREM. *A mapping f of X onto Y is locally confluent if and only if there is a number $\varepsilon > 0$ such that for each continuum Q of diameter less than ε in Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q.*

The proof is quite similar to the proof of Theorem (2.11).

Theorems (2.11) and (2.12) imply

(2.13) THEOREM. *If mappings f and g are both light locally weakly confluent (or both light locally confluent), then the mapping gf is light locally weakly confluent (locally confluent).*

Proof. Suppose that f maps X onto Y, g maps Y onto Z and f and g are both light locally weakly confluent (for locally confluent mappings the proof is quite similar). By Theorem 2.11 there are positive numbers ε_1 and ε_2 such that if Q is a continuum in Y (in Z, respectively) such that the diameter of Q is less than ε_1 (ε_2, respectively), then there is a component C of the set $f^{-1}(Q)$ ($g^{-1}(Q)$, respectively) such that $f(C) = Q$ ($g(C) = Q$, respectively). It follows from Lemma (2.8) that there is a $\delta > 0$ such that if Q is any continuum in Z of diameter less than δ, any component of $g^{-1}(Q)$ is of diameter less than ε_1. Put $\varepsilon = \min\{\delta, \varepsilon_2\}$. Let Q be an arbitrary subcontinuum of Z of diameter less than ε. Since $\operatorname{diam} Q < \varepsilon_2$, we have a component C of $g^{-1}(Q)$ such that $g(C) = Q$. Since $\operatorname{diam} Q < \delta$, we have also $\operatorname{diam} C < \varepsilon_1$. Therefore there is a component K of $f^{-1}(C)$ such that $f(K) = C$, and thus $gf(K) = g(C) = Q$. Hence we conclude that for each continuum Q in Z of diameter less than ϵ in Z there is a component K of the set $(gf)^{-1}(Q)$ such that $gf(K) = Q$. Thus, by Theorem (2.11), the mapping gf is locally weakly confluent.

We also have the following

(2.14) THEOREM. *If $f\colon X \to Y$ is a mapping, Y_1 and Y_2 are closed subsets of Y such that $Y = \operatorname{Int} Y_1 \cup \operatorname{Int} Y_2$ and the mappings $f|f^{-1}(Y_1)$ and $f|f^{-1}(Y_2)$ are locally weakly confluent (locally confluent), then f is locally weakly confluent (locally confluent, respectively).*

Proof. Put $f_i = f|f^{-1}(Y_i)$ ($i = 1, 2$) and assume that each f_i is locally weakly confluent. Let y be an arbitrary point of Y. Then we have either $y \in \operatorname{Int} Y_1$ or $y \in \operatorname{Int} Y_2$. Assume $y \in \operatorname{Int} Y_1$ (if $y \in \operatorname{Int} Y_2$, the proof is the same). Since the mapping f_1 is locally weakly confluent, there is a closed neighbourhood V_1 of y in Y_1 such that the mapping $f_1|f^{-1}(V_1) = f|f^{-1}(V_1)$ is weakly confluent. It follows from $y \in \operatorname{Int} Y_1$, that there exists a closed neighbourhood V of y in Y such that $V \subset V_1 \cap Y_1$. Therefore, by Theorem (2.6), we infer that $f|f^{-1}(V)$ is a weakly confluent mapping. Thus f is locally weakly confluent.

Assume now that each mapping f_i ($i = 1, 2$) is locally confluent and that an arbitrary point y of Y is such that $y \in \operatorname{Int} Y_1$. Then there exists a closed neighbourhood V_1 of y in Y_1 such that the mapping $f_1|f_1^{-1}(V_1) = f|f^{-1}(V_1)$ is confluent. It follows from $y \in \operatorname{Int} Y_1$ that there exists a closed neighbourhood V of y in Y such that $V \subset V_1$. Therefore, by I in [1], p. 213, we infer that $f|f^{-1}(V)$ is a confluent mapping. Thus f is a locally confluent mapping.

(2.15) Remarks. There exists a mapping f of a simple triod T onto itself and there is a decomposition $T = V_1 \cup V_2$, where V_1 and V_2 are closed sets with $T = \operatorname{Int} V_1 \cup \operatorname{Int} V_2$, such that the mappings $f|f^{-1}(V_1)$ and $f|f^{-1}(V_2)$ are weakly confluent and f is not weakly confluent. This can be seen by Example (2.3).

We have a similar theorem to Theorem (2.14) for confluent mappings, but with slightly different assumptions on Y (see [6], Theorem 5.4).

(2.16) Corollary. *If $f\colon X \to Y$ is a mapping, $Y_1, \ldots, Y_k$ are closed subsets of Y such that $Y = \bigcup_{i=1}^{k} \operatorname{Int} Y_i$ and mappings $f|f^{-1}(Y_i)$ for $i = 1, \ldots, k$ are locally weakly confluent (locally confluent), then f is locally weakly confluent (locally confluent, respectively).*

3. The invariance of the hereditarily local connectedness of continua.

(3.1) Theorem. *Locally weakly confluent images of hereditarily locally connected continua are hereditarily locally connected.*

Proof. Let f be a locally weakly confluent mapping of a hereditarily locally connected continuum X onto Y. Suppose, on the contrary, that the continuum Y is not hereditarily locally connected. Therefore there exists a subcontinuum Q of Y which is not locally connected at some point p. Thus there exists a closed neighbourhood E of p in Q such that, if C is the component of E which contains p, then p does not belong to its interior, i.e., $p \in \overline{E \setminus C}$. Let

(1) $$p = \lim_{n\to\infty} p_n ,$$

(2) $$p_n \in E \setminus C .$$

Let C_n be the component of E such that $p_n \in C_n$. It follows that

(3) $$C \cap C_n = \emptyset \quad \text{for each} \quad n = 1, 2, \ldots$$

For otherwise the set $C \cup C_n$ would be a subcontinuum of E; thus $C \cup C_n \subset C$ and we would have $p_n \in C$, contrary to (2).

Let F_1 be a closed neighbourhood of p in Q such that

(4) $$F_1 \subset \operatorname{Int}_Q(E)$$

(here $\operatorname{Int}_Q(E)$ denotes the interior of E relative to Q), and let F_2 be a closed neighbourhood of p in Y such that

(5) $$f|f^{-1}(F_2) \quad \text{is weakly confluent.}$$

Then $F = F_1 \cap F_2$ is a closed neighbourhood of p in Q such that

(4′) $$F \subset \operatorname{Int}_Q(E)$$

by (4), and

(5′) $$f|f^{-1}(F) \quad \text{is weakly confluent}$$

by (5) and by Theorem 4.7 in [6].

We may assume that $p_n \in F$ for $n = 1, 2, \ldots$ by (1). Let D_n be the component of F containing p_n. It follows from (4′) that $F \subset E$, and thus

(6) $$D_n \subset C_n .$$

We choose a convergent subsequence $\{D_{n_m}\}$ of the sequence $\{D_n\}$ (compare [4], § 42, I, Theorem 1, p. 45 and § 42, II). For each continuum D_{n_m} there exists a component R_{n_m} of the set $f^{-1}(D_{n_m})$ such $f(R_{n_m}) = D_{n_m}$ by (5)′. We choose a convergent subsequence $\{R_{n_{m_k}}\}$ of the sequence $\{R_{n_m}\}$ and define

(7) $$K = \operatorname*{Lim}_{k\to\infty} D_{n_{m_k}} ,$$

(8) $$L = \operatorname*{Lim}_{k\to\infty} R_{n_{m_k}} .$$

We have

(9) $$f(R_{n_{m_k}}) = D_{n_{m_k}} \quad \text{and} \quad f(L) = K$$

by the continuity of f.

By (1) it follows that

(10) $$p \in K \subset F.$$

Hence (compare (4′)) $K \subset E$, and since K is a continuum (compare [4], § 47, II, Theorem 4, p. 170), it follows that

(11) $$K \subset C .$$

According to Theorem 2 in [4], § 47, III, p. 172, we have $D_n \cap \operatorname{Fr}_Q(F) \neq \emptyset$, whence $K \cap \operatorname{Fr}_Q(F) \neq \emptyset$ (where $\operatorname{Fr}_Q(F)$ denotes the boundary of F in Q), and, F being a neighbourhood of p in Q, we get $p \notin \operatorname{Fr}_Q(F)$, which implies $K \neq \{p\}$.

Therefore we see by (9) that L is a non-degenerate continuum. Since $D_{n_{m_k}} \cap K \subset C_{n_{m_k}} \cap C = \emptyset$ (see (3), (6) and (11)), we infer by (9) that $R_{n_{m_k}} \cap L = \emptyset$. Hence L is a non-degenerate continuum of convergence in L, and thus X is not a hereditarily locally connected continuum (see [4], § 50, IV, Theorem 2, p. 269): a contradiction. The proof of (3.1) is complete.

Recall that a point p of a space X is called a *ramification point* (in the classical sense) if it is the common endpoint of three (or more) arcs in X whose only common point is p, i.e., if p is a top of a simple triod contained in X.

Let f be a mapping of X onto Y. We say that the mapping f *covers the ramification points* of Y provided each ramification point of Y is the image under f of a ramification point of X. Similarly, if each ramification point of Y is the image of a point of a given set Q, we say that the set of ramification points of Y is *covered* by Q under f.

We have the following

(3.2) THEOREM. *If a mapping f of a hereditarily locally connected continuum X is locally weakly confluent and f maps X onto Y, then Y is hereditarily locally connected and the set of ramification points of Y is covered by the closure of the set of ramification points of X.*

Proof. By Theorem (3.1) the continuum Y is hereditarily locally connected. Let p be a ramification point of Y and let F be a closed neighbourhood of p in Y such that the partial mapping $f|f^{-1}(F)$ is weakly confluent. Let U be an open subset of Y such that $p \in U \subset F$. It suffices to prove that there exists a ramification point q of X such that $f(q) \in U$. Let T be a simple triod with the top p, such that $T \subset U$. Denote by a, b and c the endpoints of the triod T. Take in the arc $ap \subset T$ a sequence of points $\{a_i\}$ which is convergent to a point $a_0 \neq p$ and such that $a < a_i < a_{i+1} < p$ for each $i = 1, 2, \ldots$ Similarly, take a sequence of points $\{b_i\}$ of the arc bp which is convergent to a point $b_0 \neq p$ and a sequence of points $\{c_i\}$ of the arc cp which is convergent to $c_0 \neq p$. Denote by T_i the subtriod of the triod T which has a_i, b_i and c_i as its endpoints. Since $f|f^{-1}(F)$ is a weakly confluent mapping and $T_i \subset T \subset F$ for each $i = 1, 2, \ldots$, it follows that there exist continua A_i contained in $f^{-1}(T)$ and such that $f(A_i) = T_i$. Since the set $f^{-1}(T)$ is compact, we can assume that the sequence $\{A_i\}$ is convergent (see [4], § 42, I, Theorem 1, p. 45) and $A_0 = \operatorname{Lim}_{i\to\infty} A_i$ is a continuum (see [4], § 47, II, Theorem 4, p. 170). Moreover, $f(A_0) = T_0$ by the continuity of f. Therefore there exists a number $\varepsilon > 0$ such that $\operatorname{diam} A_i > \varepsilon$ for sufficiently large i. The continuum X is hereditarily locally connected, and thus by Theorem 2 in [4], § 50, IV, p. 269, X contains no non-degenerate continuum of convergence and therefore only a finite number of components of the set $f^{-1}(T)$ have diameters larger than ε. Thereby, we can assume that all continua A_i are contained in the same component C of the set $f^{-1}(T)$.

Suppose that the continuum C fails to contain a ramification point of X. Then C is either an arc or a simple closed curve (see [4], § 51, V, pp. 291–299). For each case the set $A_i \setminus A_{i+1}$ has at most two components and the diameters of those components tend to zero as $i \to \infty$. Let ε_i be a positive number less than the diameters of arcs a_0b_0, b_0c_0 and c_0a_0 in T_0. Take a positive number δ such that, if a set K is contained in C and if $\operatorname{diam} K < \delta$, then $\operatorname{diam} f(K) < \varepsilon_1$. Take an index i such that the diameters of components of the set $A_i \setminus A_{i+1}$ are less than δ. Since the set $A_i \setminus A_{i+1}$ has at most two components and $\{a_i, b_i, c_i\} \subset f(A_i) \setminus f(A_{i+1}) = T_i \setminus T_{i+1}$, we have at least two sets from $f^{-1}(a_i)$, $f^{-1}(b_i)$ and $f^{-1}(c_i)$ have non-empty intersections with the same component of the set $A_i \setminus A_{i+1}$. If C' is a component of the set $A_i \setminus A_{i+1}$ and there are points $a_i' \in f^{-1}(a_i)$ and $b_i' \in f^{-1}(b_i)$ such that $\{a_i', b_i'\} \subset C'$, then $a_0b_0 \subset a_ib_i \subset f(C')$, where a_0b_0 and a_ib_i are arcs in T. Therefore $\operatorname{diam} f(C') > \varepsilon_1$: a contradiction. Thus the continuum C contains a ramification point of X. The proof of Theorem (3.2) is complete.

(3.3) COROLLARY. *A locally weakly confluent image of an arc (a circle) is either an arc or a circle.*

It is well known that every mapping of a continuum onto an arc-like continuum is weakly confluent (see [8], Theorem 4). Therefore, in particular, every mapping of an arc onto itself is weakly confluent. A circle does not have this property. Moreover:

(3.4) Example. There exists a locally weakly confluent mapping f of a circle onto itself such that f is not weakly confluent, and there exists a mapping g of a circle onto itself such that g is not locally weakly confluent.

The circle S will be considered as a subset of the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy. The circle S consists of all points (x, y) for which $x^2+y^2 = 1$. We define a mapping f of S onto itself as follows:

$$f\big((x, y)\big) = \Big(\cos\big(\tfrac{3}{2}\pi(x+1)\big), \sin\big(\tfrac{3}{2}\pi(x+1)\big)\Big).$$

Take $V_1 = \{(x, y) \in S\colon x \leqslant 1/2\}$ and $V_2 = \{(x, y) \in S\colon -1/2 \leqslant x\}$. Observe that for $i = 1, 2$ the partial mapping $f|f^{-1}(V_i)$ is weakly confluent, and for each point $y \in T$, we have either $y \in \operatorname{Int} V_1$ or $y \in \operatorname{Int} V_2$; therefore f is locally weakly confluent.

Let $S' = \{(x, y) \in S\colon -1/2 \leqslant y\}$. The set S' is a continuum and the set $f^{-1}(S')$ has exactly two components; namely $C_1 = \{(x, y) \in S\colon x \leqslant -2/9\}$ and $C_2 = \{(x, y) \in S\colon 2/9 \leqslant x\}$. Since $f(C_1) \neq S' \neq f(C_2)$, we conclude f is not weakly confluent.

If we take a mapping g of S onto itself defined by the formula

$$g\big((x, y)\big) = \Big(\cos\big(\pi(x+1)\big), \sin\big(\pi(x+1)\big)\Big),$$

then g is not locally weakly confluent.

The property mentioned above characterizes an arc among all locally connected continua. We have

(3.5) COROLLARY. *A locally connected continuum X is an arc if and only if each mapping from X onto itself is weakly confluent (locally weakly confluent).*

Indeed, any mapping of an arc onto an arc is weakly confluent by Theorem 4 in [8]. Conversely, suppose that X is a locally connected continuum such that each mapping f of X onto itself is locally weakly confluent. There is a mapping of X onto an arc I. Since X is locally connected, there is a mapping g of I onto X (see [4], § 50, II, Theorem 2, p. 256). Thus the mapping gf maps X onto itself and it is locally weakly confluent by assumption. Therefore, by Theorem (2.7), the mapping g is locally weakly confluent, and thus, by Corollary (3.3), X is an arc or a circle. But X is not a circle, because there is a mapping of a circle onto a circle which is not locally weakly confluent (see Example (3.4)), and this contradicts the assumption that every f of X onto itself is locally weakly confluent. Hence X must be an arc.

4. Locally weakly confluent mappings onto graphs. A continuum X is said to be a (*linear*) *graph* if X is the union of a finite number of arcs which are pairwise disjoint except for their endpoints (see [10], p. 182). We say that a continuum X is an *n-star* provided X is the union of n arcs which are pairwise disjoint except for one given point p, which is the common endpoint of these arcs; and p is called the *top* of X. We say that the space X is of *order* $\leqslant m$ at the point p provided for each $\varepsilon > 0$ there is an open set G such that

$$p \in G, \quad \operatorname{diam} G < \varepsilon \quad \text{and} \quad \operatorname{card} \operatorname{Fr}(G) \leqslant m,$$

where $\operatorname{Fr} G$ denotes the boundary of G in X (see [4], § 51, I, p. 274). The minimal cardinal number which satisfies this condition is called the *order* of X at p and is denoted by $\operatorname{ord}_p X$. If X is locally connected and n is a natural number, then $\operatorname{ord}_p X = n$ if and only if there exists an n-star in X with the top p, and X does not contain an $(n+1)$-star with the top p (Menger's theorem, the so-called "n-Beinsatz", e.g. see [4], § 51, I, p. 277). The following condition characterizes graphs: all points of X save a finite number of them are of order 2, and all points are of finite order (see [10], p. 182).

We have the following generalization of Theorem II.5 in [2].

(4.1) THEOREM. *Let f be a light locally weakly confluent mapping from the graph X onto Y. If a point p of Y is of order n, then there exists a point q in X of order m larger than or equal to n such that $f(q) = p$.*

Proof. Let ε be a positive number less than the minimal distance between the ramification points in X and also less than the minimal diameter of a simple closed curve contained in X (such an ε does exist because the graph contains only a finite number of ramification points and a finite number of simple closed curves, by definition). Since f is light, it follows from Lemma (2.8) that there exists a $\delta > 0$ such that if B is any subcontinuum of Y of diameter less than δ, any component of $f^{-1}(B)$ is of diameter less than ε. Let p be a point of Y of order n and let N be an n-star in Y with the top p and such that $\operatorname{diam} N < \delta$. We may assume that the partial mapping $f|f^{-1}(N)$ is weakly confluent, because f is locally weakly confluent (cf. Theorem (2.6)). By the choice of ε, any component of the set $f^{-1}(N)$ contains at most one ramification point and it fails to contain a simple closed curve. Therefore each component of $f^{-1}(N)$ is a point, an arc or an m-star for some $m \geqslant 3$.

Denote by $a_1, \ldots, a_n$ the endpoints of the n-star N. Take, for each $i = 1, \ldots, n$, a sequence of points $\{a_{ij}\}$ in the arc $a_i p$ in N, which is convergent to a point a_{i0} different from p and such that $a_i < a_{ij} < a_{i(j+1)} < p$ for $j = 1, 2, \ldots$ Denote by N_j the n-star contained in N with endpoints $a_{1j}, a_{2j}, \ldots, a_{nj}$. Since $f|f^{-1}(N)$ is weakly confluent and since $N_j \subset N$ for each $j = 1, 2, \ldots$, it follows that there exist continua A_j which are contained in $f^{-1}(N)$ and such that $f(A_j) = N_j$ for $j = 1, 2, \ldots$

Since the set $f^{-1}(N)$ is compact, we may assume that the sequence of continua $\{A_j\}$ is convergent (see [4], § 42, I, Theorem 1, p. 45) and $A_0 = \operatorname{Lim}_{j\to\infty} A_j$ is a continuum (see [4], § 47, II, Theorem 4, p. 170).

Moreover, $f(A_0) = N_0$ by the continuity of f. Therefore there exists a positive number ε_1 such that $\operatorname{diam} A_i > \varepsilon_1$ for sufficiently large i. The continuum X is hereditarily locally connected, and thus by Theorem 2 in [4], § 50, IV, p. 269, the continuum X does not contain a non-degenerate continuum of convergence; thus only a finite number of components of the set $f^{-1}(N)$ have diameters larger than ε_1. Hence we may assume that all continua A_i are contained in the same component C of the set $f^{-1}(N)$. The continuum C is either an arc or an m-star for $m \geqslant 3$. We will show that $m \geqslant n$.

Suppose, on the contrary, that $m < n$. The set $A_j \setminus A_{j+1}$ has at most m components of diameters tending to zero as $j \to \infty$. Let α be a positive number less than the diameters of arcs $a_{r0} a_{s0}$ for $r \neq s$ in N_0. Take a positive number β such that, if the set K is contained in C and $\operatorname{diam} K < \beta$, then $\operatorname{diam} f(K) < \alpha$. Take an index j such that the diameters of components of the set $A_j \setminus A_{j+1}$ are less than β.

Since the set $A_j \setminus A_{j+1}$ has at most m components and $f(A_j) \setminus f(A_{j+1}) = N_j \setminus N_{j+1}$, we infer that at least two sets from $f^{-1}(a_{ij})$ for $i = 1, \ldots, n$ have a non-empty intersection with the same component C' of the set $A_j \setminus A_{j+1}$. Then $\operatorname{diam} f(C') > \alpha$, which is impossible. Therefore $m \geqslant n$. Since ε may be chosen arbitrarily small, we conclude that the top q of C is such that $f(q) = p$. The proof of Theorem (4.1) is complete.

The following proposition is well known, but the author has been unable to find a reference for this result. In any case it is not difficult to prove, and the proof presented here is for completeness only.

(4.2) PROPOSITION. *A monotone image of a graph is a graph.*

Proof. Let X be a graph and let f be a monotone mapping of X onto Y. By definition, X is the union of a finite number of arcs, say $a_1a_2, a_3a_4, \ldots, a_{n-1}a_n$, which are pairwise disjoint except for their endpoints. If $y \in Y$ and $y \neq f(a_i)$ for $i = 1, \ldots, n$, then there is an open connected set V containing y and such that $\overline{V}$ does not contain any points $f(a_i)$ for $i = 1, \ldots, n$ by the local connectedness of Y. It follows from the monotoneity of f that $f^{-1}(\overline{V})$ is a continuum. Moreover, $f^{-1}(\overline{V})$ does not contain the points a_i for $i = 1, \ldots, n$, and thus $f^{-1}(\overline{V})$ is an arc contained in some arc $a_{i_0}a_{i_0+1}$. Since $f|f^{-1}(\overline{V})$ is monotone, $\overline{V} = ff^{-1}(\overline{V})$ is an arc as a monotone image of an arc (see [10], (1.1), p. 165). We infer $\mathrm{ord}_y Y \leqslant 2$. This implies that all points of Y save points $f(a_i)$ for $i = 1, \ldots, n$ are of order less than or equal to 2. Further, observe that if $m = 2n$, then $\mathrm{ord}_{f(a_i)} Y$ is less than m for $i = 1, \ldots, n$. Indeed, suppose, on the contrary, that $\mathrm{ord}_{f(a_{i_0})} Y \geqslant m+1$. Then, by Menger's theorem (e.g. see [4], § 51, I, p. 277), there is an $(m+1)$-star N in Y with the top at the point $f(a_{i_0})$. We may assume that N does not contain any points $f(a_j)$ which are different from $f(a_{i_0})$. Let $b_1f(a_{i_0}), \ldots, b_{m+1}f(a_{i_0})$ be arcs which compose N. Any set $f^{-1}(b_if(a_{i_0}) \setminus f(a_{i_0}))$ is connected by the monotoneity of f, and fails to contain any of the points a_i for $i = 1, \ldots, n$. Therefore it is contained in some arc $a_{j_i}a_{j_i+1}$. On the other hand, any arc a_ja_{j+1} contains at most two sets of the form $f^{-1}(b_if(a_{i_0}) \setminus f(a_{i_0}))$, by the monotoneity of f. Thus $m+1 \leqslant 2n$: a contradiction. This implies that all points of Y are of finite order and all points of y save $f(a_i)$ for $i = 1, \ldots, n$ are of order less than or equal to 2; thus Y is a graph (cf. [10], p. 182).

(4.3) THEOREM. *A locally weakly confluent image of a graph is a graph.*

Proof. Let a locally weakly confluent mapping f map the graph X onto Y. It follows from Theorem (2.9) that there exists a factorization of f into two mappings f_1 and f_2, i.e., $f(x) = f_2(f_1(x))$ for each $x \in X$, such that f_1 is monotone and f_2 is light locally weakly confluent. Put $X' = f_1(X)$. The continuum X' is a graph by Proposition (4.2). Let n be the largest order of ramification points of X'.

Suppose, on the contrary, that the continuum Y is not a graph. Since X' is a graph, it contains a finite number of ramification points. Therefore Y contains a finite number of ramification points by Theorem (3.2). Since, by assumption, Y is not a graph, Y must contain a ramification point p of order larger than n. Then there is an $(n+1)$-star in Y by Menger's theorem mentioned above. Since the mapping f is light locally weakly confluent, we conclude by Theorem (4.1) that there is an $(n+1)$-star in X', contrary to the choice of n. The proof of (4.3) is complete.

We will now characterize graphs which are images under locally weakly confluent mappings of an arbitrary graph. This is a partial answer to a question asked in [2] for weakly confluent mappings. To this end we firstly introduce the notions of free arcs and free simple closed curves. These notions play only an auxiliary role.

Let X be a graph and let $p_1, \ldots, p_n$ be all its ramification points and endpoints. An arc p_ip_j in X such that $p_ip_j \cap \{p_1, \ldots, p_n\} = \{p_i, p_j\}$ is called a *free arc with respect to* X. Similarly, a *simple closed curve* S contained in X is called *free with respect to* X provided $S \cap \{p_1, \ldots, p_n\}$ is a one-point set. Therefore, we have

(4.4) If the graph X is neither an arc nor a simple closed curve, then any arc A of X containing only one ramification point of X which is also an endpoint of A is contained either in some free arc or in some free simple closed curve with respect to X.

It is easy to prove the following

(4.5) LEMMA. *Let $p_1, \ldots, p_n$ be different points of a graph X and let $t_1, \ldots, t_n$ be different points of the straight line interval $I = [0, 1]$ such that $0 < t_1 < \ldots < t_n < 1$. There is a light mapping h from X onto I such that $h(p_i) = t_i$ and there is a light mapping g from I into X such that $g(t_i) = p_i$ for each $i = 1, \ldots, n$.*

We have

(4.6) LEMMA. *For each graph X' contained in a graph X there is a light mapping f from X onto X' such that $f(x) = x$ for each $x \in X'$ (i.e., f is a retraction of X onto X').*

Proof. Let X' be an arbitrary graph contained in the graph X, and let $C_1, \ldots, C_k$ denote components of the set $X \setminus X'$. Any intersection $\overline{C}_i \cap X'$ is a finite set for $i = 1, \ldots, k$. Put $\overline{C}_i \cap X' = \{p_1, \ldots, p_n\}$ and choose reals $t_1, \ldots, t_n$ such that $0 < t_1 < \ldots < t_n < 1$. It follows from Lemma (4.5) that there are light mappings h_i and g_i such that h_i maps $\overline{C}_1$ onto I and $h_i(p_j) = t_j$ for $j = 1, \ldots, n$; g_i maps I into X' and $g_i(t_j) = p_j$ for $j = 1, \ldots, n$. We define $f|C_i = g_ih_i$ for $i = 1, \ldots, k$ and $f(x) = x$ for $x \in X'$. It is easy to verify that f is continuous and satisfies the required conditions.

(4.7) THEOREM. *Let X and Y be graphs and let $p_1, \ldots, p_n$ be the ramification points of Y of orders $k_1, \ldots, k_n$, respectively. If X has ramification points $q_1, \ldots, q_n$ of orders $m_1, \ldots, m_n$, respectively, and $m_i \geqslant k_i$ for $i = 1, \ldots, n$, then there is a light locally weakly confluent mapping f from X onto Y such that $f(q_i) = p_i$ for $i = 1, \ldots, n$.*

Proof. By the assumption, the graph X contains a k_i-star M_i with the top q_i, and the graph Y contains a k_i-star N_i with the top p_i for each $i = 1, \ldots, n$. Obviously, we may assume that $M_i \cap M_j = \emptyset$ and $N_i \cap N_j = \emptyset$ for $i \neq j$ and $i, j = 1, \ldots, n$. Put $M = \bigcup_{i=1}^{n} M_i$ and $N = \bigcup_{i=1}^{n} N_i$. Obviously there is a homeomorphism h from M onto N.

Since any arc A which is free with respect to N_i has a ramification point p_i of Y as its endpoint, and since A does not contain any ramification point p_j for $j \neq i$, we have by (4.4)

(12) any arc A free with respect to N_i for $i = 1, \ldots, n$ is contained in some arc (or in a simple closed curve) which is free with respect to Y.

Moreover, since an arc (or a simple closed curve) A that is free with respect to Y must contain some ramification point p_{i_0} of Y and since N_{i_0} contains all arcs of Y with one endpoint p_{i_0} and of sufficiently small diameters, we infer that

(13) any arc (simple closed curve) free with respect to Y contains at least one arc free with respect to N_i, for some $i = 1, \ldots, n$.

We define a mapping g from N onto Y as follows. If L is a free arc with respect to N_i and is contained in the arc K which is free with respect to Y, then $g|L$ is a homeomorphism from L onto K such that $g(p_i) = p_i$. If L is a free arc with respect to N_i and L is contained in the simple closed curve K which is free with respect to Y, then, denoting by x the endpoint of L different from p_i, we put $g(p_i) = g(x) = p_i$ and we define $g|L\setminus\{p_i, x\}$ as a homeomorphism from $L\setminus\{p_i, x\}$ onto $K\setminus\{p_i\}$ such that there is a point $x' \in L\setminus\{p_i, x\}$ with the property that the image of the arc $p_ix' \subset L$ under g contains L.

It follows from the construction that

(14) g is a light locally weakly confluent mapping from N onto Y.

Let $a_1b_1, \ldots, a_rb_r$ be a finite family of arcs of X such that $a_ib_i \cap M$ is at most a one-point set for each $i = 1, \ldots, r$ and for fixed $j = 1, \ldots, n$ and the set $X' = M \cup \bigcup_{i=1}^{r} a_ib_i$ is a continuum in X. We define a mapping g_1 from X' onto Y as follows: $g_1|M = gh$, and $g_1|a_ib_i$ is a homeomorphism of the arc a_ib_i onto an arbitrary arc which has $g(h(a_i))$ and $g(h(b_i))$ as its endpoints and $g_1(a_i) = g(h(a_i))$ and $g_1(b_i) = g(h(b_i))$ for each $i = 1, \ldots, r$.

It follows from (14) and from the construction of g_1 that

(15) g_1 is a light locally weakly confluent mapping from X' onto Y.

By Lemma (4.6) there is a light mapping f_1 from X onto X' such that

(16) $$f_1(x) = x \quad \text{for each } x \in X'$$

Put $f(x) = g_1(f_1(x))$ for each $x \in X$. According to (15) and (16) we see that f is a light locally weakly confluent mapping from X onto Y. The proof of Theorem (4.7) is complete.

Theorems (4.1) and (4.7) imply the following

(4.8) Corollary. *Let X and Y be graphs and let $p_1, \ldots, p_n$ be the ramification points of Y of orders $k_1, \ldots, k_n$, respectively. There is a light locally weakly confluent mapping from X onto Y if and only if X has ramification points $q_1, \ldots, q_n$ of orders, $m_1, \ldots, m_n$, respectively and $m_i \geqslant k_i$ for $i = 1, \ldots, n$.*

A graph which does not contain a simple closed curve is called a *finite dendrite*. It follows from Theorem (4.7) that

(4.9) Corollary. *For each graph X there is a finite dendrite Y such that there is a light locally weakly confluent mapping from X onto Y and there is a light locally weakly confluent mapping from Y onto X.*

(4.10) Remarks. If we know the number of ramification points of an arbitrary graph X and the orders of those points, then by Corollary (4.8) we can find the number lwc(X) of non-homeomorphic images of X under light locally weakly confluent mappings. For example, if X is an n-star for some $n \geqslant 3$, then

$$\mathrm{lwc}(X) = (n-1) + \sum_{k=2}^{n} r_k,$$

where

$$r_k = \begin{cases} k/2 & \text{if } k \text{ is an even number,} \\ (k-1)/2 & \text{if } k \text{ is an odd number.} \end{cases}$$

Thus if X is a simple triod, then lwc(X) is equal to 4; namely, a non-homeomorphic images of a simple triod under light locally weakly confluent mappings are: an arc, a simple closed curve, a simple triod and a union of an arc and a simple closed curve which are disjoint except for one point, which is an endpoint of that arc.

The proof of Theorem (3.1) partially coincides with the proof of Theorem 1 in [4], § 49, VI, p. 246; and the proofs of Theorems (3.2) and (4.1) are almost the same as the proofs of Theorems II.1 and II.5 in [2], but we obtain more general results.

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INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

Accepté par la Rédaction le 24. 1. 1974

A theory of absolute proper retracts

by

R. B. Sher* (Greensboro, N. C.)

Abstract. We construct a theory of absolute *proper* retracts (APR's) for locally compact metric spaces analogous to the usual theory, only requiring that all maps be proper. The APR's are shown to be the non-compact ANR's having property SUV$^\infty$. We obtain the standard extension theorems and a result characterizing the APR's by a property of their Freudenthal compactification.

1. Introduction. In this paper it is our aim to lay the foundation for a study of absolute *proper* retracts and absolute neighborhood *proper* retracts. The basic idea is to modify the definition of absolute retract and absolute neighborhood retract by requiring that all maps be proper.

Rather than concentrating at this time on the general properties of absolute proper retracts and absolute neighborhood proper retracts, we shall limit ourselves to the basic definitions and facts, and to the problem of identifying the absolute proper retracts and absolute neighborhood proper retracts among the ANR's. For absolute neighborhood proper retracts, the result is essentially trivial (and well-known). However, we include it here for completeness. It is that, for the class of spaces under consideration, X is an absolute neighborhood proper retract if and only if $X \in$ ANR. However, for absolute proper retracts, the situation is more complicated, and we show that X is an absolute proper retract if and only if X is non-compact, $X \in$ ANR, and X has a certain geometric property called *property* SUV$^\infty$. As a tool, we obtain a result about the Freudenthal compactification of ANR's having property SUV$^\infty$ which is of interest in its own right.

2. Absolute proper retracts and absolute neighborhood proper retracts. A map $f\colon X \to Y$ is said to be *proper* if $f^{-1}(C)$ is compact for each compact set $C \subset Y$. Proper maps seem to make good geometric sense as a vehicle for the study of locally compact metric spaces (e.g., see the results of [2]), and throughout this paper we shall restrict our attention to this class of

* The author gratefully acknowledges the support of the National Science Foundation.