THEOREM DL. *Let P be a $\vee$-semilattice with a partial $\wedge$ taking g.l.b. as values such that the existence of $\alpha\wedge\beta$ implies $(\alpha\vee\gamma)\wedge(\beta\vee\gamma)=(\alpha\wedge\beta)\vee\gamma$ for every γ. Then P is embedded in the distributive lattice F it freely generates, and every congruence (for both the total and partial operation) of P is the restriction of a lattice congruence on F.*

This furnishes in particular a solution for the distributive case of Problem 20 in [G′].

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Topologically nondegenerate functions

by

Marston Morse (Princeton, N. J.)

Abstract. Let M_n be a compact, connected topological manifold and F a continuous real valued function on M_n that is topologically nondegenerate in the sense of Morse [12]. Let c be an arbitrary value of F and set

$$F_c=\{p\in M_n|\ F(p)\leqslant c\}.$$

The "topological critical points" of F on F_c are finite in number and can be related to the invariants of the homology groups of F_c as in the differentiable case (Morse and Cairns [14]). F-deformations and F-tractions make this possible. F-tractions are our extensions of retracting deformations of Borsuk [1]. Kirby and Siebenmann in [7] have affirmed the existence of topologically nondegenerate functions on M_n when $n\neq 4$ or 5. For the differentiable case see [15], Milnor [9] and Cerf [3]. Paper [16] reorganizes the classical group structure of the singular homology theory of Eilenberg [5] for use in this paper.

Introduction. This paper is concerned with *continuous*, real-valued, *topologically* nondegenerate functions F, as distinguished from *differentiably* nondegenerate functions. (See § 1 for definitions.) The domain of F is taken as a compact topological manifold M_n. The paper [14] of Morse and Cairns is here extended from the differentiable case to the topological case. A brief abstract of this paper is found in [13].

Singular homology theory is used of the type first introduced by Eilenberg in 1944. See reference [6]. No "triangulations" are needed. Deformations termed "tractions", are fundamental; they relax the conditions on "retracting deformations" as commonly defined. For original concepts see Borsuk [10]. The theorem of Kirby and Siebenmann on the existence of topologically nondegenerate functions, when $n\neq 4$ or 5, is a starting point. This paper draws heavily on Morse [12] in which topologically nondegenerate functions were first defined. Paper [16] reorganizes the classical group structure for use in the necessary homology theory.

To avoid complexity in a first treatment this study has been subjected to many restrictions that can be readily removed. In particular, one could greatly lighten the condition that the manifold be compact. One could also remove the condition that the topological critical values be of *singleton* type in the sense of § 0.

For a general approach to the *differential* case the reader is referred to the treatises by J. Cerf [3] and J. Milnor [9]. Fundamental applications have been announced by S. Smale.

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§ 0. Program. This paper is concerned with a compact, connected, topological manifold M_n on which there exists a continuous real-valued function F which is TND (¹) in the sense of § 1 of [12]. T-critical points and T-critical values are defined in [12] and in § 1 of this paper. As [12] shows, there exists a *finite* set of T-critical points on M_n. Given a value $c \in R$ we shall set

(0.1) $$F^c = \{p \in M_n | \ F(p) = c\},$$

(0.2) $$F_c = \{p \in M_n | \ F(p) \leqslant c\}$$

and term F^c and F_c c-*level* subsets and c-*sublevel* subsets, respectively, of M_n. If c is not a value of F these sets may be empty.

Given a TND function F on M_n, we shall show in § 1 that F can be infinitesimally modified to yield a TND function F on M_n such that there is just one T-critical point p_a at each T-critical level a. Once this is proved in § 1, we shall thereafter assume that T-critical values a of F have this property. Such T-critical values a of F and the corresponding T-critical points p_a will be called *singleton* T-critical values a and points p_a, respectively.

The major differences between this paper and [14] arise from the fact that "retracting deformations", so essential in [14], were defined with the aid of orthogonal trajectories of the level manifolds f^c of the ND function f given in [14]. The level manifolds F^c of this paper, in general, possess no such orthogonal trajectories.

Use will be made, as in [14], of the singular homology theory of Eilenberg [5] over the ring Z of rational integers.

Part I. F-tractions. This paper is in three parts. Part I is largely an adaptation for the needs of this paper of the "f-deformations" introduced in [12]. We shall replace the "retracting deformations" of [14] by special F-deformations, termed F-*tractions*. F-tractions, as we shall see, have the fundamental property, that an F-traction of a subset χ of M_n into a subset $\chi' \subset \chi$ of M_n, implies an isomorphism

(0.3) $$H_q(\chi, Z) \approx H_q(\chi', Z) \qquad (q = 0, 1, \ldots)$$

of the qth homology groups over Z of χ and χ'.

(¹) T abbreviates topological; ND abbreviates nondegenerate.

Part II. The (¹) FG **of the homology groups $H_q(F_c, Z)$.** Part II establishes the basic theorem, that if c is an arbitrary value of F, the homology groups $H_q(F_c, Z)$ are FG. The singleton critical values of F are listed as a sequence

(0.4) $$a_0 < a_1 < a_2 < \ldots < a_\nu .$$

It is found that as c increases from a_0 to a_ν, the groups $H_q(F_c, Z)$ remain isomorphic, except at most when c increases from a value in an interval $[a_{r-1}, a_r)$ to a_r. In Part II we show that as c increases, $H_q(F_c, Z)$ remains FG without exception. One shows, by an inductive proof, that each of the groups $H_q(F_{a_r}, Z)$ is FG.

Part III. Critical invariants of T-critical points p_a of F. Part III is concerned with the changes in the integral invariants (²) of the groups $H_q(F_c, Z)$ as c changes from a_0 to a_ν. We shall associate a finite set of integers with each singleton critical value a in the list (0,4) and term these integers *critical invariants* of p_a and F_a. Foremost among these critical invariants of p_a are its T-index k_a, its "free index" $s^a \geqslant 0$, and its positive "torsion (³) index" t^a, defined only when $s^a = 0$ and $a > a_0$. Free indices and torsion indices are defined in [16]. Paper [16] gives the group theoretic background of this paper, while [12] gives its more geometric background.

Given an arbitrary value c of F, the theorems of Part III show that the integral invariants of the groups $H_q(F_c, Z)$ are determined by the critical invariants of the largest of the T-critical values $a = a_r \leqslant c$ in the list (0.4) and the integral invariants of $H_q(\dot{F}_a, Z)$ where $\dot{F}_a = F_a - p_a$.

The invariance of critical invariants. Let h be a homeomorphism $p \rightarrow h(p)$ of the manifold M_n onto a manifold $\hat{M}_n$. Let $\hat{F}$ be a continuous mapping of $\hat{M}_n$ into R such that $F(p) \equiv \hat{F}(h(p))$ for $p \in M_n$. Our definitions will show the following. $\hat{F}$ is TND if and only if F is TND. If p_a is a T-critical or T-ordinary point of F, then $h(p_a)$ will be, respectively, a T-critical or T-ordinary point of $\hat{F}$. Each critical invariant N of p_a will be a critical invariant of $h(p_a)$.

The existence of TND functions on M_n. If M_n is a compact differentiable manifold of class at least C^2, the existence of a TND function on M_n was made clear by Morse in 1927. See [10] and § 6 of [15]. Eells and Kuiper in [4] have gone beyond the differentiable case and established the existence of TND functions on combinatorial manifolds. More generally Kirby and Siebenmann have affirmed the existence of TND functions

(¹) FG abbreviates "finite generation" or "finitely generated."

(²) By the integral invariants of a homology group are meant its Betti number and torsion coefficients.

(³) The torsion index t^a of a T-critical point p_a is not to be confused with a torsion coefficient of $H_q(F_a, Z)$ or $H_q(\dot{F}_a, Z)$.

on each topological manifold M_n for which n is neither 4 nor 5. This affirmation is found in a deep study [7] (1) of the classification theorems of metrizable topological manifolds of finite dimension. TND functions on a compact topological manifold were first defined and studied in [12]. We here continue this study, seeking to make clear the implications of the existence of TND functions in homotopy and homology theory. The general concepts and theorems of Cerf [3] are part of this evolving theory.

Noncompact manifolds. The restriction of this paper to compact manifolds is for simplicity only. One could obtain similar results on a connected, noncompact manifold provided with a TND function F such that for each value a of F, F_a is compact. Cf. Theorem 23.5 of [15] for the differentiable case. In this more general case there may be a countably infinite set of T-critical points.

Part I. F-tractions on M_n

§ 1. TND-functions F on M_n. We shall recall the definition of a TND function F on M_n, as given in [12].

Let $z_1, \ldots, z_n$ be coordinates of a point $z \in R^n$. Let Z_ϱ be an origin-centered closed n-ball in R^n of radius ϱ, and $\mathring{Z}_\varrho$ its interior. Let $\mathbf{0}$ be the null n-tuple $(0, \ldots, 0)$.

Definition 1.1. A point $q \in M_n$ will be called a *T-ordinary point* of F, if there exists an injective homeomorphism

$$z \to \Psi_q(z)\colon\ \mathring{Z}_1 \to M_n \tag{1.1}$$

such that $\Psi_q(\mathbf{0}) = q$ and for some sufficiently small scalar $\varrho_q > 0$

$$F(\Psi_q(z)) \equiv F(q) + z_n \varrho_q \qquad (z \in \mathring{Z}_1) \tag{1.2}$$

where z_n is the nth of the coordinates $z_1, \ldots, z_n$ of z.

The condition (1.2) implies that the level sets of F in the neighborhood $\Psi_q(\mathring{Z}_1)$ of q are images under Ψ_q of open sets on a set of parallel $(n-1)$-planes.

Definition 1.2. *T-critical points of F.* A point σ of M_n which is not a T-ordinary point of F will be called a *T-critical point* of F.

Let k be an integer on the range $0, 1, \ldots, n$. Set

$$-z_1^2 - \ldots - z_k^2 + z_{k+1}^2 + \ldots + z_n^2 = Q_k(z) \qquad (z \in R^n)\,. \tag{1.3}$$

A T-critical point σ of F will be said to have a *T-index* k if there exists a homeomorphism (termed canonical) of form

$$z \to \Phi_\sigma(z)\colon\ \mathring{Z}_2 \to M_n \tag{1.4}$$

(1) Our reference to Kirby and Siebenmann is to one of a series of papers by these authors on topological manifolds. It was obtained by writing to Kirby and Siebenmann.

into M_n, such that $\Phi_\sigma(\mathbf{0}) = \sigma$ and for fixed σ and some sufficiently small scalar ϱ_σ

$$F(\Phi_\sigma(z)) - F(\sigma) \equiv Q_k(z)\varrho_\sigma \qquad (z \in \mathring{Z}_2)\,. \tag{1.5}$$

We set

$$\Phi_\sigma(\mathring{Z}_1) = N_\sigma\,, \qquad \Phi_\sigma(\mathring{Z}_2) = N_\sigma^* \tag{1.6}$$

as (1) in [12] and term N_σ and N_σ^* *inner* and *outer canonical* neighborhoods of σ on M_n.

When $k = 0$, (1.5) shows that $F(p) \geqslant F(\sigma)$ for $p \in N_\sigma^*$. When $k = n$, $F(p) \leqslant F(\sigma)$ for $p \in N_\sigma^*$.

Definition 1.3. TND *functions F.* The function F will be *termed* TND, if corresponding to each T-critical point σ of F, there exists a canonical homeomorphism Φ_σ of $\mathring{Z}_2$ into M_n, conditioning F as in (1.5) and defining a T-index k of σ.

The condition (1.5) requires that the level sets of F in the neighborhood $\Phi_\sigma(\mathring{Z}_2)$ of σ be the T-images of level sets of the quadratic form Q_k in a neighborhood of the origin in R^n.

We assume that F is TND.

The set of T-ordinary points of F in M_n is clearly open. The set of T-critical points of F is accordingly closed in M_n and hence compact, since M_n is compact. If σ is a T-critical point of F, one shows readily that each point in N_σ^* other than σ, is T-ordinary. As isolated points in the compact set of T-critical points of F, T-critical points of F are finite in number.

A metric on M_n. As a compact topological manifold, M_n admits a metric which induces the topology with which M_n is given.

T-critical values. If σ is a T-critical point of F, $F(\sigma)$ will be called a *T-critical value* of F. A value of F which is not T-critical will be called *T-ordinary*. If c is a T-ordinary value of F, the level manifold F^c is a compact $(n-1)$-manifold without boundary. If c is a T-critical value of F, the deletion of T-critical points of F on F^c from F^c will yield an open $(n-1)$-manifold on M_n.

Singleton T-critical points and values of F have been defined in § 0. A singleton critical value a will be assigned a T-index $k = k_a$ equal to the T-index of the unique T-critical point p_a for which $F(p_a) = a$. We shall prove the following.

Theorem 1.1. *If there exists a* TND *function F on M_n there exists a* TND *function $\hat{F}$ on M_n each of whose T-critical values is a singleton critical value.*

(1) Conditions (1.2) and (1.4) in reference [12], multiplied on the right by scalars ϱ_q and ϱ_σ, respectively, should hold for some choice of these scalars as positive numbers.

Moreover, if F is given, $\hat{F}$ can be defined so as to have the same T-critical points as F with the same T-indices and with singleton critical values which differ arbitrarily little from the T-critical values which they replace. The following lemma implies this. In this lemma, as elsewhere, CX will denote the complement in M_n of a subset X of M_n.

LEMMA 1.1. *Let F be a* TND*-function on M_n and σ a T-critical point of F for which $F(\sigma)$ is not a singleton T-critical value of F. Let ε be a prescribed positive constant.*

It is possible to redefine F in a closed subneighborhood H of the neighborhood N_σ^* of σ, leaving $F(p)$ unchanged for $p \in CH$, so as to replace F by a TND*-function G on M_n such that*

(i) *σ is the only T-critical point of G in H.*

(ii) *The T-index k of σ, relative to F, is the T-index of σ relative to G.*

(iii) *$G(\sigma)$ is a singleton critical value of G such that*

$$0 < |G(\sigma) - F(\sigma)| < \varepsilon . \tag{1.7}$$

Proof. One readily defines a C^∞-mapping

$$z \to \lambda(z)\colon\ R^n \to R \tag{1.8}$$

such that $\lambda(z) \equiv 1$ for $\|z\| \leqslant 1$ and $\lambda(z) \equiv 0$ for $\|z\| \geqslant \frac{3}{2}$. If $e > 0$ is sufficiently small the mapping $z \to \mu(z)$ with values

$$\mu(z) \equiv -z_1^2 - \ldots - z_k^2 + z_{k+1}^2 + \ldots + z_n^2 + e\lambda(z) \qquad (z \in R^n) \tag{1.9}$$

has no critical point other than $z = 0$.

DEFINITION OF G. We refer to the homeomorphism Φ_σ of Z_2 onto N_σ^* introduced in (1.4) and set

$$H = \Phi_\sigma(Z_\varrho) \qquad (\varrho = \tfrac{3}{2}) . \tag{1.10}$$

Then H is a compact subset of N_σ^* and CH is *open* in M_n. Set

$$G(p) = F(p) \qquad (p \in CH) , \tag{1.11}$$

$$G(p) = F(\sigma) + \mu(z)\varrho_\sigma \qquad (p \in N_\sigma^*) \tag{1.12}$$

subject to the condition, $p = \Phi_\sigma(z)$ for $\|z\| < 2$. The sets CH and N_σ^* are open subsets of M_n whose union is M_n and whose intersection is $N_\sigma^* - H$. Both (1.11) and (1.12) define $G(p)$ on this intersection, but consistently [1].

From (1.12) and (1.5) we infer that

$$G(p) \equiv F(p) + e\varrho_\sigma \qquad (p \in N_\sigma) , \tag{1.13}$$

taking account of the fact that $\lambda(z) \equiv 1$ in (1.9) when $\|z\| \leqslant 1$.

We continue with a proof of the following.

[1] Since $\lambda(z) \equiv 0$ in (1.9), when $\|z\| \geqslant \frac{3}{2}$.

(α) *The mapping G is* TND *on M_n.*

To verify (α) set $N_\sigma^* - \sigma = \dot{N}_\sigma^*$. Note that the subsets

$$N_\sigma,\ CH,\ \dot{N}_\sigma^* \tag{1.14}$$

of M_n are open and have the union M_n. We shall show that the restrictions of G to each of these sets is TND. This term is defined on *open* subsets of M_n as on M_n. That $G|N_\sigma$ is TND follows from (1.13). That $G|CH$ is TND follows from (1.11). That each point of $\dot{N}_\sigma^*$ is T-ordinary relative to G follows from (1.12), subject to the condition $p = \Phi_\sigma(z)$, and from the fact that the mapping $z \to \mu(z)$ of R^n into R has no critical point other than $z = \mathbf{0}$.

Thus (α) is true.

Statement (i) now follows from (1.12), while (ii) follows from (1.13). Statement (iii) will be true, in accord with (1.13), if e is sufficiently small.

This completes the proof of Lemma 1.1. Theorem 1.1 follows.

Singleton notation. It is assumed that F is a TND-function on M_n whose T-critical values a are singleton values, and are listed in (0.4). With each such value a there is associated the unique T-critical point p_a at the F-level a. The T-index of p_a will be denoted by $k = k_a$ and will be termed the *T-index* of a, as well as the T-index of p_a. If $\sigma = p_a$, the canonical homeomorphism Φ_σ associated with σ as in (1.5), when $k = k_a$ will be denoted by Φ_σ^a. The canonical neighborhoods N_σ and N_σ^* of σ introduced in (1.6), will be denoted by N_σ^a and N_σ^{*a}, respectively.

§ 2. F-deformations.

F-deformations will be defined after we have defined deformations of a more general character.

Let t be a real variable termed the *time*. Let $I = [0, 1]$ denote an interval for t. With us a deformation of a subset A of a topological space χ is a continuous mapping

$$(p, t) \to D(p, t)\colon\ A \times I \to \chi \tag{2.0}$$

such that $D(p, 0) \equiv p$ for each $p \in A$. We shall denote $D(p, t)$ by p_t. Given $p \in A$ we say that p_t *replaces* p under D *at the time* t. We set $D(p, 1) = D_1(p)$ and term $D_1(p)$ the *final* image of p under D.

By the carrier $|D|$ of D is meant the image of $A \times I$ under D. We say that D *deforms* A on a subset X of χ if $|D| \subset X$.

Retracting deformations. A deformation D of A is said to be a *deformation retracting* A onto $D_1(A)$, if D deforms A on A and leaves each point of $D_1(A)$ fixed.

In case χ is a differentiable manifold M_n, retracting deformations which are adequate for our purposes are relatively easy to define. When M_n is no longer differentiable, retracting deformations of the desired types may fail to exist. There is, however, a larger class of deformations which serve our purposes and will now be defined.

Definition 2.1. *Tractions.* A deformation D of a subset A of χ will be termed a *traction* of A *into* a subset B of A (possibly A) if D deforms A on A into B and deforms B on B.

Each "deformation retracting A onto B" is a "traction of A into B" but a traction of A into B is not, in general, a deformation retracting A onto B. However a "traction of A into B" shares with a "deformation retracting A onto B" a fundamental property: there exist isomorphisms (1)

$$H_q(A, Z) \approx H_q(B, Z) \quad (q = 0, 1, \ldots, n)\,. \tag{2.1}$$

See Theorem 3.1.

The following definition is given in [12], p. 192. A related definition is given in [11], p. 30.

Definition 2.2. *An F-deformation of A on M_n.* A deformation of A on M_n which replaces each point $p \in A$ by a point p_t at the time t is called an *F-deformation*, if $F(p) \geqslant F(p_t)$ for each $t \in [0, 1]$ and $p \in A$, and is called a *proper* F-deformation of A, if in addition

$$F(p_t) < F(p) \quad (\text{whenever } p_t \neq p)\,. \tag{2.2}$$

An F-deformation which is a traction is called an *F-traction*.

Recall that a T-critical value a of F is, by hypothesis, a "singleton value", assumed at just one T-critical point of F denoted by p_a. See singleton notation at end of § 1.

Definition 2.3. *The T-(k-disc), K^a.* For each integer k such that $0 \leqslant k \leqslant n$, we introduce the open (2) k-disc,

$$\omega_k = \{z \in R^n \mid z_1^2 + \ldots + z_k^2 < 4;\ z_{k+1} = \ldots = z_n = 0\} \tag{2.3}$$

in R^n. Note that $\omega_0 = \mathbf{0}$ and that ω_n is an origin-centered n-ball in R^n on which $\|z\| < 2$. Let a be a T-critical value and $\sigma = p_a$, the T-critical point at the F-level a. Let Φ_σ^a be the canonical homeomorphism Φ_σ of $\mathring{Z}_2$ onto N_σ^*, of Definition (1.6) when $\sigma = p_a$ and $k = k_a$, and denote this set N_σ^* by N_σ^{*a}. Using the same homeomorphism Φ_σ^a, we introduce a T-(k-disc), the subset

$$K^a = \Phi_\sigma^a(\omega_k) \quad (k = k_a) \tag{2.4}$$

of N_σ^{*a}. It is clear that N_σ^{*a} and its subset K^a satisfy the relation

$$K^a \subset N_\sigma^{*a} \cap F_a\,. \tag{2.5}$$

(1) With us an "isomorphism" is understood, a priori, to be surjective unless the contrary is noted.

(2) That a 0-disc $\mathbf{0}$ is open is a convention.

We say that K^a and N_σ^{*a} are *related* by Φ_σ^a. Note that K^a is a T-(k-disc) included in N_σ^{*a}, meeting F^a in just one point σ.

The following lemma is basic. Its formulation differs trivially from the formulation of Lemma (1) 2.1 of [12]. It is proved in [12].

Lemma 2.1. *Let $\sigma = p_a$ be an arbitrary T-critical point of F with T-index $k = k_a$. Let N_σ^a, N_σ^{*a} be canonical neighborhoods of σ, defined as in* (1.6) *when $\sigma = p_a$ and $k = k_a$. Corresponding to σ there exists a proper F-deformation Λ_σ of M_n with the following properties.*

I. *Λ_σ leaves σ and CN_σ^{*a} pointwise fixed.*

II. *Λ_σ displaces each point of $N_\sigma^{*a} - \sigma$.*

III. *Λ_σ deforms N_σ^a on N_σ^a into the T-(k-disc) K^a, defined in* (2.4).

IV. *Λ_σ deforms K^a on itself onto K^a.*

The F-level sections of K^a, when $k = k_a$. The definitions of ω_k in (2.3), and of K^a in (2.4), show the following. If $1 < k \leqslant n$ and if c is a value of F on $K^a - \sigma$, the section of K^a at the F-level c is a topological $(k-1)$-sphere. If $k = 1$, this section is a pair of distinct points. These sections of K^a vary continuously with c and shrink to σ as c increases. When $k = 0$, $K^a = \sigma$.

In § 3 of [12] the following lemma is established as Lemma 3.1.

Lemma 2.2. *There exists a proper F-deformation Δ of M_n onto M_n which leaves each T-critical point of F fixed and displaces each other point of M_n.*

The following lemma refers to a T-critical value a of F and to the subset

$$F_{a-} = \{p \in M_n \mid F(p) < a\}\ (2) \tag{2.6}$$

of M_n. Theorem 3.1 of [12] is established with the aid of the deformation Δ of Lemma 2.2 and implies the following.

Lemma 2.3. *Let a be a T-critical value of F and $(a, b]$ an interval free of T-critical values of F. If $\sigma = p_a$, there exists an F-traction D of F_b into $\sigma \cup F_{a-}$* (3).

Let N_σ^a be a canonical neighborhood N_σ of $\sigma = p_a$, introduced in § 1. Lemma 2.3 has the following corollary.

(1) In ref. [12], T abbreviates topological. The following table of notational errata of [12] is appended:

Page	189,	190,	190,	193,	194,	195,	197
Line	—5,	3,	8,	—4,	14,	—13,	6,8
Symbol	Ψ,	F,	Φ,	η,	Z,	$p\Re\, \epsilon_q^*$,	E
Replacement	Ψ_q,	M,	Φ_σ,	η_r,	Z^l,	$p \in \Re_q^*$,	E^1

(2) In [12] F_{a-} is denoted by S_{a-}.

(3) F_{a-} is empty when $a = a_0$.

COROLLARY 2.1. *Under the conditions of Lemma 2.3 the following is true. If $c \in R$ is such that $a-c$ is sufficiently small and positive, then D is an F-traction of F_b into $N_\sigma^a \cup F_c$ if $k_a \geqslant 0$, and into $\sigma \cup F_c$ if $k_a = 0$. If $a = a_0$, F_c is empty.*

The F-deformation D of Lemma 2.3 of F_b is into a final image $D_1(F_b) \subset \sigma \cup F_{a-}$. This final image is closed and is accordingly included in $N_\sigma^a \cup F_c$ for a suitable choice of c, with $a-c$ sufficiently small and positive. When $k_a = 0$, $N_\sigma^a \cap F_a = \sigma$; hence $D_1(F_b) \subset \sigma \cup F_c$ when $k_a = 0$. (Cf. (1.5) and (1.6).)

Preparation for Theorem 2.1. The formulation of Theorem 2.1 requires the definition of a *T-saddle* of F at each T-critical point p_a for which $k_a > 0$. The set K^a, defined in (2.4), is on F_a, and below the F-level a, except for the point σ. K^a will serve our purposes as a T-saddle of p_a if we cut off from K^a all points definitely below a suitably chosen F-level $c < a$. With this understood we give the following definition.

DEFINITION 2.4. *A T-saddle $L^{a,c}$ of F at p_a.* Our T-saddle is defined only when $k = k_a > 0$. Let c be a value of F on N_σ^a such that $a > c > \alpha$, where α is the T-critical value next below a. We set

$$L^{a,c} = \{p \in K^a \mid a \geqslant F(p) \geqslant c\} \tag{2.7$'$}$$

and term $L^{a,c}$ a *T-saddle* of F at $\sigma = p_a$.

It is a consequence of (2.4), (2.7)$'$ and the choice of c that

$$L^{a,c} \subset N_\sigma^a \cap F_a \tag{2.7$''$}$$

and that $L^{a,c}$ meets F^a only in $\sigma = p_a$.

Corollary 2.1 leads to the following basic theorem.

THEOREM 2.1. *Let $[\alpha, \beta]$ be an interval in which a is the only T-critical value. Let γ be a value in $[\alpha, \beta]$. If $c < a$ and $a-c$ is sufficiently small, there exists an F-traction d of F_γ into $L^{a,c} \cup F_c$ when $k_a > 0$, and into $\sigma \cup F_c$ when $k_a = 0$.*

Proof. By Corollary 2.1, there exists an F-traction D of F_γ into $N_\sigma^a \cup F_c$ if $k = k_a > 0$, and if $a-c$ is sufficiently small and positive. If Δ_σ^a is the F-deformation Δ_σ of Lemma 2.1 when $\sigma = p_a$, then by III of Lemma 2.1 the product F-deformation $d = \Delta_\sigma^a D$ will satisfy Theorem 2.1 when $k > 0$ and $a-c$ is sufficiently small.

In case $k = 0$, D of Corollary 2.1 will serve as d of Theorem 2.1. Thus Theorem 2.1 is true.

Permanent notation. If $\sigma = p_a$ is a T-critical point with T-index $k = k_a > 0$, a deletion of σ from F_a, or from the sets K^a and $L^{a,c}$ included in F_a, will yield sets to be denoted by

$$\dot{F}_a,\ \dot{K}^a,\ \dot{L}^{a,c}, \tag{2.8}$$

respectively. $\dot{F}_a$ is similarly defined and non-empty when $k_a = 0$, provided a is not the minimum T-critical value a_0.

The following theorem supplements Theorem 2.1.

THEOREM 2.2. *Let a and b, with $a < b$, be T-critical values of F such that (a, b) is an interval of T-ordinary values of F. There then exists an F-traction of $\dot{F}_b$ into F_a.*

The proof of Theorem 2.2 is begun by verifying the following statement.

(i) *If b is given as in Theorem 2.2 and if $c \in R$ is such that $b-c$ is sufficiently small and positive, there exists an F-traction d of $\dot{F}_b$ into F_c.*

Proof of (i). Let $k = k_b$ be the T-index of $\sigma = p_b$. Suppose that $k = k_b > 0$. It follows from Theorem 2.1, with α and γ of Theorem 2.1 both taken as b of Theorem 2.2, that if $b-c$ is sufficiently small and positive, there exists an F-traction D of $\dot{F}_b$ into a set of form $\dot{L}^{b,c} \cup F_c$. It is trivial that there exists an F-deformation D' of $\dot{L}^{b,c} \cup F_c$ onto F_c, retracting $\dot{L}^{b,c} \cup F_c$ onto F_c, so that $D'D$ is an F-traction d of $\dot{F}_b$ into F_c. Thus (i) is true when $k_b > 0$.

When $k = k_b = 0$, statement (i) follows from Lemma 2.2, since $\dot{F}_b$ is compact when $k_b = 0$ and contains no T-critical point at the F-level b.

It follows from Theorem 2.1 with α, β of Theorem 2.1 taken as a, c, where $a < c < b$, that there exists an F-deformation D'' of F_c into F_a. Hence the product deformation $D''d$ is an F-traction of $\dot{F}_b$ into F_a.

Thus Theorem 2.2 is true.

Introduction to Lemma 2.4. Let $\sigma = p_a$ be a T-critical point of F with a *positive* T-index $k = k_a$. We seek a neighborhood X of σ relative to F_a, such that there exists an F-deformation of X which retracts X into a T-saddle $L^{a,c}$ of F at $\sigma = p_a$. Taking account of the definition of $L^{a,c}$ in (2.7)$'$ and of K^a in (2.4), one sees that a neighborhood of p_a relative to F_a, with the desired property, is defined by the union $X^{a,c}$ of all sections of N_σ^a with F-levels in the interval $[c, a]$. We suppose that $a-c < 1$ and note that

$$X^{a,c} = N_\sigma^a \cap (F_a - F_{c-}) = \Phi_\sigma^a(A) \quad (\text{cf. } (2.4)) \tag{2.9}$$

where

$$A = \{z \in Z_1 \mid 0 \geqslant \varrho_\sigma Q_k(z) \geqslant c-a\} \quad (\text{cf. } (1.5)).$$

LEMMA 2.4. *When $k = k_a > 0$, there exists an F-deformation d of $X^{a,c}$, retracting $X^{a,c}$ onto the T-saddle $L^{a,c}$ of F at $\sigma = p_a$.*

Let B be the subset of A of points $(z_1, \ldots, z_k, 0, \ldots, 0) \in A$. Then

$$X^{a,c} = \Phi_\sigma^a(A); \quad L^{a,c} = \Phi_\sigma^a(B) \quad (\text{cf. } (2.7)'). \tag{2.10}$$

One readily shows that there exists an F-deformation retracting A onto B, in which a point $z \in A$ is "replaced" by a point $z(t) \in A$ as t increases from 0 to 1, with $Q_k(z) \geqslant Q_k(z(t))$. Lemma 2.4 is satisfied by a deformation d in which, as t increases from 0 to 1, each point $\Phi_\sigma^a(z)$ in $X^{a,c}$ is replaced by $\Phi_\sigma^a(z(t))$ as z ranges over A.

§ 3. Relevant theorems in singular homology theory. We shall be concerned with singular homology on a topological space χ. Use will be made of Eilenberg's definition in [5] of singular r-cells. No triangulations of χ are presupposed. See also § 26 of [15].

Given χ and the ring $\boldsymbol{Z}$ of rational integers, the singular r-cells on χ are combined linearly, with coefficients in $\boldsymbol{Z}$, to define a $\boldsymbol{Z}$-module, denoted by $C_r(\chi, \boldsymbol{Z})$. The singular r-cells on χ form a *basis* for $C_r(\chi, \boldsymbol{Z})$ in the sense of Bourbaki [2], p. 11. The elements of $C_r(\chi, \boldsymbol{Z})$ are termed *r-chains*. The homology groups $H_r(\chi, \boldsymbol{Z})$ are well-defined for each rational integer r. They are trivial if $r < 0$.

The following notational innovation is very useful.

DEFINITION 3.0. $((u^r, \chi))$. If u^r is an r-cycle of $C_r(\chi, \boldsymbol{Z})$ then $((u^r, \chi))$ shall denote the subset of r-cycles in $C_r(\chi, \boldsymbol{Z})$ which are homologous to u^r on χ. One can regard $((u^r, \chi))$ as an element in $H_r(\chi, \boldsymbol{Z})$.

DEFINITION 3.1. *The chain-transformation $\widehat{\varphi}$* (Eilenberg). Let there be given a continuous mapping $\varphi\colon \chi \to \chi'$ of a topological space χ into a topological space χ'. A singular q-cell σ^q on χ is defined by the class of "equivalent" mappings τ of vertex-ordered euclidean q-simplices into χ. Cf. § 26 of [15]. In a chain-transformation,

$$(3.0) \qquad \widehat{\varphi}\colon\ C_q(\chi, Z) \to C_q(\chi', Z) \qquad (q = 0, 1, 2, \ldots),$$

"induced by φ", the image $\widehat{\varphi}\sigma^q$ on χ' of a q-cell σ^q on χ is defined by the compositions $\varphi \circ \tau$ with φ of the equivalent mappings τ into χ which define σ^q. The mappings $\widehat{\varphi}$, so defined for cells σ^q, are extended linearly over $\boldsymbol{Z}$ to define the mappings (3.0). Eilenberg shows that $\widehat{\varphi}$ is permutable with the boundary operator δ. Natural homomorphisms

$$(3.1) \qquad \widehat{\varphi}_*\colon\ H_q(\chi, Z) \to H_q(\chi', Z) \qquad (q = 0, 1, \ldots)$$

are induced by $\widehat{\varphi}$.

Let z be a q-cycle (over $\boldsymbol{Z}$) on χ and d a deformation of χ on χ. If d_1 is the terminal mapping of d, the homology $z \sim \widehat{d}_1 z$ is valid on the image under d of any carrier $|z|$ of z. By a *carrier* $|z|$ of z is understood any subset of χ on which the cycle z is well-defined. Cf. Corollary 27.1 of [15].

The F-"tractions" defined in § 2 induce isomorphisms as follows.

THEOREM 3.1. *Let χ and χ' be topological spaces, with χ' a subspace of χ, and let d be a traction of χ into χ'. Isomorphisms*

$$(3.2) \qquad G_q\colon\ H_q(\chi', \boldsymbol{Z}) \underset{\sim}{\to} H_q(\chi, \boldsymbol{Z}) \qquad (q = 0, 1, \ldots)$$

are induced in which, for each q-cycle z on χ', $((z, \chi'))$ is mapped onto $((z, \chi))$.

The homomorphism G_q of Theorem 3.1 which we have affirmed to be an isomorphism of $H_q(\chi', \boldsymbol{Z})$ onto $H_q(\chi, \boldsymbol{Z})$ is induced by the inclusion mapping i of χ' into χ. It is clearly an isomorphism if the following is true.

(a) Each q-cycle on χ is homologous on χ to a q-cycle on χ'.

(b) Each q-cycle on χ' which is bounding on χ is bounding on χ'.

In fact G_q is surjective if (a) holds and has a null kernel if (b) holds.

Proof of (a). Since d deforms χ on χ into χ', (a) is clearly true.

Proof of (b). Let d_1 be the terminal mapping of d. As is well-known, one can associate with d a linear homomorphism (see § 27 of [15])

$$\boldsymbol{d}\colon\ C_r(\chi, \boldsymbol{Z}) \to C_{r+1}(\chi, \boldsymbol{Z}) \qquad (r = 0, 1, \ldots)$$

such that for each r-chain $z \in C_r(\chi, \boldsymbol{Z})$

$$(3.3) \qquad \partial \boldsymbol{d} z = \widehat{d}_1 z - z - \boldsymbol{d}\partial z\,.$$

Moreover, the definition of $\boldsymbol{d}$ in [15] is such that a carrier $|\boldsymbol{d}z|$ exists on a subset X of χ if d deforms $|z|$ on X. (Lemma 27.1 of [15].)

As in (b), let $u_{\chi'}$ be a q-cycle on χ' such that

$$(3.4)' \qquad u_{\chi'} = \partial y_\chi\,,$$

where y_χ is a $(q+1)$-chain on χ. By virtue of (3.3)

$$(3.4)'' \qquad \partial \boldsymbol{d} y_\chi = \widehat{d}_1 y_\chi - y_\chi - \boldsymbol{d}\partial y_\chi\,.$$

On applying ∂ to the members of (3.4)″ and making use of (3.4)′, we find that

$$(3.4)''' \qquad 0 = \partial\widehat{d}_1 y_\chi - u_{\chi'} - \partial \boldsymbol{d} u_{\chi'}\,.$$

By hypothesis both $\widehat{d}_1 y_\chi$ and $\boldsymbol{d}u_{\chi'}$ are on χ', so that (3.4)‴ implies that $u_{\chi'} \sim 0$ on χ', confirming (b).

Theorem 3.1 follows from (a) and (b).

Relevant homologies over $\boldsymbol{Z}$. Relative homologies were introduced by Lefschetz. Suitably modified, relative homologies will serve in § 4 to characterize the effect on a group $H_q(\dot{F}_a, \boldsymbol{Z})$ of replacing $\dot{F}_a$ by F_a. Here a is a T-critical value of F, with T-index $k = k_a > 0$.

Given a topological space χ, a subspace $A \neq \chi$ of χ is taken as a "modulus" and the pair (χ, A) termed *admissible*. The qth relative homology group is denoted by $H_q(\chi, A, \boldsymbol{Z})$. On passing from a field $\mathcal{K}$ to the ring $\boldsymbol{Z}$, Theorem 28.4 of [15] leads to the following theorem. The proof is similar to that in [15].

THEOREM 3.2. *Let (χ, A) and (χ', A') be admissible pairs with $\chi' \subset \chi$ and $A' \subset A$. Let d be a deformation retracting χ onto χ' and A onto A', with d_1 the terminal mapping of d.*

Corresponding to the inclusion mapping i of (χ', A') into (χ, A) the chain transformation $\hat{i}$ induces isomorphisms

$$H_q(\chi', A', Z) \underset{\sim}{\to} H_q(\chi, A, Z) \qquad (q = 0, 1, \ldots) \tag{3.5}$$

under whose respective inverses the rel. homology class on χ of a q-cycle $z_\chi \bmod A$, corresponds to the rel. homology class on χ' of $\hat{d}_1 z_\chi \bmod A'$.

The proof of Theorem 28.4 in [15] shows the following:

(**a**) A q-cycle $z_\chi \bmod A$ is homologous on $\chi \bmod A$ to the q-cycle $\hat{d}_1 z_\chi$ on $\chi' \bmod A'$.

(**b**) A q-cycle $u_{\chi'}$ on $\chi' \bmod A'$ which is bounding on $\chi \bmod A$ is bouding on $\chi' \bmod A'$.

Statement (**a**) implies that the inclusion induced homomorphisms $(\hat{i})_*$ are surjective, while (**b**) implies that the kernels of these mappings are null. Theorem 3.2 follows.

Excision. Among the axioms of Eilenberg and Steenrod, formulated on page 11 of [6], is found the Excision Axiom. Our next theorem formulates a simplified version of the Excision Axiom adequate for our purposes.

Theorem 3.3. *Excision. Let χ be a metric space, A a proper subspace of χ and A^* a subspace of A such that for some positive e*

$$(\chi - A)_e \subset \chi - A^*, \tag{3.6}$$

where $(\chi - A)_e$ is the open e-neighborhood of $\chi - A$, relative to χ.

There then exist isomorphisms,

$$H_q(\chi - A^*, A - A^*, Z) \underset{\sim}{\to} H_q(\chi, A, Z), \qquad (q = 0, 1, \ldots) \tag{3.7}$$

induced by the inclusion mapping

$$i\colon (\chi - A^*, A - A^*) \to (\chi, A).$$

The proof of Theorem 28.3 of [15] yields a proof of the above Excision Theorem, provided, of course, the field $\mathfrak{K}$ of [15] is replaced by the ring Z of integers.

Definition 3.2. *$\#$-mappings J_q^a.* The mappings of homology groups into homology groups which have been introduced in this section have all been isomorphisms. We shall now define an "inclusion induced" homomorphism

$$J_q^a\colon H_q(\dot{F}_a, Z) \to H_q(F_a, Z) \qquad (a > a_0;\ q = 0, 1, \ldots) \tag{3.8}$$

which may or may not be an isomorphism, depending on the value of q, on the T-index k_a of a and other integral critical invariants to be introduced.

Let i_a be the inclusion mapping of $\dot{F}_a$ into F_a and $\hat{i}_a$ the corresponding chain-transformation

$$\hat{i}_a\colon C_q(\dot{F}_a, Z) \to C_q(F_a, Z) \qquad (q = 0, 1, \ldots). \tag{3.9}$$

For each q there is thereby induced a homomorphism of the form (3.8). One sets $(\hat{i}_a)_*$ equal to J_q^a. We term J_q^a a *$\#$-mapping*. Whether J_q^a is an isomorphism or not, it is of basic importance. If z is a q-cycle in $C_q(\dot{F}_a, Z)$, then in the notation of Definition 3.0

$$J_q^a\big(\big((z, \dot{F}_a)\big)\big) = \big((z, F_a)\big) \tag{3.10}$$

in accord with the definition of $(\hat{i}_a)_*$.

Part II. The finite generation of the groups $H_q(F_c, Z)$

§ 4. The homology groups $H_q(F_a, \dot{F}_a, Z)$. We shall characterize the groups $H_q(F_a, \dot{F}_a, Z)$ when $a > a_0$. It will then be relatively easy to give an inductive proof that for each T-critical value $a > a_0$ of F in the list (0.4), $H_q(F_a, Z)$ is FG and to conclude that $H_q(F_c, Z)$ is FG for each value c of F on M_n.

The neighborhoods $X^{a,c}$ of $\sigma = \varrho_a$ relative to F_a when $k = k_a > 0$. $X^{a,c}$ was defined in (2.9). Note the inclusions

$$F_a \supset X^{a,c} \supset L^{a,c} \qquad \text{(cf. (2.7)')}. \tag{4.0}$$

Deleting σ from each of these sets one finds that

$$\dot{F}_a \supset \dot{X}^{a,c} \supset \dot{L}^{a,c}. \tag{4.1}$$

Preparation for Theorem 4.1.

Lemma 4.1. *If i is the inclusion mapping,*

$$i\colon (X^{a,c}, \dot{X}^{a,c}) \to (F_a, \dot{F}_a) \qquad (k = k_a > 0)$$

then $\hat{i}$ induces the isomorphisms,

$$(\hat{i})_*\colon H_q(X^{a,c}, \dot{X}^{a,c}, Z) \underset{\sim}{\to} H_q(F_a, \dot{F}_a, Z) \qquad (q = 0, 1, \ldots). \tag{4.2}$$

That the homomorphism $(\hat{i})_*$ induced by the chain-transformation $\hat{i}$ is an isomorphism is a consequence of Excision Theorem 3.3. One identifies (χ, A) of the Excision Theorem with $(F_a, \dot{F}_a)$ and sets $F_a - X^{a,c} = A^*$. With this understood

$$A^* \subset A, \qquad X^{a,c} = \chi - A^*, \qquad \dot{X}^{a,c} = A - A^*. \tag{4.3}$$

The excision condition (3.6) is satisfied, since a sufficiently small e-neighborhood, relative to F_a of $\chi - A = p_a$, is included in $\chi - A^* = X^{a,c}$ when $k = k_a > 0$.

The isomorphism (4.2) follows from Theorem 3.3.

According to Lemma 2.4 there exists an F-deformation retracting $X^{a,c}$ onto the T-saddle $L^{a,c}$, holding $\sigma = p_a$ fast. It follows from Theorem 3.2 that if i is the inclusion mapping

$$(4.4)' \qquad i\colon (L^{a,c}, \dot{L}^{a,c}) \to (X^{a,c}, \dot{X}^{a,c}) \qquad (k_a > 0)$$

the chain transformation $\hat{i}$ induces isomorphisms,

$$(4.4)'' \qquad H_q(L^{a,c}, \dot{L}^{a,c}, \boldsymbol{Z}) \underset{\sim}{\to} H_q(X^{a,c}, \dot{X}^{a,c}, \boldsymbol{Z}) \qquad (q = 0, 1, \ldots).$$

From this result and from Lemma 4.1 we infer the following.

THEOREM 4.1. *Let $\sigma = p_a$ be a T-critical point with positive T-index $k = k_a$. If J is the inclusion mapping*

$$(4.5)' \qquad J\colon (L^{a,c}, \dot{L}^{a,c}) \to (F_a, \dot{F}_a) \qquad (k_a > 0)$$

the chain transformation $\hat{J}$ induces isomorphisms,

$$(4.5)'' \qquad H_q(L^{a,c}, \dot{L}^{a,c}, \boldsymbol{Z}) \underset{\sim}{\to} H_q(F_a, \dot{F}_a, \boldsymbol{Z}) \qquad (q = 0, 1, \ldots).$$

Preparation for Theorem 4.2. Theorem 4.2 will make clear what are the invariants of the Abelian groups (4.5)''. By virtue of Definition 2.4 of the T-saddle $L^{a,c}$, one sees that there exists a homeomorphism,

$$\Theta_k\colon L^{a,c} \to \Delta_k \qquad (k = k_a > 0)$$

of $L^{a,c}$ onto an origin-centered k-ball Δ_k in R^k, with $\Theta_k(p_a) = \boldsymbol{0}$. Set $\dot{\Delta}_k = \Delta_k - \boldsymbol{0}$. Under Θ_k, $\dot{L}^{a,c}$ is mapped homeomorphically onto $\dot{\Delta}_k$. A classical theorem then implies the following. (Cf. Theorem 28.1 of [15].)

LEMMA 4.2. *The chain-transformation $\hat{\Theta}_k$ induces isomorphisms,*

$$(4.6) \qquad H_q(L^{a,c}, \dot{L}^{a,c}, \boldsymbol{Z}) \underset{\sim}{\to} H_q(\Delta_k, \dot{\Delta}_k, \boldsymbol{Z}) \qquad (q = 0, 1, \ldots;\ k = k_a > 0).$$

The Abelian group $H_q(\Delta_k, \dot{\Delta}_k, \boldsymbol{Z})$, $k > 0$, is free, as is readily shown, and has a base of dimension δ_k^q. This group is trivial except when $q = k$, and when $q = k$ has a base which consists of a single element of infinite order. Because of the isomorphisms (4.5)'' and (4.6) we infer the following.

THEOREM 4.2. *The homology groups (4.5)'' are free. They are trivial except when $q = k$. When $q = k > 0$, a base for these groups consists of a single element of infinite order.*

Preparation for Theorem 4.3. According to Theorem 4.2 when the T-index of $\sigma = p_a$ is a positive integer $k = k_a$, a base of the free group $H_k(F_a, \dot{F}_a, \boldsymbol{Z})$ consists of a single element of infinite order. Such an element is the homology class of special k-cycles $\varkappa_a$ on $F_a \bmod \dot{F}_a$ which we shall term *saddle* k-cycles of $\sigma = p_a$, and shall now characterize.

Three definitions are required.

DEFINITION 4.1. *A prebase for a relative homology group.* Given a rel. homology group $H_q(\chi, A, \boldsymbol{Z})$ which is free and has a finite base, a set of non-trivial relative q-cycles, one from each relative homology class in a base for $H_q(\chi, A, \boldsymbol{Z})$, will be called a *prebase* for H_q. A prebase may be empty.

We seek a prebase for the group $H_k(F_a, \dot{F}_a, \boldsymbol{Z})$, $k > 0$. According to Theorem 4.2 it will consist of one k-cycle on F_a, $\bmod \dot{F}_a$. We shall define a prebase which is given by a single singular k-cell (taken $\bmod \dot{F}_a$) which is *simply carried* by F_a in the sense of the following definition.

DEFINITION 4.2. *Simply-carried singular q-cells.* A singular q-cell on M_n is defined by an equivalence class (Eilenberg) of mappings $\tau\colon s \to M_n$ of vertex-ordered q-simplices s into M_n. If the mappings τ are homeomorphisms of their domains s onto their images $\tau(s)$, the resultant singular q-cell on M_n will be said to be *simply-carried*.

We give a fundamental definition.

DEFINITION 4.3. *A saddle k-cell $\varkappa_a$.* If $k = k_a > 0$, a singular k-cell which is "simply-carried" on some T-saddle $L^{a,c}$ of p_a with p_a an interior point of the carnier of $\varkappa_a$ will be called a *saddle k-cell* of p_a. Taken $\bmod \dot{L}^{a,c}$, $\varkappa_a$ will be called a *saddle k-cycle* on $L^{a,c} \bmod \dot{L}^{a,c}$.

THEOREM 4.3. *If a T-critical point $\sigma = p_a$ has a positive T-index $k = k_a$ the following is true.*

(i) *A saddle k-cell $\varkappa_a$ of p_a which is simply-carried by a T-saddle $L^{a,c}$ of p_a and is taken $\bmod \dot{L}^{a,c}$, is a prebase of $H_k(L^{a,c}, \dot{L}^{a,c}, \boldsymbol{Z})$.*

(ii) *Such a saddle k-cell taken $\bmod \dot{F}_a$, is a prebase of $H_k(F_a, \dot{F}_a, \boldsymbol{Z})$.*

Proof of (i). To prove (i) use will be made of the isomorphism (4.6), supplemented by the following affirmation.

(α) *Let y^k, $k > 0$ be a singular k-cell simply-carried by Δ_k with the center $\boldsymbol{0}$ of Δ_k in the interior of the carrier $|y^k|$ of y^k. Taken $\bmod \dot{\Delta}_k$, y^k is a k-cycle on $\Delta_k \bmod \dot{\Delta}_k$ which is a prebase of $H_k(\Delta_k, \dot{\Delta}_k, \boldsymbol{Z})$.*

The proof of (α) is elementary and will be left to the reader. One should note that the carrier $|\partial y^k|$ is a topological $(k-1)$-sphere ([1]) on $\dot{\Delta}_k$ whose "Jordan" interior on Δ_k contains $\boldsymbol{0}$. The readers will find Lemma 29.0 of [15] useful in proving (α).

Granting the truth of (α), Theorem 4.3 (i) follows from Lemma 4.2. For the isomorphism (4.6) is induced by the homeomorphism Θ_k of $L^{a,c}$ onto Δ_k. Under the inverse of $\hat{\Theta}_k$ the cycle y^k on $\Delta_k \bmod \dot{\Delta}_k$ of (α) goes into a saddle k-cycle $\varkappa_a$ of p_a on $L^{a,c} \bmod \dot{L}^{a,c}$, which is a prebase of $H_k(L^{a,c}, \dot{L}^{a,c}, \boldsymbol{Z})$.

Proof of (ii). If J is the inclusion mapping (4.5)' and $\varkappa_a$ a saddle k-cell on $L^{a,c}$, which, taken $\bmod \dot{L}^{a,c}$, is a prebase of $H_k(L^{a,c}, \dot{L}^{a,c}, \boldsymbol{Z})$,

([1]) A topological 0-sphere is understood to be a pair of points.

then by Theorem 4.1 $\hat{\mathcal{J}}\varkappa_a$ will be a prebase of $H_k(F_a, \dot{F}_a, \boldsymbol{Z})$. Statement (ii) follows, since $\hat{\mathcal{J}}\varkappa_a = \varkappa_a \bmod \dot{F}_a$.

T-saddles $L^{a,c}$ are a means to an end, the definition of saddle k-cells of p_a on F_a. In the following corollary of Theorem 4.3 the ends rather than the means come to the fore.

Corollary 4.1. *If $\varkappa_a(1)$ and $\varkappa_a(2)$ are two rel. saddle k-cycles on F_a of the same T-critical points $\sigma = p_a$ with T-index $k = k_a$, then for some choice of e as* 1 *or* -1

$$\varkappa_a(1) \sim e\varkappa_a(2) \qquad (\textit{on } F_a \bmod \dot{F}_a) \tag{4.7}$$

and consequently,

$$\partial\varkappa_a(1) \sim e\partial\varkappa_a(2) \qquad (\textit{on } \dot{F}_a)\,. \tag{4.8}$$

Proof of (4.7). According to Theorem 4.3 both $\varkappa_a(1)$ and $\varkappa_a(2)$ are prebases of the *free* Abelian group $H_k(F_a, \dot{F}_a, \boldsymbol{Z})$. The relative homology (4.7) is implied.

Proof of (4.8). The homology (4.7) implies that

$$\varkappa_a(1) - e\varkappa_a(2) = \partial c_+^{k+1} + c_-^k \qquad (k = k_a) \tag{4.9}$$

where c_+^{k+1} and c_-^k are integral chains on F_a and $\dot{F}_a$ respectively. The application of ∂ to the members of (4.9) yields (4.8).

A critical homology class $((\partial\varkappa_a, \dot{F}_a))$. Corresponding to a saddle k-cell $\varkappa_a$ of a T-critical point p_a of positive T-index $k = k_a$, the homology class of $\partial\varkappa_a$ on $\dot{F}_a$ is denoted by $((\partial\varkappa_a, \dot{F}_a))$ and termed a *critical homology* class of $\dot{F}_a$. (Cf. Definition 3.0). It may be regarded as an element in $H_{k-1}(\dot{F}_a, \boldsymbol{Z})$. According to (4.8) any other critical homology class of $\dot{F}_a$ has the form $e((\partial\varkappa_a, \dot{F}_a))$ where $e = -1$. As an element of $H_{k-1}(\dot{F}_a, \boldsymbol{Z})$, the order of $((\partial\varkappa_a, \dot{F}_a))$ may be finite or infinite. We now define a basic invariant $\boldsymbol{t}^a$.

Definition 4.4. $\boldsymbol{t}^a$. *The torsion index $\boldsymbol{t}^a$ of p_a when $k_a > 0$.* The order of $((\partial\varkappa_a, \dot{F}_a))$ in $H_{k-1}(\dot{F}_a, \boldsymbol{Z})$, when finite, will be denoted by $\boldsymbol{t}^a$ and termed the *torsion*[1] *index* of p_a. No definition of $\boldsymbol{t}^a$ is given when the order of $((\partial\varkappa_a, \dot{F}_a))$ in $H_{k-1}(\dot{F}_a, \boldsymbol{Z})$ is infinite.

If $\boldsymbol{t}^a$ exists, it is positive and for each saddle k-cell $\varkappa_a$ of p_a

$$\boldsymbol{t}^a\partial\varkappa_a \sim 0 \qquad (\text{on } \dot{F}_a)\,. \tag{4.10}$$

If $\mu \neq 0$ is an integer such that

$$\mu\partial\varkappa_a \sim 0 \qquad (\text{on } \dot{F}_a) \tag{4.11}$$

then $\boldsymbol{t}^a$ exists and $\mu = m\boldsymbol{t}^a$ for some integer $m \neq 0$. This is an elementary result in the theory of cyclic groups.

[1] The torsion index $\boldsymbol{t}^a$ is not to be confused with a torsion coefficient of $H_{k-1}(\dot{F}_a, \boldsymbol{Z})$.

Definition 4.5. λ_a. *A k-cycle on F_a, $\boldsymbol{t}^a$-fold linking.* When $k = k_a > 0$ and $\boldsymbol{t}^a$ exists, (4.10) holds and there accordingly exists a k-chain c_-^k on $\dot{F}_a$ such that

$$\partial\boldsymbol{t}^a\varkappa_a = \partial c_-^k \tag{4.12}$$

and hence a k-cycle

$$\lambda_a = \boldsymbol{t}^a\varkappa_a - c_-^k \qquad (\text{on } F_a)\,. \tag{4.13}$$

We term λ_a a k-cycle which is $\boldsymbol{t}^a$*-fold linking* on F_a, which *belongs* to p_a and is *associated* with $\varkappa_a$.

The following lemma is a consequence of Corollary 4.1.

Lemma 4.3. (i) *Any two $\boldsymbol{t}^a$-fold linking k-cycles $\lambda_a(1)$ and $\lambda_a(2)$ on F_a satisfy a relative homology*

$$\lambda_a(1) \sim e\lambda_a(2) \qquad (\textit{on } F_a \bmod \dot{F}_a) \tag{4.14}$$

where e has one of the values ± 1.

(ii) *If λ_a is a $\boldsymbol{t}^a$-fold linking k-cycle on F_a, then $\mu\lambda_a \sim 0$ on $F_a \bmod \dot{F}_a$, for no nonnull integer μ.*

Proof of (i). The relative homology (4.14) follows from (4.7) and (4.13).

Proof of (ii). It follows from (4.13) that

$$\lambda_a \sim \boldsymbol{t}^a\varkappa_a \qquad (\text{on } F_a \bmod \dot{F}_a)\,. \tag{4.15}$$

Moreover $\varkappa_a$, taken $\bmod \dot{F}_a$, is a prebase of the free group $H_k(F_a, \dot{F}_a, \boldsymbol{Z})$ in accord with Theorem 4.3 (ii), so that $\mu\varkappa_a \sim 0$ on $F_a \bmod \dot{F}_a$ for no non-null integer μ. Reference to (4.15) shows that (ii) is true.

This completes the proof of Lemma 4.3.

Theorem 4.4 distinguishes between the cases in which a torsion index $\boldsymbol{t}^a$ of p_a exists or does not exist.

Theorem 4.4. *If the T-index $k = k_a$ of a T-critical point p_a is positive, the following is true.*

(i) *In case p_a has a torsion index $\boldsymbol{t}^a$ and λ_a is a $\boldsymbol{t}^a$-fold linking k-cycle on F_a associated with p_a, then if e_+^k is a k-cycle on F_a*

$$e_+^k \sim m\lambda_a \qquad (\textit{on } F_a \bmod \dot{F}_a) \tag{4.16}$$

for some integer m (possibly zero).

(ii) *If no torsion index of p_a exists, then if e_+^k is a k-cycle on F_a*

$$e_+^k \sim 0 \qquad (\textit{on } F_a \bmod \dot{F}_a)\,. \tag{4.17}$$

In both cases (i) and (ii), Theorem 4.3 (ii) implies that for some integer μ (possibly zero)

$$e_+^k = \mu\varkappa_a + \partial e_+^{k+1} \qquad (\text{on } F_a \bmod \dot{F}_a) \tag{4.18}$$

for a suitably chosen chain e_+^{k+1} on F_a. From (4.18) we infer that in both cases (i) and (ii)

$$\mu\partial\varkappa_a \sim 0 \quad (\text{on } \dot{F}_a). \tag{4.19}$$

Proof of (i). If $\mu = 0$ in (4.18), (4.18) implies (4.16) with $m = 0$. If $\mu \neq 0$, (4.19) implies that μ is an integral multiple mt^a of t^a, since t^a is finite. In this case (4.16) follows from (4.18) and (4.15).

Proof of (ii). In the case of (ii), (4.19) implies that $\mu = 0$ in (4.19). Otherwise t^a would exist contrary to the hypothesis of (ii). When $\mu = 0$, (4.18) implies (4.17).

Thus theorem 4.4 is true.

We shall now establish a corollary of Theorem 4.2 which will be useful both in Part II and Part III. Here the T-index $k_a \geqslant 0$.

COROLLARY 4.2. *Concerning the* $\#$*-mapping* J_q^a, $a > a_0$, *of Definition* 3.2, *the following is true.*

(i) $\operatorname{Ker} J_q^a = 0$ *when* $q \neq k_a - 1$.

(ii) J_q^a *is surjective when* $q \neq k_a$.

(iii) *When q is neither k_a nor $k_a - 1$, J_q^a is an isomorphism of* $H_q(\dot{F}_a, \boldsymbol{Z})$ *onto* $H_q(F_a, \boldsymbol{Z})$.

Proof of (i). If c_-^q [1] is a q-cycle on $\dot{F}_a$ such that $c_-^q \nsim 0$ on $\dot{F}_a$, we shall show that $c_-^q \nsim 0$ on F_a when $q \neq k_a - 1$, implying thereby that $\ker J_q^a = 0$ when $q \neq k_a - 1$ (cf. Definition 3.2).

Suppose on the contrary that there exists a $(q+1)$-chain c_+^{q+1} such that $c_-^q = \partial c_+^{q+1}$. The chain c_+^{q+1} is then a cycle on $F_a \bmod \dot{F}_a$. Since $q+1 \neq k$ by hypothesis of (i), Theorem 4.2 implies that $c_+^{q+1} \sim 0$ on $F_a \bmod \dot{F}_a$, or equivalently

$$c_+^{q+1} = \partial c_+^{q+2} + c_-^{q+1}. \tag{4.20}$$

Since $\partial c_+^{q+1} = c_-^q$ by hypothesis of this paragraph, (4.20) implies that $c_-^q \sim 0$ on $\dot{F}_a$, contrary to the hypothesis of the preceding paragraph.

We infer that (i) is true.

Proof of (ii). It is sufficient to show that if c_+^q is a q-cycle on F_a, if $q \neq k_a$ and if $a > a_0$, then for some q-cycle c_-^q on $\dot{F}_a$,

$$c_+^q \sim c_-^q \quad (\text{on } F_a). \tag{4.21}$$

The homology is trivial when $k_a = 0$ and $a > a_0$, since F_a is then the union of sets [2] $\dot{F}_a$ and p_a whose closures are disjoint.

When $k = k_a > 0$ and $q \neq k_a$ it follows from Theorem 4.2 that

$$c_+^q = \partial e_+^{q+1} + e_-^q \quad (\text{on } F_a) \tag{4.22}$$

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(1) Subscripts $-$ or $+$ will indicate that the chain or cycle is on $\dot{F}_a$ or F_a respectively.

(2) Strictly p_a should be denoted by (p_a) when considered a set.

for suitable chains e_+^{q+1} and e_-^q. An application of ∂ to the members of (4.22) shows that e_-^q is a q-cycle. With this understood, (4.22) implies (4.21), on taking c_-^q as e_-^q. Thus (ii) is true.

Proof of (iii). Statement (iii) follows immediately from (i) and (ii). Thus Corollary 4.2 is true.

§ 5. Proof of finite generation.

The principal theorem of this section follows.

THEOREM 5.1. *If γ is any value of F on M_n, the homology groups* $H_q(F_\gamma, \boldsymbol{Z})$ *are* FG.

The proof of Theorem 5.1 is inductive in character. To make this clear each T-critical value a we shall set

$$H_q(F_a, \boldsymbol{Z}) = H_q^a \tag{5.0$'$}$$

$$H_q(\dot{F}_a, \boldsymbol{Z}) = \dot{H}_q^a \qquad (q = 0, 1, \ldots) \tag{5.0$''$}$$

and corresponding to the listing (0.4) of the T-critical values a_r of F, shall list the homology groups

$$H_q^{a_0}, H_q^{a_1}, \ldots, H_q^{a_\nu} \quad (q = 0, 1, \ldots). \tag{5.1}$$

It is trivial that the groups $H_q^{a_0}$ are FG. We shall give an inductive proof of the following.

THEOREM 5.2. *Each homology group in the list* (5.1) *is* FG.

Before coming to the proof of Theorem 5.2 note that Theorem 5.2 implies Theorem 5.1 by virtue of the following lemma.

LEMMA 5.1. *If γ is an ordinary value of F and if a is the maximum of the T-critical values of F less than γ, then if i is the inclusion mapping of F_a into F_γ, the corresponding chain-transformation $\widehat{i}$ induces isomorphisms,*

$$H_q(F_a, \boldsymbol{Z}) \underset{\sim}{\to} H_q(F_\gamma, \boldsymbol{Z}) \quad (q = 0, 1, \ldots). \tag{5.2}$$

Proof of Lemma 5.1. It is a corollary of Theorem 2.1 that there exists an F-traction of F into F_a, so that by Theorem 3.1, (5.2) holds as stated.

Proof of Theorem 5.2. It is sufficient to prove Lemmas 5.2 and 5.3.

LEMMA 5.2. *If a_r is a T-critical value in the list* (0.4) *with $a_r > a_0$, then if $H_q^{a_{r-1}}$ is* FG, $\dot{H}_q^{a_r}$ *is* FG.

Proof of Lemma 5.2. According to Theorem 2.2 there exists an F-traction of $\dot{F}_{a_r}$ into $F_{a_{r-1}}$. Hence by Theorem 3.1 there is an isomorphism

$$H_q^{a_{r-1}} \underset{\sim}{\to} \dot{H}_q^{a_r}. \tag{5.3}$$

Hence $\dot{H}_q^{a_r}$ is FG if $H_q^{a_{r-1}}$ is FG. Thus Lemma 5.2 is true.

LEMMA 5.3. *If a_r is a T-critical value in the list* (0.4) *with $a_r > a_0$, then if $\dot{H}_q^{a_r}$ is* FG, *$H_q^{a_r}$ is* FG.

Proof of Lemma 5.3. Set $a = a_r$. Let $k = k_a$ the T-index of a. Three mutually exclusive cases arise:

Case I. $k = 0$,

Case II. $k > 0$, $q \neq k$,

Case III. $k > 0$, $q = k$.

Proof in Case I. In this case F_a is the union of two disjoint closed sets $\sigma = p_a$ and $\dot{F}_a$. Lemma 5.2 follows trivially in Case I.

Proof in Case II. J_q^a is a homomorphism. According to Corollary 4.2 (ii) J_q^a maps $\dot{H}_q^a$ onto H_q^a when $q \neq k$. A finite base for $\dot{H}_q^a$, accordingly goes under J_q^a onto a set of generators of H_q^a. Thus H_q^a is FG in Case II.

Case III. We shall make use of Theorem 4.4, distinguishing between the cases in which a torsion index t^a of p_a exists and does not exist.

According to (i) of Theorem 4.4, when t^a *exists*, an arbitrary k-cycle e_+^k on F_a is such that

$$e_+^k \sim m\lambda_a \quad (\text{on } F_a \operatorname{mod} \dot{F}_a) \tag{5.4}$$

for some integer m, or equivalently

$$e_+^k = m\lambda_a + \partial e_+^{k+1} + e_-^k \tag{5.5}$$

for suitable chains e_+^{k+1} and e_-^k on F_a and $\dot{F}_a$, respectively. It follows from (5.5) that e_-^k is a k-cycle on $\dot{F}_a$. We draw the following conclusion from (5.5): if $y_1, \ldots, y_r$ is a prebase for $\dot{H}_k^a$ the homology classes (Def. 3.0)

$$\big((\lambda_a, F_a)\big), \big((y_1, F_a)\big), \ldots, \big((y_r, F_a)\big) \tag{5.6}$$

generate H_k^a.

According to (ii) of Theorem 4.4 when t^a *does not exist*, an arbitrary k-cycle e_+^k on F_a is such that

$$e_+^k \sim 0 \quad (\text{on } F_a \operatorname{mod} \dot{F}_a) \tag{5.7}$$

and one concludes again that H_k^a is FG.

Thus Lemma 5.3 *is true and, together with Lemma* 5.2, *implies Theorem* 5.2. *Theorem* 5.1 *follows, as stated previously.*

We add a theorem which is related to the theorems of this section but which is not needed to prove Theorem 5.1.

THEOREM 5.3. *If c is any value of F on M_n and if $q > n$, the homology group $H_q(F_c, Z)$ is trivial.*

It follows from Lemma 5.1 that Theorem 5.3 is true if and only if each homology group in the sequence (5.1) is trivial for $q > n$.

The isomorphism (5.3) is valid for any T-critical value $a_r > a_0$. Moreover, inclusion induced isomorphisms,

$$\dot{H}_q^{a_r} \rightleftarrows H_q^{a_r} \quad (q > n) \tag{5.8}$$

are valid by (iii) of Corollary 4.2. For $q > n$ each homology group in the sequence

$$H_q^{a_0}\colon\ \dot{H}_q^{a_1}, H_q^{a_1}\colon\ \ldots\colon\ \dot{H}_q^{a_\nu}, H_q^{a_\nu} \tag{5.9}$$

is accordingly trivial, since $H_q^{a_0}$ is trivial.

Theorem 5.3 follows.

Part III. Critical invariants of T-critical points p_a

§ 6. Program for Part III. We are concerned with relations between successive groups in the sequence

$$H_q^{a_0}\colon\ \dot{H}_q^{a_1}, H_q^{a_1}\colon\ \dot{H}_q^{a_2}, H_q^{a_2}\colon\ \ldots\colon\ \dot{H}_q^{a_\nu}, H_q^{a_\nu}. \tag{6.1}$$

Here q is on the range $0, 1, \ldots$ Groups in this sequence, which are separated by a colon, admit an inclusion induced isomorphism of form (5.3). Let a be any one of the T-critical values $a_1, \ldots, a_\nu$. In Definition 3.2 we have introduced an inclusion induced homomorphism

$$J_q^a\colon\ \dot{H}_q^a \to H_q^a \quad (a > a_0;\ q = 0, 1, \ldots). \tag{6.2}$$

It follows from Corollary 4.2 (iii) that J_q^a is an isomorphism if q is neither k_a nor $k_a - 1$. In § 7 our attention will be restricted to the case, $q = k_a$, while in § 9 we shall study the case, $q = k_a - 1$.

Each group H_q^a is a direct sum

$$H_q^a = \mathfrak{B}_q^a \oplus \mathfrak{J}_q^a \tag{6.3}$$

of its "torsion subgroup" $\mathfrak{J}_q^a$ and a complementary free subgroup $\mathfrak{B}_q^a$, termed a "Betti subgroup" of H_q^a. The decomposition (6.3) is possible, since the group H_q^a has been shown to be FG. We shall similarly represent $\dot{H}_q^a$ as a direct sum

$$\dot{H}_q^a = \dot{\mathfrak{B}}_q^a \oplus \dot{\mathfrak{J}}_q^a \tag{6.4}$$

of its torsion subgroup and a complementary Betti subgroup $\dot{\mathfrak{B}}_q^a$.

Of particular interest is the sequence,

$$\mathfrak{B}_q^{a_0}\colon\ \dot{\mathfrak{B}}_q^{a_1}, \mathfrak{B}_q^{a_1}\colon\ \ldots\colon\ \dot{\mathfrak{B}}_q^{a_\nu}, \mathfrak{B}_q^{a_\nu}, \tag{6.5}$$

of Betti subgroups and the sequence,

$$\mathfrak{J}_q^{a_0}\colon\ \dot{\mathfrak{J}}_q^{a_1}, \mathfrak{J}_q^{a_1}\colon\ \ldots\colon\ \dot{\mathfrak{J}}_q^{a_\nu}, \mathfrak{J}_q^{a_\nu}, \tag{6.6}$$

of torsion subgroups. Groups separated by a colon are isomorphic because

(5.3) holds. Groups in these sequences separated by a comma are isomorphic, except at most when $q = k_a$ or $k_a - 1$, in accord with Corollary 4.2.

The dimension of a Betti group $\mathfrak{B}_q^a$ is termed the qth *Betti number* of F_a. Similarly the dimension of a Betti group $\dot{\mathfrak{B}}_q^a$ is termed the qth Betti number of $\dot{F}_a$. The *torsion coefficients* of the groups $\mathfrak{J}_q^a$ and $\dot{\mathfrak{J}}_q^a$ are defined in the usual way and termed torsion coefficients of dimension q of F_a and $\dot{F}_a$, respectively. We shall show how these invariants change (if at all) as one passes from $\dot{F}_a$ to F_a. The only dimensions q for which there can be a change as one passes from $\dot{F}_a$ to F_a are the dimensions $q = k_a$ or $k_a - 1$.

Critical invariants. Program. The data needed to determine the changes in the torsion coefficients and Betti numbers as one passes from $\dot{F}_a$ to F_a are certain integers termed *critical invariants*. They are associated with each T-critical point p_a.

If $k = k_a > 0$ and one compares $\dot{\mathfrak{B}}_k^a$ with $\mathfrak{B}_k^a$ and $\dot{\mathfrak{J}}_k^a$ with $\mathfrak{J}_k^a$ (as in § 7) the critical invariants are the T-index $k = k_a$ and the torsion index t^a of p_a, introduced in Definition 4.4. If $k_a = 0$ a torsion index t^a is not defined.

If $k = k_a > 0$ and one compares $\dot{\mathfrak{B}}_{k-1}^a$ with $\mathfrak{B}_{k-1}^a$ and $\dot{\mathfrak{J}}_{k-1}^a$ with $\mathfrak{J}_{k-1}^a$ (as in § 9) the above critical invariants must be supplemented by other critical invariants, including integers $s^a \geqslant 0$ defined in § 9. As will be seen, these invariants are determined for each T-critical point p_a for which $k = k_a > 0$ by the critical homology classes $\pm((\partial \varkappa_a, \dot{F}_a))$ in $H_{k-1}(\dot{F}_a, \boldsymbol{Z})$, introduced in § 4. Here $\varkappa_a$ is the "saddle k-cell" of Definition 4.3. The integers s^a were first defined in [14].

It follows from the definition of each of these so-called "critical invariants" that they are unchanged if M_n is mapped by a homeomorphism h onto another manifold $\hat{M}_n$ and F replaced by a mapping $\hat{F}$ of $\hat{M}_n$ into R such that $F(p) = \hat{F}(q)$ when $q = h(p)$.

§ 7. From $\dot{H}_k^a$ to H_k^a; $a > a_0$; $k = k_a$. Use will be made of the inclusion-induced homomorphism

$$J_k^a\colon \dot{H}_k^a \to H_k^a \qquad \text{(Definition 3.2)} \tag{7.1}$$

in case $a > a_0$ and $k = k_a$. As we shall see in Theorem 7.2, the homomorphism (7.1) may not be an isomorphism. However, the restriction $J_k^a|\dot{\mathfrak{J}}_k^a$ is an isomorphism, as the following theorem explicitly affirms.

THEOREM 7.1. *If a is a T-critical value of F such that $a > a_0$, then for $k = k_a$, the torsion subgroup $\dot{\mathfrak{J}}_k^a$ of $\dot{\mathcal{H}}_k^a$ is mapped isomorphically onto the torsion subgroup $\mathfrak{J}_k^a$ of H_k^a by the restriction $\hat{J}_k^a$ of J_k^a to $\dot{\mathfrak{J}}_k^a$.*

If $\dot{\mathfrak{J}}_k^a$ is trivial we understand the theorem to affirm that $\mathfrak{J}_k^a$ is trivial.

Since J_k^a is a homomorphism, $\hat{J}_k^a$ is a homomorphism

$$\hat{J}_k^a\colon \dot{\mathfrak{J}}_k^a \to \mathfrak{J}_k^a \tag{7.2}$$

into $\mathfrak{J}_k^a$. By hypothesis of this section $k = k_a$ and $a > a_0$. Hence $\ker J_k^a = 0$ by Corollary 4.2 (i), so that $\ker \hat{J}_k^a = 0$, and $\hat{J}_k^a$ is an injective isomorphism. It remains to show that $\hat{J}_k^a$ maps $\dot{\mathfrak{J}}_k^a$ *onto* $\mathfrak{J}_k^a$. To that end we prove the following.

(α) *Corresponding to each nontrivial k-cycle e_+^k on F_a such that*

$$((e_+^k, F_a)) \in \mathfrak{J}_k^a \qquad (k = k_a,\ a > a_0) \tag{7.3}$$

there exists a k-cycle e_-^k on $\dot{F}_a$ such that $e_+^k \sim e_-^k$ on F_a.

The case $k = k_a = 0$. In this case $\mathfrak{J}_a^k$ is trivial, so that (α) is trivially true.

The case $k = k_a > 0$. Since $k_a > 0$, Theorem 4.4 can be applied to prove (α). Two subcases are distinguished.

Case I. $k_a > 0$ and a torsion index t^a of p_a fails to exist.

Case II. $k_a > 0$ and a torsion index t^a of p_a exists.

Proof in Case I. In this case (4.17) of Theorem 4.4 (ii) holds and implies that $e_+^k \sim e_-^k$ on F_a for some k-cycle e_-^k on $\dot{F}_a$. Thus (α) is true in Case I.

Proof in Case II. In Case II it follows from Theorem 4.4 (i) that (4.16) holds. By hypothesis (7.3), $re_+^k \sim 0$ on F_a for some non-null integer r. The homology (4.16) then shows that

$$0 \sim rm\lambda_a \qquad (\text{on } F_a \operatorname{mod} \dot{F}_a)\,. \tag{7.4}$$

However, (ii) of Lemma 4.3 implies that (7.4) is possible only if $m = 0$. From (4.16), with $m = 0$ therein, we infer that $e_+^k \sim e_-^k$ on F_a for some k-cycle e_-^k on $\dot{F}_a$. Thus (α) is true in Case II, as well as in Case I.

Completion of proof that $\hat{J}_k^a$ maps $\dot{\mathfrak{J}}_k^a$ onto $\mathfrak{J}_k^a$. We refer to e_+^k and e_-^k of (α). Since J_k^a is induced by the inclusion mapping of $\dot{F}_a$ into F_a,

$$J_k^a((e_-^k, \dot{F}_a)) = ((e_-^k, F_a)) \qquad (\text{by } (3.10))\,. \tag{7.5$'$}$$

One infers from (α) that

$$((e_-^k, F_a)) = ((e_+^k, F_a)) \in \mathfrak{J}_k^a\,. \tag{7.5$''$}$$

Since $\ker J_k^a = 0$, it follows from (7.5) that $((e_-^k, \dot{F}_a)) \in \dot{\mathfrak{J}}_k^a$.

Thus $J_k^a(\dot{\mathfrak{J}}_k^a) = \mathfrak{J}_k^a$ so that Theorem 7.1 is true.

We state the following corollary of Theorem 7.1.

COROLLARY 7.1. *If $a > a_0$ is a T-critical value of F with T-index $k = k_a$, then H_k^a is free if $\dot{H}_k^a$ is free. More generally, the torsion coefficients of $\dot{H}_k^a$ and H_k^a (if any exist) are identical.*

Betti subgroups. It remains to show how Betti subgroups $\dot{\mathfrak{B}}_k^a$ of $\dot{H}_k^a$ are related to Betti subgroups $\mathfrak{B}_k^a$ of H_k^a when $a > a_0$ and $k = k_a$.

Notation. To that end let

$$u_1, \ldots, u_\beta \qquad (\beta \geqslant 0) \tag{7.6}$$

be a base for $\mathfrak{B}_k^a$. When $k_a = 0$, let $\boldsymbol{p}_a$ be the 0-cycle with carrier p_a. When $k_a \geqslant 0$ and $a > a_0$, let

$$u_1^{\#}, \ldots, u_\beta^{\#} \tag{7.7}$$

be the images of the respective elements (7.6) under the $\#$-homomorphisms J_k^a: (7.1).

If $k_a > 0$ and if a torsion index t^a of p_a exists, a t^a-fold linking k-cycle λ_a exists in accord with Definition 4.5. Consider the homology class $((\lambda_a, F_a))$ of λ_a on F_a. The second principal theorem of this section follows.

THEOREM 7.2. (i) *When $k = k_a > 0$ and a torsion index t^a of p_a fails to exist, the elements* (7.7) *give a base for a Betti group $\mathfrak{B}_k^a$ of H_k^a.*

(ii) *When $k = k_a > 0$ and a torsion index t^a of p_a exists, the elements*

$$((\lambda_a, F_a)), u_1^{\#}, \ldots, u_\beta^{\#}, \tag{7.8}$$

give a base for a Betti group $\mathfrak{B}_k^a$.

(iii) *When $k = k_a = 0$ and $a > a_0$ the elements,*

$$((\boldsymbol{p}_a, F_a)), u_1^{\#}, \ldots, u_\beta^{\#}, \tag{7.9}$$

give a base for a Betti group $\mathfrak{B}_k^a = \mathfrak{B}_0^a$.

Proof of (i). Statement (i) is a consequence of the following lemma.

LEMMA 7.1. *When $k = k_a > 0$ and no torsion index t^a of p_a exists, the homomorphism J_k^a: (7.1) is an isomorphism.*

Under the hypotheses of Lemma 7.1, Theorem 4.4 (ii) implies that J_k^a is surjective. Moreover $\ker J_k^a = 0$ when $a > a_0$ and $k = k_a$ by Corollary 4.2 (i), so that J_k^a is an isomorphism when t^a fails to exist. The lemma follows and implies (i) of Theorem 7.2.

Proof of (ii). Let $\hat{H}_k^a$ denote the image of $\dot{H}_k^a$ under J_k^a. Since $\ker J_k^a = 0$ the homomorphism (1)

$$J_k^a: \dot{H}_k^a \to \hat{H}_k^a \tag{7.10}$$

is an isomorphism, so that the set of elements (7.7) is a base for a Betti subgroup $\hat{\mathfrak{B}}_k^a$ of $\hat{H}_k^a$.

Since a torsion index t^a of p_a exists by hypothesis of (ii), a t^a-fold linking k-cycle λ_a exists. The relation

$$H_k^a = \{((\lambda_a, F_a)), \hat{H}_k^a\}\ (^2) \tag{7.11}$$

follows from (4.16) and the isomorphism (7.10). That is, when t^a exists, $((\lambda_a, F_a))$ and $\hat{H}_k^a$ generate H_k^a.

(1) Strictly, not J_k^a, but a homomorphism, say $\hat{J}_k^a$ induced by J_k^a.

(2) If an Abelian group A is generated by the elements in subsets $A_1, \ldots, A_m$ of A, one writes $A = \{A_1, \ldots, A_m\}$.

Statement (ii) follows if H_k^a is a direct sum,

$$H_k^a = \{((\lambda_a, F_a))\} \oplus \hat{H}_k^a. \tag{7.12}$$

That (7.12) is true is a consequence of (7.11) and (ii) of Lemma 4.3, which implies that for no integer $\mu \neq 0$ is $\mu((\lambda_a, F_a))$ an element in $\hat{H}_k^a$.

Proof of (iii). Statement (iii) is consequence of the fact that when $k = k_a = 0$ and $a > a_0$, F_a is the union of disjoint closed sets, p_a and $\dot{F}_a$. This completes the proof of Theorem 7.2.

§ 8. Elementary group quotients A/W. Certain general theorems on Abelian groups presented in [16] will be recalled and will be applied in § 9. For references to relevant books on groups see [16].

OBJECTIVES OF § 8. *There is given a FG Abelian group A, together with a cyclic subgroup $W = \{w\}$ of A generated by an element $w \in A$. One seeks to determine the Betti numbers and torsion coefficients of A/W in terms of minimal data on A and W. We shall describe such data.*

Recall that Abelian group A which is FG is a direct sum

$$A = \mathfrak{B} \oplus \mathfrak{J} \tag{8.1}$$

of its uniquely determined *torsion* subgroup $\mathfrak{J}$ and a "complementary" free subgroup $\mathfrak{B}$ of A, termed a Betti subgroup of A. $\mathfrak{B}$ has a base

$$u_1, \ldots, u_\beta \quad (\dim \mathfrak{B} = \beta, \text{ possibly } 0) \tag{8.2}$$

consisting of β elements of A, every non-trivial linear combination of which (over $\boldsymbol{Z}$) has an infinite order in A. In general $\mathfrak{B}$ is not uniquely determined by $\mathfrak{J}$, nor the base (8.2) uniquely determined by $\mathfrak{B}$. However the number β is independent of the choice of the free group complementary to $\mathfrak{J}$ and of the choice of a base of $\mathfrak{B}$. We term β the *Betti number* of A.

The torsion coefficients of $\mathfrak{J}$. It is a classical theorem that a finite, non-trivial Abelian group, $\mathfrak{J}$, is a direct sum of a finite set of cyclic subgroups which can be canonically arranged so as to have orders $q_1, q_2, \ldots, q_m$ exceeding 1, each of which, except q_m, is divisible by its successor. These integers are uniquely determined by $\mathfrak{J}$ and are termed the *torsion coefficients* (1) of $\mathfrak{J}$.

Elementary divisors ED **of** $\mathfrak{J}$. If is known that a finite, non-trivial group, $\mathfrak{J}$, is a direct sum $g_1 \oplus \ldots \oplus g_e$ of cyclic subgroups g_i such that the order of g_i is a power $p_i^{e_i}$ of a prime p_i and g_i is a subgroup of no cyclic subgroup of $\mathfrak{J}$ whose order is a higher power of p_i. Such a direct sum is

(1) When $\mathfrak{J}$ is the torsion subgroup of A, torsion coefficients and ED of $\mathfrak{J}$ will be called torsion coefficients and ED of A.

called a *cyclic primary decomposition* (abbreviated CPD) of $\mathfrak{J}$. The prime powers

$$p_1^{e_1}, \dots, p_\varrho^{e_\varrho} \quad (e_i > 0;\ i = 1, \dots, \varrho) \tag{8.3}$$

which are the orders of the respective summands in a CPD of $\mathfrak{J}$ are called *elementary divisors*, ED of $\mathfrak{J}$. The ED of $\mathfrak{J}$ are said to be *normally arranged* if $p_1 \geqslant p_2 \geqslant \dots \geqslant p_r$ and if, when $p_i = p_{i+1}$, then $e_i \geqslant e_{i+1}$. $\mathfrak{J}$ uniquely determines a set of normally ordered ED's.

We state a classical lemma.

LEMMA 8.1. *Canonically ordered torsion coefficients of a* FG *non-trivial Abelian group* $\mathfrak{J}$, *determine and are uniquely determined by normally ordered* ED *of* $\mathfrak{J}$. (See [8], p. 147.)

DEFINITION 8.1. *A basis of a* FG *Abelian group* A. Suppose that A has a torsion subgroup with a CPD

$$\mathfrak{J} = \{x_1\} \oplus \dots \oplus \{x_\varrho\} \quad (x_i \in \mathfrak{J})\,. \tag{8.4}$$

Let $\mathfrak{B}$ be a Betti subgroup of A with a base $(u_1, \dots, u_\beta)$. The set of elements

$$u_1, \dots, u_\beta;\ x_1, \dots, x_\varrho \tag{8.5}$$

of A is called a *basis* for A.

An arbitrary element $w \in A$ has the form,

$$w = \mu_1 u_1 + \dots + \mu_\beta u_\beta + m_1 x_1 + \dots + m_\varrho x_\varrho \tag{8.6}$$

where μ_i is an integer uniquely determined by w and the choice of the basis (8.5), while each m_j is uniquely determined by w and the choice of the CPD (8.4), provided m_j is restricted to integral values such that

$$0 \leqslant m_j < \operatorname{order} x_j \quad (j = 1, 2, \dots, \varrho)\,. \tag{8.7}$$

Minimal data on A and W. In I, II, III, IV we present data adequate for meeting the objectives of § 8 outlined above. These data follow.

I. A basis of A (Definition 8.1) of form,

$$u_1, \dots, u_\beta;\ x_1, \dots, x_\varrho\,.$$

II. A normally ordered set

$$n_1, \dots, n_\varrho \tag{8.8}$$

of ED of A of form (8.3).

III. A generator w of the cyclic subgroup (¹) W of A and a profile of w, that is, a set

$$\mu_1, \dots, \mu_\beta;\ m_1, \dots, m_\varrho \tag{8.9}$$

(¹) We term W the *critical* cyclic subgroup of A.

of coefficients in an admissible representation (8.6) of w, subject to (8.7).

IV. An integer $s \geqslant 0$, termed the *free index* of W, defined as the GCD of the integers $\mu_1, \dots, \mu_\beta$ of (8.9), zero, if these integers all vanish.

We shall give another, but equivalent, definition of s.

DEFINITION 8.2. *The free index s of W.* We set $s = 0$ if and only if W has a finite order. If W has an infinite order, s is finite and positive, with a value defined by the following lemma.

LEMMA 8.2. *With a "critical" cyclic subgroup W of A of infinite order there can be associated a positive integer s which is unique among positive integers with the following property.*

If w is an arbitrary generator of W and B an arbitrary Betti subgroup of A, there exists a basis of B with a first element u_B such that

$$w = s u_B \quad (\operatorname{mod} \mathfrak{J})\ (^1)\,. \tag{8.10}$$

Lemma 8.2 follows from Lemma 3.1 of [16] and its proof.

Theorem 3.3 of [16] gives a first indication of the meaning of the free index s of W. It may be restated as follows.

THEOREM 8.0. *Suppose that A is torsion free and that the free index of W is s. Then A/W is torsion free unless $s > 1$, and when $s > 1$, the first and only torsion coefficient of A/W is s.*

It will be noted that the minimal data contained in I and II depend upon A alone, while the data contained in III and IV depend upon both A and W. In our application of this section in § 9, A and W will be replaced by $\dot{H}^a_{k-1}$ and W^a_{k-1}, respectively, where W^a_{k-1} is the critical cyclic subgroup of $\dot{H}^a_{k-1}$ to be introduced in Definition 9.1. The analogue of the integers m_i in III and s in IV, thereby appearing in § 9, will be called "critical invariants" of the T-critical point p_a.

In the first of two principal theorems of this section we relate the Betti number of A to that of A/W. In the second of our two principal theorems we show that the above minimal data on A and W enable us to evaluate the torsion coefficients of A/W. The minimal data on A include the normally ordered ED of A, or equivalently by Lemma 8.1, the torsion coefficients of A.

THEOREM 8.1 (i) *If the free index s of W is positive, A/W has a Betti number one less than that of A.*

(ii) *If the free index $s = 0$, A/W has a Betti number equal to that of A.*

Theorem 8.1 (i) is included in Theorem 3.2 of [16]. Theorem 8.1 (ii) is included in Lemma 4.1 of [16].

(¹) Given x and y in A we write $x = y \operatorname{mod} \mathfrak{J}$ if $x - y \in \mathfrak{J}$, where $\mathfrak{J}$ is the torsion subgroup of A.

To formulate Theorem 8.2 we introduce a $\varrho+1$ square matrix

$$\|a_{ij}\| = \left\| \begin{matrix} n_1 & & & & \\ & n_2 & & & \\ & & \ddots & & \\ & & & n_\varrho & \\ m_1 & m_2 & \ldots & m_\varrho & s \end{matrix} \right\|$$

in which the elements in the diagonal are the ED (8.8) of $\mathfrak{J}$ followed by the free index s of W. The elements $m_1, \ldots, m_\varrho$ in the last row are taken from the *profile* of a generator w of W, as presented in (8.9). Elements in the matrix $\|a_{ij}\|$, other than those in the diagonal and last row, are zero. The integers m_j are subject to the condition (8.7). The rank of this matrix is $\varrho+1$ or ϱ, according as $s>0$ or $s=0$.

The second principal theorem of this section follows.

THEOREM 8.2. *The invariant factors exceeding* 1 *of the above* $(\varrho+1)$-*square matrix* $\|a_{ij}\|$, *if properly ordered, give the torsion coefficients of* A/W.

Theorem 8.2 is proved in [16]. It is formulated separately in [16] as Corollary 3.1, when $s>0$, and Corollary 4.1 when $s=0$.

We add Theorem 3.1 of [16].

THEOREM 8.3. *If the free index* s *of* W *is* 1 *the torsion subgroup of* A/W *is isomorphic to the torsion subgroup of* A *and the Betti number of* A/W *is one less than that of* A.

§ 9. From $\dot H^a_{k-1}$ to H^a_{k-1}: $k=k_a>0$. We shall apply the theorems of § 8 on A/W, setting

$$A = \dot H^a_{k-1};\; W = W^a_{k-1} \qquad (k=k_a>0) \tag{9.1}$$

where W^a_{k-1} is a cyclic subgroup of $\dot H^a_{k-1}$ now to be defined.

DEFINITION 9.1. *The critical cyclic subgroup* W^a_{k-1} *of* $\dot H^a_{k-1}$. Let $\varkappa_a$ be a saddle k-cell of p_a (Definition 4.3). Then $\partial\varkappa_a$ is a $(k-1)$-cycle on $\dot F_a$ whose carrier is a topological $(k-1)$-sphere. We shall set

$$W^a_{k-1} = \{((\partial\varkappa_a, \dot F_a))\} \qquad (k=k_a>0) \tag{9.2}$$

and term W^a_{k-1} the *critical* cyclic subgroup of $\dot H^a_{k-1}$. According to (4.8) of Corollary 4.1, the pair of homology classes $\pm((\partial\varkappa_a, \dot F_a))$ in $\dot H^a_k$ is independent of the choice of $\varkappa_a$ as a saddle k-cell of p_a.

A principal property of a saddle k-cell $\varkappa_a$ on F_a is that, taken $\operatorname{mod}\dot F_a$, it is a rel. k-cycle and, as such, is a prebase for the rel. homology group $H^a_k(F_a, \dot F_a, \boldsymbol{Z})$. Cf. Theorem 4.3 (ii).

We shall prove the following theorem.

THEOREM 9.0. *The critical cyclic subgroup* W^a_{k-1} *of* $\dot H^a_{k-1}$ *is the kernel of the inclusion induced* $\#$-*homomorphism*,

$$J^a_{k-1}\colon \dot H^a_{k-1} \to H^a_{k-1},$$

of Definition 3.2.

Proof of Theorem 9.0. We must show that both of the inclusions (a) $W^a_{k-1} \subset \ker J^a_{k-1}$; (b) $\ker J^a_{k-1} \subset W^a_{k-1}$ are valid.

Proof of (a). It suffices to show that a generator

$$w = ((\partial\varkappa_a, \dot F_a)) \text{ of } W^a_{k-1} \qquad (\text{see } (9.2))$$

annihilates J^a_{k-1}. Now w and $J^a_{k-1}(w)$, by definition are the homology classes of $\partial\varkappa_a$ on $\dot F_a$ and F_a, respectively. Since $\partial\varkappa_a \sim 0$ on F_a, $w \in \ker J^a_{k-1}$. The inclusion (a) is thus valid.

Proof of (b). It suffices to show that if a $(k-1)$-cycle a^{k-1}_- on $\dot F_a$ bounds on F_a, then

$$((a^{k-1}_-, \dot F_a)) \in W^a_{k-1}. \tag{9.3}$$

Proof of (9.3). By hypothesis on a^{k-1}_-, there exists a k-chain e^k_+ on F_a such that

$$\partial e^k_+ = a^{k-1}_-. \tag{9.4}$$

The k-chain e^k_+ is thus a k-cycle on $F_a \operatorname{mod} \dot F_a$. As a prebase of $H_k(F_a, \dot F_a, \boldsymbol{Z})$, $\varkappa_a \operatorname{mod}\dot F_a$ is such that there exists an integer μ, a chain e^{k+1}_+ on F_a and a chain e^k_- on $\dot F_a$ such that

$$e^k_+ = \mu\varkappa_a + \partial e^{k+1}_+ + e^k_-. \tag{9.5}$$

The application of ∂ to both members of (9.5) shows (with the aid of (9.4)) that

$$a^{k-1}_- = \mu\partial\varkappa_a + \partial e^k_- \tag{9.6}$$

from which (9.3) follows.

Thus the inclusion (b), as well as the inclusion (a) holds, and Theorem 9.0 follows.

Theorem 9.0 has the following corollary.

COROLLARY 9.1. *The natural homomorphism*

$$\dot H^a_{k-1}/W^a_{k-1} \to H^a_{k-1} \qquad (k=k_a>0) \tag{9.7}$$

induced by J^a_{k-1} *is an isomorphism.*

That the mapping (9.7) is biunique follows from Theorem 9.0. That it is surjective follows from (ii) of Corollary 4.2. Thus Corollary 9.1 is true.

Applications of the theorems on W/A of § 8. The objective of § 9 is to relate the invariants (¹) of $\dot{H}^a_{k-1}$ to the corresponding invariants of H^a_{k-1}, understanding always that $k = k_a > 0$. The special case in which $k = k_a = 0$ is considered in § 7 but not in § 9. See scheme outlined in § 6. Here as elsewhere a is a critical value of F. In § 9, $a > a_0$.

Suppose then that $k = k_a > 0$. In this case the sequence of homology groups

$$\dot{H}^a_{k-1}, \quad \dot{H}^a_{k-1}/W^a_{k-1}, \quad H^a_{k-1} \tag{9.8}$$

will serve our purpose of relating $\dot{H}^a_{k-1}$ to H^a_{k-1}, with $\dot{H}^a_{k-1}/W^a_{k-1}$ serving as a mediator, A/W, between $\dot{H}^a_{k-1}$ and H^a_{k-1}. According to Corollary 9.1, the last two groups in the sequence (9.8) are isomorphic. The data needed to relate the invariants of the first two groups in the sequence (9.8) include the following.

DEFINITION 9.2. *The free index* s^a. When $k = k_a > 0$ one identifies $\dot{H}^a_{k-1}$ and W^a_{k-1} with A and W of § 8. The *free index* of $W = W^a_{k-1}$, given by Definition 8.2, is denoted by s^a. There are two cases. In both cases $k = k_a > 0$.

Case I. Order W^a_{k-1} finite. In this case $s^a = 0$ and t^a exists by Definitions 8.2 and 4.4 respectively.

Case II. Order W^a_{k-1} infinite. In this case s^a is the finite positive value s associated with $W = W^a_{k-1}$ in Lemma 8.2. A torsion index t^a fails to exist by Definition 4.4.

Theorem 8.0 gives the following first indication of the meaning of s^a when $k = k_a > 0$.

THEOREM 9.1. *Suppose that* $k = k_a > 0$ *and that* $\dot{H}^a_{k-1}$ *is torsion free. Then* H^a_{k-1} *is torsion free unless* $s^a > 1$ *and when* $s^a > 1$, *has a unique torsion coefficient* s^a.

Proof. Theorem 8.0 implies that when $k = k_a > 0$ and $\dot{H}^a_{k-1}$ is torsion free, then $\dot{H}^a_{k-1}/W^a_{k-1}$ and hence its isomorph H^a_{k-1} (Corollary 9.1) is torsion free unless $s^a > 1$, and when $s^a > 1$, has a unique torsion coefficient s^a.

With a similar use of Corollary 9.1 and of the three groups (9.8) we infer the following from Theorem 8.1. Here $k = k_a > 0$.

THEOREM 9.2 (i). *If the free index* s^a *of* W^a_{k-1} *is positive,* H^a_{k-1} *has a Betti number which is one less than that of* $\dot{H}^a_{k-1}$.

(ii) *If the free index* $s^a = 0$, H^a_{k-1} *has a Betti number equal to that of* $\dot{H}^a_{k-1}$.

Similarly if $\varkappa_a$ is a "saddle k-cell" of p_a (Definition 4.3), Theorem 8.2 implies the following.

(¹) By the invariants of a finitely generated Abelian group we have meant its Betti number and torsion coefficients. See Lemma 8.1.

THEOREM 9.3. *If* $n_1, \ldots, n_\varrho$ *are the elementary divisors of* $\dot{H}^a_{k-1}$, *if the integers* $m_1, \ldots, m_\varrho$ *are taken from the profile* (8.9) *of a generator* $((\partial\varkappa_a, \dot{F}_a))$ *of* W^a_{k-1} *and if* $s = s^a$ *is the free index of* W^a_{k-1}, *then the torsion coefficients of* H^a_{k-1} *are such of the invariant factors of the matrix* $\|a_{ij}\|$ *of* § 8 *as exceed* 1.

Theorem 8.3 similarly implies the following.

THEOREM 9.4. *If* $k = k_a > 0$ *and if the free index* s^a *of* $\dot{W}^a_{k-1}$ *is* 1, *the torsion subgroup of* $\dot{H}^a_{k-1}$ *is isomorhpic to the torsion subgroup of* H^a_{k-1}, *and the* $(k-1)$*st Betti number of* F_a *is one less than that of* $\dot{F}_a$.

§ 10. The existence of T-critical points. We are supposing that M_n is a compact, connected topological n-dimensional manifold upon which a TND function F is defined with critical values

$$a_0 < a_1 < \ldots < a_\nu, \tag{10.0}$$

of singleton type. If a is any one of these values, p_a denotes the corresponding T-critical point. As we have seen, the T-index $k = k_a$ of p_a is on the range $0, 1, \ldots, n$. Let m_k be the number of T-critical points with T-index k (Definition 1.2). For $q = 0, 1, \ldots$, let $\beta_q(M_n)$ be the qth Betti number of M_n.

We shall prove the following. Cf. Theorem 30.1 of [15].

THEOREM 10.1 (i) *The Betti numbers* $\beta_q(M_n)$ *are finite and vanish for* $q > n$.

(ii) *The following relations are valid:*

$$\begin{aligned} m_0 &\geqslant \beta_0(M_n), \\ m_1 - m_0 &\geqslant \beta_1(M_n) - \beta_0(M_n), \\ m_2 - m_1 + m_0 &\geqslant \beta_2(M_n) - \beta_1(M_n) + \beta_0(M_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ m_n - m_{n-1} + m_{n-2} - \ldots (-1)^n m_0 &= \beta_n(M_n) - \beta_{n-1}(M_n) + \ldots (-1)^n \beta_0(M_n). \end{aligned} \tag{10.1}$$

(iii) *The relations* (10.1) *imply the inequalities,*

$$m_k \geqslant \beta_k(M_n) \qquad (k = 0, \ldots, n). \tag{10.2}$$

Proof of (i). $\beta_q(M_n)$ is finite, since $H^{a_\nu}_q$ is FG (Theorem 5.1). Moreover $\beta_q(M_n) = 0$ when $q > n$, by Theorem 5.3.

The following two lemmas aid in proving the relations (10.1) of (ii).

LEMMA 10.1. *Let* a *be a* T*-critical value of a* T*-critical point* p_a *of positive index* $k = k_a$. *Then*

$$\beta_{k-1}(F_a) - \beta_{k-1}(\dot{F}_a) = 0 \text{ or } -1, \tag{10.3}$$

$$\beta_k(F_a) - \beta_k(\dot{F}_a) = 1 \text{ or } 0, \tag{10.4}$$

according as $s^a = 0$ *or* $s^a > 0$. *Moreover*

$$\beta_q(F_a) = \beta_q(\dot{F}_a) \quad (k \neq q \text{ or } q+1)\,. \tag{10.5}$$

Theorem 9.2 implies (10.3).

According to Definition 9.2 when $k = k_a > 0$, $s^a = 0$ or $s^a > 0$, according as the torsion index t^a exists or fails to exist. With this understood Theorem 7.2, (i) and (ii), imply (10.4). Moreover (10.5) follows from Corollary 4.2 (iii). Thus Lemma 10.1 is true.

A second lemma is needed to prove the relation (10.1). It makes use of integers a_q and γ_q defined for $q = 0, 1, 2, \ldots$ as follows.

I. *When* $q > n$, $a_q = \gamma_q = 0$.

II. *When* $q = 1, 2, \ldots, n$, a_q *and* γ_q equal the number of T-critical points p_a of T-index q with $s^a = 0$ and $s^a > 0$, respectively.

III. *When* $q = 0$, a_q is the number of T-critical points with T-index 0, and $\gamma_q = 0$.

Lemma 10.2. *The Betti number*

$$\beta_q(M_n) = a_q - \gamma_{q+1} \quad (q = 0, 1, 2, \ldots)\,. \tag{10.6}$$

Proof of (10.6). We refer to the T-critical values $a_0 < a_1 < \ldots < a_\nu$ of F and, for $i = 1, \ldots, \nu$ and $q = 0, 1, \ldots$ set

$$\Delta_q^i = \beta_q(\dot{F}_{a_i}) - \beta_q(F_{a_{i-1}})\,, \tag{10.7$'$}$$

$$D_q^i = \beta_q(F_{a_i}) - \beta_q(\dot{F}_{a_i}) \tag{10.7$''$}$$

so that

$$\Delta_q^i + D_q^i = \beta_q(F_{a_i}) - \beta_q(F_{a_{i-1}})\,.$$

Hence for $q = 0, 1, 2, \ldots$

$$\beta_q(M_n) - \beta_q(F_{a_0}) = (\Delta_q^1 + D_q^1) + \ldots + (\Delta_q^\nu + D_q^\nu)\,. \tag{10.8}$$

Now each value $\Delta_q^i = 0$ in (10.8), since there exists an F-traction of $\dot{F}_{a_i}$ into $F_{a_{i-1}}$, by Theorem 2.2, and hence an isomorphic mapping of $H_q(\dot{F}_{a_i}, \boldsymbol{Z})$ onto $H_q(F_{a_{i-1}}, \boldsymbol{Z})$, by Theorem 3.1. We infer from (10.8) that

$$\beta_q(M_n) - \beta_q(F_{a_0}) = D_q^1 + D_q^2 + \ldots + D_q^\nu\,. \tag{10.9}$$

The difference D_q^i has the value

$$\beta_q(F_{a_i}) - \beta_q(\dot{F}_{a_i}) \quad (i = 1, 2, \ldots, \nu) \tag{10.10}$$

and by Lemma 10.1, equals 1 when $q = k_{a_i}$ and $s^{a_i} = 0$. It equals -1 when $q+1 = k_{a_i}$ and $s^{a_i} > 0$. Otherwise the difference (10.10) is zero. Relation (10.6) follows.

Proof of (ii) of Theorem 10.1. For each k on the range $0, 1, \ldots, n$, set $\varepsilon_k = m_k - \beta_k(M_n)$. The relations $m_k = a_k + \gamma_k$, are valid for $k = 0, 1, \ldots, n$ and with the relations (10.6) imply that $\varepsilon_k = \gamma_k + \gamma_{k+1}$. Since $\gamma_0 = 0$

$$\varepsilon_k - \varepsilon_{k-1} + \varepsilon_{k-2} \ldots (-1)^k \varepsilon_0 = \gamma_{k+1} \quad (k = 0, 1, \ldots, n). \tag{10.11}$$

The inequalities follow from the relations (10.11). The final equality in (10.1) is a consequence of (10.11) when $k = n$ and the vanishing of γ_{n+1}.

Proof of (iii) of Theorem 10.1. The relations (10.2) are a trivial consequence of the relations (10.1).

Extension I of Theorem 10.1. Theorem 10.1 remains valid if in the formulation of Theorem 10.1 one replaces M_n by F_c, where c is any value of F and m_k denotes the number of critical points of F on F_c of index k. The proof of this extension is similar to the proof given when $F_c = M_n$. In making this extension the main body of the paper is altered only by replacing M_n by F_c.

Extension II. In this extension one drops the condition that the critical values be *singleton*. Theorem 10.1 remains valid. The main body of the paper requires simple but non-trivial modifications in which the critical points at each critical level are ordered.

Extension III. In this extension one replaces $\boldsymbol{Z}$ by an arbitrary field. The resultant homology groups are free. One omits § 8 and the theorems in § 9 that concern torsion groups. This modification is trivial and simple to apply.

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INSTITUTE FOR ADVANCED STUDY
Princeton, N. J.

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P-ideals and F-ideals in rings of continuous functions

by

David Rudd (Norfolk, Virginia)

Abstract. A ring of continuous functions is a ring of the form $C(X)$, the ring of all continuous real-valued functions on a completely-regular Hausdorff space X.

The author defines two classes of ideals in $C(X)$, P-ideals and F-ideals, which are analogs of P-spaces and F-spaces. He then discusses properties of these ideals, such as their structure spaces and zero-sets of their members, and characterizes those spaces X for which there exist P-ideals (or F-ideals) in $C(X)$.

Introduction. If X is a space so that every prime ideal in $C(X)$ is maximal, then X is said to be a P-*space*. We extend this concept to ideals in rings of continuous functions by defining a non-zero ideal I to be a P-ideal if every proper prime ideal in I is a maximal ideal in I. It is known [2, 14.29] that $C(X)$ is a P-ideal, i.e. X is a P-space, if and only if its real structure space (vX) is a P-space. We show that a modified version of this theorem holds for P-ideals. We also characterize those spaces whose rings of continuous functions possess a P-ideal.

If X is a space so that mM $(=\{f|\ f\in fM\})$ is prime for every maximal ideal M in $C(X)$, then X is said to be an F-*space*. We extend this concept also to ideals, by defining a non-zero ideal I to be an F-ideal if mM is prime whenever $M \not\supseteq I$ and M is a maximal ideal in $C(X)$. We are then able to show that I is an F-ideal if and only if its structure space is an F'-space, an analog to the theorem that X is an F-space if and only if βX is an F-space. We are also able to characterize those spaces whose rings of continuous functions possess an F-ideal.

Preliminaries and notations. The reader is referred to section 2 in [4] for most of the preliminaries. Familiarity with [2] is also assumed.

If $f\in C(X)$, then $Z(f)=\{x|\ f(x)=0\}$, $\operatorname{pos} f=\{x|\ f(x)>0\}$, and $\operatorname{neg} f=\{x|\ f(x)<0\}$. If $f\in C^*(X)$ (i.e. f is bounded), then $\hat{f}$ denotes the extension of f to βX. In general $Z(\hat{f})\supseteq Z(f)^\beta$ $\big(=\operatorname{cl}_{\beta X}Z(f)\big)$ but $\operatorname{int}_{\beta X}Z(\hat{f})=\operatorname{int}_{\beta X}Z(f)^\beta$.

We shall use the letter M for maximal ideals of $C(X)$, and $M_x=\{f|\ f(x)=0\}$.

We regard βX as the structure space of $C(X)$. Thus if U is open in βX, $U=\ \sim\{M|\ M\supseteq I\}$ for some ideal I in $C(X)$.