# Core structures for theories *

by

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**Abstract.** A core structure for a theory $T$ is a structure which is isomorphic to exactly one substructure of each model of $T$. Our basic result is a definability characterization of such structures, which implies A. Robinson's result on defining the elements of a core model of a strongly convex theory. A further application is a characterization of the sentences axiomatizing convex theories. Intersections of models and their connections with core structures are also investigated.

This paper is mainly concerned with structures that are isomorphic to exactly one substructure of every model of a theory $T$. Such structures, which we call core structures for $T$, are generalizations of core models of strongly convex theories. The basic result of this paper is Theorem 2.1, which gives a definability characterization of such structures. As an easy corollary we obtain A. Robinson's result on defining the elements of a core model of a strongly convex theory. We also derive other properties of core structures. In Section three we apply Theorem 2.1 to obtain a sort of preservation theorem characterizing the sentences and sets of sentences which define convex theories. Section four concerns intersections of the models of a theory and, in particular, their connection with core structures. Further generalizations are given in section five.

**1. Preliminaries.** In this paper we consider a first-order finitary predicate logic $L$ with identity, which we allow to have any number of non-logical symbols. Structures for $L$ are denoted by $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, and their universes will be $A$, $B$, and $C$ respectively. If $a_0, \ldots, a_n$ are elements of $A$ then $(\mathfrak{A}, a_0, \ldots, a_n)$ is a structure for the language formed by adding $n+1$ new individual constants to $L$.

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The cardinality of a set $X$ will be denoted by $|X|$. By convention, $|L|$ is the cardinality of the set of formulas of $L$. Therefore every structure has some elementary submodel $\mathfrak{A}$ with $|A| \leqslant |L|$.

We write the formula $\varphi$ as $\varphi(x_1, \ldots, x_n)$ in order to indicate that $\varphi$ has at most the variables $x_1, \ldots, x_n$ free. Such a sequence of variables is sometimes abbreviated as $\vec{x}$ to save space, and we will use expressions like $\exists\vec{x}\varphi(\vec{x}, y)$. $\bigvee_{1\leqslant i\leqslant n} \varphi_i$ and $\bigwedge_{1\leqslant i\leqslant n} \varphi_i$ are abbreviations for $\varphi_1\vee\ldots\vee\varphi_n$ and $\varphi_1\wedge\ldots\wedge\varphi_n$ respectively. In addition, for a set $\Phi$ of formulas, $\bigwedge\Phi$ and $\bigvee\Phi$ are the possibly infinite conjunction and disjunction of all the formulas in $\Phi$. If $\Phi = \{\varphi_i\colon i \in I\}$ these will also be written as $\bigwedge_{i\in I} \varphi_i$ and $\bigvee_{i\in I} \varphi_i$. $\exists^{\leqslant n}x\varphi$ and $\exists^{\geqslant n}x\varphi$ are abbreviations for formulas saying "at most $n$ $x$'s satisfy $\varphi$" and "at least $n$ $x$'s satisfy $\varphi$". $\exists^{=n}x\varphi$ is $\exists^{\geqslant n}x\varphi \wedge \exists^{\leqslant n}x\varphi$, and $\exists^{<\omega}x\varphi$ is $\bigvee_{n\in\omega} \exists^{\leqslant n}x\varphi$. Notice that by compactness, if $T \models \exists^{<\omega}x\varphi$ then $T \models \exists^{\leqslant n}x\varphi$ for some $n$. We similarly use $\exists^{=n}\langle y_1, \ldots, y_m\rangle\varphi$ to mean "exactly $n$ sequences $\langle y_1, \ldots, y_m\rangle$ satisfy $\varphi$".

A formula is existential if it is $\exists\vec{x}\alpha$ for some quantifier-free $\alpha$, universal if it is $\forall\vec{x}\alpha$ for some quantifier-free $\alpha$, and $\forall\exists$ if it is $\forall\vec{x}\alpha$ for some existential $\alpha$. Notice that if $\varphi$ is existential then $\exists^{\leqslant n}x\varphi$ is universal and $\exists^{\geqslant n}x\varphi$ is existential. We therefore have the following

LEMMA 1.1. *Assume that $\mathfrak{A} \subseteq \mathfrak{B}$ and that both $\mathfrak{A}$ and $\mathfrak{B}$ are models of $\exists^{=n}x\varphi$ where $\varphi(x)$ is existential. Then*

$$\{a \in A\colon \mathfrak{A} \models \varphi[a]\} = \{b \in B\colon \mathfrak{B} \models \varphi[b]\}.$$

We will at times require structures with special properties and deal with theories satisfying certain model-theoretic conditions. In the remainder of this section we define the properties we will use and state some elementary facts concerning them. For more information see [2] and [9].

We write $\mathfrak{A}\exists\mathfrak{B}$ to mean that $\mathfrak{B}$ satisfies every existential sentence true on $\mathfrak{A}$. $\mathrm{Th}(\mathfrak{C})$, the complete theory of $\mathfrak{C}$, is the set of all sentences true on $\mathfrak{C}$.

DEFINITION. $\mathfrak{A}$ is *sufficiently saturated* if whenever $|B| \leqslant |L|$ and $(\mathfrak{B}, b_0, \ldots, b_n) \exists (\mathfrak{A}, a_0, \ldots, a_n)$ then $\mathfrak{B}$ can be isomorphically embedded in $\mathfrak{A}$ by a map $h$ such that $h(b_i) = a_i$ for all $i = 0, \ldots, n$.

LEMMA 1.2. *Every structure has a sufficiently saturated elementary extension.*

DEFINITION. $\mathfrak{A}$ is a *universal model of* $T$ if $\mathfrak{A}$ is a model of $T$ and every model of $T$ whose universe has cardinality at most $|L|$ can be embedded in $T$.

DEFINITION. $T$ satisfies *joint embedding* if for any two models of $T$ there is some third model of $T$ in which the first two can both be isomorphically embedded.



LEMMA 1.3. *$T$ satisfies joint embedding if and only if it has a universal model.*

DEFINITION. $T$ is *complete for existential* sentences if for any existential sentence $\sigma$ either $T \models \sigma$ or $T \models \neg\sigma$.

A theory that is complete for existential sentences satisfies joint embedding, but not conversely.

Finally, there is the following important concept due to A. Robinson (see [9], p. 80).

DEFINITION. $T$ is *convex* if for any model $\mathfrak{A}$ of $T$ and any collection $\{\mathfrak{B}_i\colon i \in I\}$ of substructures of $\mathfrak{A}$ which are models of $T$, the intersection $\bigcap_{i\in I} \mathfrak{B}_i$ is a model of $T$, provided it is non-empty. If in addition such an intersection is never empty, then $T$ is called *strongly convex*.

Convex theories are those with the following important algebraic property: every non-empty subset of a model of $T$ generates a unique substructure which is a model of $T$ (namely the intersection of all models of $T$ contained in the given model which contain the set). If $T$ is strongly convex then the intersection of all models of $T$ contained in a given model of $T$ is also a model of $T$. This intersection is called a *core model* of $T$ (since every model of $T$ contains a core model, and a core model does not properly contain any models of $T$).

If $T$ satisfies joint embedding in addition to being strongly convex, then the core model of $T$ is unique up to isomorphism. It then has the property (see the proof of Corollary 2.2) that it is isomorphic to exactly one substructure of every model of $T$, and it is uniquely determined as the largest structure with this property. It is this aspect of the core model which we generalize in the next section.

## 2. Core structures.

The following notion of core structure is intended to generalize one aspect of the core model of a strongly convex theory.

DEFINITION. $\mathfrak{C}$ is a *core structure for* $T$ if $\mathfrak{C}$ is isomorphic to exactly one substructure of every model of $T$.

Notice especially that a core structure for $T$ need not be a model of $T$, and that the same theory (even if complete) may have many different non-isomorphic core structures. A core structure for $T$ is one that can be picked out uniquely inside every model of $T$, and thus at least informally is definable within every model of $T$. The following result precisely characterizes the core structures as those whose elements are definable in a particular fashion.

THEOREM 2.1. *For any $T$ the following conditions are equivalent:*

(1) *$\mathfrak{C}$ is a core structure for $T$.*

(2) *$\mathfrak{C}$ is a model of every universal sentence consistent with $T$, and there are existential formulas $\varphi_i(x)$ and $k_i \in \omega$, for $i \in I$, such that*

$$\mathfrak{C}, T \models \exists^{=k_i} x\varphi_i \text{ for all } i \in I ,$$

*and*

$$\mathfrak{C} \models \forall x \bigvee_{i \in I} \varphi_i .$$

Proof. We first show that (2) implies (1). Let $\mathfrak{B}$ be a model of $T$ and let $\mathfrak{A}$ be a sufficiently saturated elementary extension of $\mathfrak{B}$. Then (2) implies that $|C| \leqslant |L|$ and $\mathfrak{C} \exists \mathfrak{A}$, hence $\mathfrak{C} \cong \mathfrak{C}'$ for some $\mathfrak{C}' \subseteq \mathfrak{A}$. Now,

$$C' = \{c \in C' \colon \ \mathfrak{C}' \models \bigvee_{i \in I} \varphi_i[c]\}$$

where the $\varphi_i$ are existential formulas such that $\mathfrak{C}'$, $\mathfrak{A}$, and $\mathfrak{B}$ are models of $\exists^{=k_i} x\varphi_i$. By Lemma 1.1, therefore,

$$C' = \{a \in A \colon \ \mathfrak{A} \models \bigvee_{i \in I} \varphi_i[a]\} = \{b \in B \colon \ \mathfrak{B} \models \bigvee_{i \in I} \varphi_i[b]\} .$$

Hence $\mathfrak{C}' \subseteq \mathfrak{B}$ and is in fact uniquely determined. Therefore $\mathfrak{C}$ is a core structure for $T$.

To show that (1) implies (2), let $\mathfrak{C}$ be a core structure for $T$. Then $\mathfrak{C}$ is a model of every universal sentence consistent with $T$ since it can be embedded in every model of $T$. Also, $|C| \leqslant |L|$ since $T$ has models of cardinality at most $|L|$. We will be finished if we can find, for every $c_0 \in C$, an existential formula $\varphi_0(x)$ and an integer $k_0$ such that

$$\mathfrak{C} \models \varphi_0[c_0] \quad \text{and} \quad \mathfrak{C}, T \models \exists^{=k_0} x\varphi_0 .$$

So, let $c_0 \in C$ and let $\Phi$ be the set of all existential formulas $\varphi(x)$ satisfied by $c_0$ in $\mathfrak{C}$. We first show the following:

(i) If $\mathfrak{A}$ is a model of $T$ containing $\mathfrak{C}$ and $a \in A$ satisfies every formula of $\Phi$, then $a \in C$ and $\mathfrak{C} \models \varphi[a]$ for all $\varphi \in \Phi$.

We may assume $\mathfrak{A}$ is sufficiently saturated. If $a$ satisfies every $\varphi(x) \in \Phi$, then $(\mathfrak{C}, c_0) \exists (\mathfrak{A}, a)$ and so there is an isomorphism $h$ of $\mathfrak{C}$ into $\mathfrak{A}$ such that $h(c_0) = a$. Since $\mathfrak{C}$ is a core structure for $T$, $h$ is actually an automorphism of $\mathfrak{C}$ and thus $a \in C$.

Using (i) we next show

(ii) $T \models \exists^{<\omega} x \bigwedge \Phi$.

If this were false then by compactness we could find a model of $T$ containing $\mathfrak{C}$ which also contained more than $|L|$ elements satisfying every formula of $\Phi$. This contradicts the assertion in (i) that every such element belongs to $C$.

It follows that $\mathfrak{C} \models \exists^{<\omega} x \bigwedge \Phi$ too. Let $k_0$ be the integer such that

$$\mathfrak{C} \models \exists^{=k_0} x \bigwedge \Phi .$$



Then by (i) every model of $T$ containing $\mathfrak{C}$ also has exactly $k_0$ elements satisfying every formula of $\Phi$, so we have

(iii) $T \models \exists^{=k_0} x \bigwedge \Phi$.

Finally, by compactness we can find a finite $\Phi_0 \subseteq \Phi$ such that $T \models \exists^{\leqslant k_0} x \bigwedge \Phi_0$. Let $\varphi_0$ be $\bigwedge \Phi_0$. Then

$$\models \forall x (\bigwedge \Phi \rightarrow \varphi_0) ,$$

so by (iii) we in fact have $T \models \exists^{=k_0} x\varphi_0$ and $\mathfrak{C} \models \exists^{=k_0} x\varphi_0$. Hence this $\varphi_0$ is as desired. (Notice that for this argument to work we cannot apply compactness after (ii) but need to wait until we know (iii)).

The following corollary shows the precise relation between this notion of core structure and core models of strongly convex theories.

COROLLARY 2.2. *$\mathfrak{C}$ is a core structure for some theory $T$ if and only if it is a core model of some strongly convex theory $T_0$.*

Proof. First, assume $T_0$ is strongly convex and that $\mathfrak{C}$ is a core model of $T_0$. Let $T$ be $T_0$ together with all the existential sentences true on $\mathfrak{C}$. Let $\mathfrak{A}$ be any model of $T$ and $\mathfrak{A}_0$ a sufficiently saturated elementary extension of $\mathfrak{A}$. Then $\mathfrak{C} \exists \mathfrak{A}_0$, hence $\mathfrak{C} \cong \mathfrak{C}_1$ for some $\mathfrak{C}_1 \subseteq \mathfrak{A}_0$. Since no proper substructure of $\mathfrak{C}$ is a model of $T_0$ we must have

$$\mathfrak{C}_1 = \bigcap \{\mathfrak{B} \colon \ \mathfrak{B} \subseteq \mathfrak{A}_0 \text{ and } \mathfrak{B} \models T\} .$$

In particular $\mathfrak{C}_1 \subseteq \mathfrak{A}$. If $\mathfrak{C}$ is isomorphic to some other $\mathfrak{C}_2 \subseteq \mathfrak{A}$ then the same argument shows that $\mathfrak{C}_1 = \mathfrak{C}_2$. Therefore $\mathfrak{C}$ is a core structure for $T$.

For the other direction, let $\mathfrak{C}$ be a core structure for $T$, and let $T_0$ be the theory of all the existential and universal sentences true on $\mathfrak{C}$. Condition (2) of Theorem 2.1 still holds for $\mathfrak{C}$ and $T_0$, so $\mathfrak{C}$ is a core structure for $T_0$. Let $\mathfrak{A}$ be any model of $T_0$. $\mathfrak{C}$ is isomorphic to exactly one $\mathfrak{C}' \subseteq \mathfrak{A}$ and can also be embedded in every other model of $T_0$, hence

$$\mathfrak{C}' = \bigcap \{\mathfrak{B} \colon \ \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T_0\} .$$

It follows that $T_0$ is strongly convex and that $\mathfrak{C}$ is a core model of $T_0$, as claimed.

Notice that we have shown that if $T$ is strongly convex and has exactly one core model (in particular if $T$ satisfies joint embedding), then this core model is also a core structure for $T$.

We can therefore apply Theorem 2.1 to core models of strongly convex theories and obtain the following result of Robinson.

COROLLARY 2.3 ([9] Theorem 6.4.1). *Let $\mathfrak{C}$ be a core model of a strongly convex theory $T$. Then there are existential formulas $\varphi_i(x)$, for $i \in I$, such that*

$$T \models \exists^{<\omega} x\varphi_i \quad \text{for all } i \in I ,$$

*and*

$$\mathfrak{C} \models \forall x \bigvee_{i \in I} \varphi_i .$$

Proof. Let $T'$ be $T$ together with all existential sentences true on $\mathfrak{C}$. The proof of Corollary 2.2 shows that $\mathfrak{C}$ is a core structure for $T'$. Let $c_0 \in C$. Then there is, by Theorem 2.1, an existential formula $\psi_0(x)$ and an integer $k_0$ such that $\mathfrak{C} \models \psi_0[c_0]$ and $T' \models \exists^{=k_0} x\psi_0$. By compactness there is an existential sentence $\sigma$ true on $\mathfrak{C}$ such that

$$T \models \sigma \to \exists^{=k_0} x\psi_0 .$$

Therefore $\sigma \wedge \psi_0$ is equivalent to some existential $\varphi_0(x)$ such that $\mathfrak{C} \models \varphi_0[c_0]$ and

$$T \models \exists x\varphi_0 \to \exists^{=k_0} x\varphi_0 ,$$

in particular $T \models \exists^{<\omega} x\varphi_0$.

Other examples of core structures are given by models rigidly embeddable in all models of $T$. Recall that $\mathfrak{C}$ is *rigidly embeddable* in $\mathfrak{A}$ if there is exactly one isomorphism of $\mathfrak{C}$ into $\mathfrak{A}$. It is then clear that $\mathfrak{C}$ is rigidly embeddable in every model of $T$ if and only if $\mathfrak{C}$ is a core structure for $T$ and has no proper automorphisms. We therefore immediately obtain the following result of Kreisel.

COROLLARY 2.4. ([4]). *$\mathfrak{C}$ is rigidly embeddable in every model of $T$ if and only if condition* (2) *of Theorem* 2.1 *holds with $k_i = 1$ for all $i \in I$.*

It should be remarked that, if there is any core structure for $T$, then there is a unique *maximal core structure* for $T$ (that is, a core structure in which every other core structure for $T$ can be embedded), namely the union of all the core structures contained in some model of $T$. If $T$ is strongly convex and satisfies joint embedding then the core model of $T$ is the maximal core structure.

Structures which are core structures for some $T$ have a number of interesting properties which follow from Theorem 2.1. We collect them together in the next theorem. Notice that by Theorem 2.1 if $\mathfrak{C}$ is a core structure for some $T$ then it is a core structure for $\mathrm{Th}(\mathfrak{C})$.

THEOREM 2.5. *Let $\mathfrak{C}$ be a core structure for some theory. Then the following hold.*

(a) *$\mathfrak{C}$ can be embedded in any model of all the existential and universal sentences true on $\mathfrak{C}$.*

(b) *$\mathfrak{C}$ completes* $\mathrm{Th}(\mathfrak{C})$ (*that is, if $\mathfrak{C} \subseteq \mathfrak{A}$ and $\mathfrak{C} \equiv \mathfrak{A}$ then $\mathfrak{C} \prec \mathfrak{A}$*).

(c) *$\mathfrak{C}$ is a prime model of* $\mathrm{Th}(\mathfrak{C})$.

(d) *If $\mathfrak{C} \subseteq \mathfrak{A}$ and $\mathfrak{A} \exists \mathfrak{C}$ then for every existential formula $\varphi(x_1, \ldots, x_n)$ and every $c_1, \ldots, c_n \in C$,*

$$\mathfrak{C} \models \varphi[c_1, \ldots, c_n] \quad \textit{if and only if} \quad \mathfrak{A} \models \varphi[c_1, \ldots, c_n] .$$

(e) *$\mathfrak{C}$ is prehomogeneous* (*as defined in* [3]).



(f) *$\mathfrak{C}$ is $T_0$-generic (in the sense of finite forcing) and $T_0^f = \mathrm{Th}(\mathfrak{C})$ for any consistent theory $T_0$ containing all the universal and existential sentences true on $\mathfrak{C}$.*

Proof. (a) This was shown in the course of proving Corollary 2.2.

(b) Let $\mathfrak{C} \subseteq \mathfrak{A}$ and $\mathfrak{C} \equiv \mathfrak{A}$. There is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{C} \simeq \mathfrak{C}'$ for some $\mathfrak{C}' \prec \mathfrak{B}$. Since $\mathfrak{C}$ is a core structure for $\mathrm{Th}(\mathfrak{C})$ we must have $\mathfrak{C} = \mathfrak{C}'$, and hence $\mathfrak{C} \prec \mathfrak{A}$.

(c) This is immediate from (a) and (b).

(d) For every $c \in C$ the existential formula $\psi(x)$, given by Theorem 2.1, which defines $c$ is such that $\mathfrak{C} \models \psi[c]$ and whenever $\theta(x)$ is existential and $\mathfrak{C} \models \exists x(\psi \wedge \theta)$ then $\mathfrak{C} \models \forall x(\psi \to \theta)$. It follows that for $c_1, \ldots, c_n \in C$ there is an existential formula $\psi(x_1, \ldots, x_n)$ such that $\mathfrak{C} \models \psi[\vec{c}]$ and whenever $\varphi(x_1, \ldots, x_n)$ is existential and $\mathfrak{C} \models \exists \vec{x}(\psi \wedge \varphi)$, then $\mathfrak{C} \models \forall \vec{x}(\psi \to \varphi)$. So assume that $\mathfrak{A} \models \varphi[c_1, \ldots, c_n]$ and $\varphi$ is existential. Then $\mathfrak{A} \models \exists \vec{x}(\psi \wedge \varphi)$, and so $\mathfrak{C} \models \exists \vec{x}(\varphi \wedge \psi)$ by hypothesis. Therefore $\mathfrak{C} \models \forall \vec{x}(\psi \to \varphi)$, so in particular $\mathfrak{C} \models \varphi[c_1, \ldots, c_n]$.

(e) It is sufficient to show that given $c_1, \ldots, c_n \in C$ there is an existential formula $\psi(x_1, \ldots, x_n)$ such that $\mathfrak{C} \models \psi[c_1, \ldots, c_n]$ and whenever $\mathfrak{C} \models \psi[c_1', \ldots, c_n']$ there is an automorphism $h$ of $\mathfrak{C}$ such that $h(c_i) = c_i'$, for $i = 1, \ldots, n$. The formula $\psi$ used in (d) is easily seen to have this property.

(f) By (b), $\mathfrak{C}$ completes $\mathrm{Th}(\mathfrak{C})$. Hence by Corollary 4.10 of [1], $(\mathrm{Th}(\mathfrak{C}))^f = \mathrm{Th}(\mathfrak{C})$, and by Theorem 3.4 of [1] $\mathfrak{C}$ is $\mathrm{Th}(\mathfrak{C})$-generic. By Corollary 4.3(1) of [1], $T_0^f = (\mathrm{Th}(\mathfrak{C}))$ and so $\mathfrak{C}$ is also $T_0$ generic.

If $\mathfrak{C}$ is the maximal core structure for $T$, there in general need not be any model $\mathfrak{A}$ of $T$ such that

$$\mathfrak{C} = \bigcap \{\mathfrak{B}\colon \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\} .$$

Equivalently, for many theories $T$ there is no model $\mathfrak{A}$ of $T$ such that $\bigcap \{\mathfrak{B}\colon \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}$ is either empty or a core structure for $T$. An example is given in section four. Conditions under which a core structure is given by such an intersection are given in Theorem 4.3 and Corollary 4.4.

We close this section with several examples concerning core structures.

(1) Let $T$ be the theory of algebraically closed fields of characteristic 0 (in the language $+, \cdot, 0, 1,$). Then $T$ is complete and strongly convex. Let $\mathfrak{C}_0$ be the field of (real) rationals and $\mathfrak{C}_1$ the field of (complex) algebraic numbers. Then $\mathfrak{C}_0$ and $\mathfrak{C}_1$ are both core structures for $T$. $\mathfrak{C}_0$ is the largest structure rigidly embeddable in every model of $T$ (Kreisel's "hard core"), and $\mathfrak{C}_1$ is the maximal core structure for $T$, the core model of $T$.

(2) Theorem 2.5(a) cannot be improved, since there is a core structure $\mathfrak{C}$ which cannot be embedded in every $\mathfrak{A}$ such that $\mathfrak{C} \exists \mathfrak{A}$. Let $\mathfrak{C} = \langle \omega, S, 0\rangle$ where $S$ is the relation of immediate successor, that is,

$S(m,n)$ holds if and only if $m+1=n$. $\mathfrak{C}$ is easily seen to be a core structure for $\mathrm{Th}(\mathfrak{C})$. Let $\mathfrak{A}$ be $\langle A, S', (0,0)\rangle$ where

$$A=\{(i,j):\ i,j\in\omega \text{ and } i\leqslant j\}$$

and $S'((i,j),(i',j'))$ holds if and only if $i+1=i'$ and either $j=j'$ or $j=0$. Then in $\mathfrak{A}$ there are chains under $S'$ of every finite length starting at $(0,0)$, but none of infinite length. It follows that $\mathfrak{C}$ cannot be embedded in $\mathfrak{A}$, but every finite substructure of $\mathfrak{C}$ can be embedded in $\mathfrak{A}$, hence $\mathfrak{C}\exists\mathfrak{A}$. (Notice that $\mathfrak{A}$ is not a core structure for $\mathrm{Th}(\mathfrak{A})$.)

(3) We have shown that if $\mathfrak{C}$ is a core structure for some theory, then $\mathfrak{C}$ can be embedded in every model of all the universal and existential sentences true on $\mathfrak{C}$, and whenever $\mathfrak{C}_0\subseteq\mathfrak{C}$ and $\mathfrak{C}\exists\mathfrak{C}_0$ then $\mathfrak{C}=\mathfrak{C}_0$. These conditions, however, are not sufficient to imply that $\mathfrak{C}$ is a core structure. For example, let $\mathfrak{C}$ be $\langle Z, S, P\rangle$ where $Z$ is the set of all (negative and non-negative) integers, $S(i)=i+1$, and $P(i)=i-1$ for all $i\in Z$. $\mathfrak{C}$ is easily seen to satisfy the stated conditions, but it is not a core structure for any theory.

**3. Axiomatizing convex theories.** A. Robinson showed (see [9], Theorem 3.5.1) that if $T$ is a convex theory then $T$ can be axiomatized by a set $\Sigma$ of $\forall\exists$ sentences. The converse to this certainly fails since there are $\forall\exists$ sentences which do not define a convex theory. The question remained, therefore, of precisely which $\forall\exists$ sentences and sets of $\forall\exists$ sentences define convex theories. Rabin [8] gave an involved characterization of such sentences and sets of sentences. In this section we use Theorem 2.1 to give a very different characterization, which is more like one announced by Rabin [7] but never published. We will utilise Robinson's results in restricting our attention to $\forall\exists$ sentences.

DEFINITION. Let $\sigma$ be an $\forall\exists$ sentence, say $\forall\vec{x}\exists y_1\dots y_m\alpha$ where $\alpha(\vec{x},y_1,\dots,y_m)$ is quantifier-free. Then a *convexization* of $\sigma$ is any sentence $\sigma^c$ of the form

$$\forall\vec{x}\bigvee_{1\leqslant i\leqslant n}\theta_i\wedge\bigwedge_{1\leqslant i\leqslant n}\forall\vec{x}\,[\theta_i\to\exists^{=k_i}\langle y_1,\dots,y_m\rangle(\chi_i\wedge\alpha)]$$

where $\theta_i(\vec{x})$ is universal, $\chi_i(\vec{x},\vec{y})$ is existential, and $k_i$ is a positive integer, for $i=1,\dots,n$.

Thus, $\sigma^c$ says the following: for any $\vec{x}$ one of $n$ cases holds (each case determined by a universal formula); if the $i$th case holds, then there are exactly $k_i$ $m$-tuples $\langle y_1,\dots,y_m\rangle$ satisfying $\alpha$ and an additional condition (given by an existential formula). The following lemma gives some easy properties of convexizations.

LEMMA 3.1. *Let $\sigma$ be an $\forall\exists$ sentence and $\sigma^c$ some convexization of $\sigma$. Then the following hold.*



(a) $\models\sigma^c\to\sigma$.

(b) *If $\mathfrak{A}$, $\mathfrak{B}_i\models\sigma^c$ and $\mathfrak{B}_i\subseteq\mathfrak{A}$ for all $i\in I$, then $\bigcap_{i\in I}\mathfrak{B}_i\models\sigma$, unless the intersection is empty.*

(c) *If $\models\sigma\to\sigma^c$ then $\sigma$ defines a convex theory.*

Proof. We show (b), from which (a) and (c) follow. Assume that $\bigcap_{i\in I}\mathfrak{B}_i$ is non-empty and let $\vec{b}$ be in the intersection. In $\mathfrak{A}$, $\vec{b}$ satisfies some $\theta_j$, hence the same $\theta_j$ is satisfied by $\vec{b}$ in every $\mathfrak{B}_i$ since $\theta_j$ is universal. Therefore, in $\mathfrak{A}$ and every $\mathfrak{B}_i$ there are exactly $k_j$ tuples $\langle a_1,\dots,a_m\rangle$ satisfying $\chi_j\wedge\alpha$ with $\vec{b}$. It follows that exactly the same such tuples belong to every $\mathfrak{B}_i$, since $\chi_j$ is existential, and therefore they belong to the intersection. Hence there is some $\langle a_1,\dots,a_m\rangle$ in the intersection satisfying $\alpha$ with $\vec{b}$. Therefore $\bigcap_{i\in I}\mathfrak{B}_i\models\sigma$, as desired.

Our theorem gives essentially the converse to part (c) of Lemma 3.1.

THEOREM 3.2. (a) *If $T$ is finitely axiomatizable then $T$ is convex if and only if $T$ can be axiomatized by some $\forall\exists$ sentence $\sigma$ such that $\models\sigma\leftrightarrow\sigma^c$ for some convexization $\sigma^c$ of $\sigma$.*

(b) *$T$ is convex if and only if $T$ can be axiomatized by a set $\Sigma$ of $\forall\exists$ sentences such that $\Sigma$ is equivalent to some set $\Sigma^c$ of convexizations of all the sentences in $\Sigma$.*

Proof. The easy directions, showing the conditions are sufficient for $T$ to be convex, follow from Lemma 3.1. For the other directions we assume $T$ is convex and show that if $\sigma$ is an $\forall\exists$ consequence of $T$ then there is some convexization $\sigma^c$ of $\sigma$ such that $T\models\sigma^c$. Since $T$ is equivalent to the set of its $\forall\exists$ consequences this yields the desired results.

Let $T\models\sigma$ where $\sigma$ is $\forall\vec{x}\exists y_1\dots y_m\alpha$ where $\alpha(\vec{x},y_1,\dots,y_m)$ is quantifier-free (we assume some variable is universally quantified in $\sigma$). Let $\mathfrak{A}$ be a model of $T$ and let $\vec{a}$ be in $A$. We will find a universal $\theta(\vec{x})$, an existential $\chi(\vec{x},y_1,\dots,y_m)$, and a positive $k$ such that

$$\mathfrak{A}\models\theta[\vec{a}]\quad\text{and}\quad T\models\forall\vec{x}\,[\theta\to\exists^{=k}\langle y_1\dots y_m\rangle(\chi\wedge\alpha)]\,.$$

Let $\vec{e}$ be a sequence of new constants and consider the theory

$$T^*=T\cup\{\theta(\vec{e}):\ \theta(\vec{x}) \text{ is universal and } \mathfrak{A}\models\theta[\vec{a}]\}\,.$$

Then $T^*$ satisfies joint embedding and is strongly convex. Hence $T^*$ has a core model $\mathfrak{C}^*$ which we may take to be a substructure of $\mathfrak{A}^*=(\mathfrak{A},\vec{a})$. Thus $\mathfrak{C}^*=(\mathfrak{C},\vec{a})$ for some model $\mathfrak{C}$ of $T$, and we must have

$$\mathfrak{C}^*\models\exists y_1\dots\exists y_m\alpha(\vec{e},y_1,\dots,y_m)\,.$$

Let $b_1,\dots,b_m\in C$ be such that $\mathfrak{C}^*\models\alpha[\vec{a},b_1,\dots,b_m]$. Since $\mathfrak{C}^*$ is a core structure for $T^*$, Theorem 2.1 implies that there are existential formulas

$\chi_1(\vec{x}, y_1), \ldots, \chi_m(\vec{x}, y_m)$ of $L$ and positive integers $j_1, \ldots, j_m$ such that

$$\mathfrak{C} \models \bigwedge_{1\leq i\leq m} \chi_i[\vec{a}, b_i] \quad \text{and} \quad T^* \models \bigwedge_{1\leq i\leq m} \exists^{=j_i} y_i \chi(\vec{e}, y_i) .$$

Let $\chi(\vec{x}, y_1, \ldots, y_m)$ be $\bigwedge_{1\leq i\leq m} \chi_i$ and let $k$ be the number of $m$-tuples satisfying $\alpha(\vec{e}, \vec{y}) \wedge \chi(\vec{e}, \vec{y})$ in $\mathfrak{C}^*$. Precisely the same $m$-tuples satisfy $\alpha \wedge \chi$ in $\mathfrak{A}^*$ and in $\mathfrak{C}^*$, because precisely the same elements satisfy $\chi_i$ in $\mathfrak{A}^*$ and in $\mathfrak{C}^*$ for each $i$ (by Lemma 1.1). Therefore there are exactly $k$ such $m$-tuples in every model of $T^*$ (since every universal sentence true of $\mathfrak{A}^*$ is a consequence of $T^*$), that is

$$T^* \models \exists^{=k}\langle y_1, \ldots, y_m\rangle [\chi(\vec{e}, \vec{y}) \wedge \alpha(\vec{e}, \vec{y})] .$$

Applying compactness to this we get some universal $\theta(\vec{x})$ such that $\theta(\vec{e}) \in T^*$, that is

$$\mathfrak{A} \models \theta[\vec{a}] \quad \text{and} \quad T \models \forall \vec{x} [\theta(\vec{x}) \rightarrow \exists^{=k}\langle y_1, \ldots, y_m\rangle (\chi \wedge \alpha)] .$$

Thus, these $\theta$, $\chi$, and $k$ are precisely as desired.

Now we repeat this argument for all $\vec{a}$ in every model $\mathfrak{A}$ of $T$, and obtain $\theta_{\mathfrak{A},\vec{a}}$, $\chi_{\mathfrak{A},\vec{a}}$ and $k_{\mathfrak{A},\vec{a}}$ with the above properties. It follows that in every model of $T$ every tuple satisfies some $\theta_{\mathfrak{A},\vec{a}}$, so by compactness there are a finite number of them — say $\theta_1, \ldots, \theta_n$ — such that

$$T \models \forall \vec{x} (\theta_1 \vee \ldots \vee \theta_n) .$$

If $\chi_1, \ldots, \chi_n$ and $k_1, \ldots, k_n$ are the corresponding $\chi$'s and $k$'s, then we have

$$T \models \forall \vec{x} \bigvee_{1\leq i\leq n} \theta_i \wedge \bigwedge_{1\leq i\leq n} \forall \vec{x} [\theta_i \rightarrow \exists^{=k_i}\langle y_1, \ldots, y_m\rangle (\chi \wedge \alpha)] .$$

That is, $T \models \sigma^c$, as desired.

The main point in the above argument is that Theorem 2.1 tells us exactly how the elements of the core model of a strongly convex theory satisfying joint embedding are defined. This information in fact yields the following characterization of convex theories.

COROLLARY 3.3. *$T$ is convex if and only if for every model $\mathfrak{A}$ of $T$ and every non-empty $X \subseteq A$,*

$$\mathfrak{A} \upharpoonright D_T^*(\mathfrak{A}, X) \models T ,$$

*where $D_T^*(\mathfrak{A}, X)$ is the set of all $a \in A$ such that for some existential $\varphi(x_1, \ldots, x_n, y)$, some $k \in \omega$, and some $b_1, \ldots, b_n \in X$, $\mathfrak{A} \models \varphi[\vec{b}, a]$ and whenever $\mathfrak{B} \subseteq \mathfrak{A}$ is a model of $T$ containing $X$, then in $\mathfrak{B}$ exactly $k$ elements $b'$ satisfy $\varphi[\vec{b}, b']$; and $\mathfrak{A} \upharpoonright Y$ is the substructure of $\mathfrak{A}$ whose universe is $Y$.*

Notice that in Theorem 3.2(b) we only assert that $\Sigma$ and $\Sigma^c$ are equivalent as sets of sentences; we do not assert that for every $\sigma \in \Sigma$ there is some $\sigma^c \in \Sigma^c$ such that $\models \sigma \leftrightarrow \sigma^c$. This is in general false, as is shown by Rabin's example in [8] of a convex theory which cannot be axiomatized by any set $\Sigma$ of sentences such that each $\sigma \in \Sigma$ also defines a convex theory.



**4. Intersections of models.** As was remarked in section two, there need be no model $\mathfrak{A}$ of a theory $T$ such that

$$\bigcap \{\mathfrak{B}\colon \ \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}$$

is a core structure for $T$. In this section we consider such intersections, motivated by the hope that, at least for universal models $\mathfrak{A}$, such intersections behave analogous to core structures. The notion we investigate, $I_T(\mathfrak{A})$, defined below, agrees with the above intersection on universal models. Theorem 4.2 is a definability result for $I_T(\mathfrak{A})$ analogous to Theorem 2.1 and also has Robinson's result, Corollary 2.3, as an immediate consequence. We use Theorem 4.2 in proving Theorem 4.3, which gives conditions under which the above intersection is a core structure for $T$. Theorem 4.2, in slightly different notation, was announced in [6].

DEFINITION. (a) $I_T(\mathfrak{A})$ is the set of all elements $a$ of $A$ such that

$$a \in \bigcap \{B\colon \ \mathfrak{B} \subseteq \mathfrak{A}' \text{ and } \mathfrak{B} \models T\}$$

for every model $\mathfrak{A}'$ of $T$ containing $\mathfrak{A}$.

(b) $C_T(\mathfrak{A}) = \bigcup \{C\colon \ \mathfrak{C} \subseteq \mathfrak{A},\ \mathfrak{C}$ is a core structure for $T\}$.

Notice that $C_T(\mathfrak{A})$ is the universe of the maximal core structure contained in $\mathfrak{A}$. If $T$ is strongly convex then $I_T(\mathfrak{A})$ is the universe of the core model of $T$ contained in $\mathfrak{A}$; if $T$ also satisfies joint embedding then $I_T(\mathfrak{A}) = C_T(\mathfrak{A})$. The following lemma gives some basic properties of $I_T$.

LEMMA 4.1. (a) *$I_T(\mathfrak{A}) \subseteq I_T(\mathfrak{B})$ whenever $\mathfrak{A} \subseteq \mathfrak{B}$.*

(b) *$I_T(\mathfrak{A}) = I_T(\mathfrak{B})$ whenever $\mathfrak{A} \prec \mathfrak{B}$.*

(c) *$I_T(\mathfrak{A}) = \bigcap \{B\colon \ \mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} \models T\}$ whenever $\mathfrak{A}$ is a universal model of $T$.*

Proof. (a) This is clear from the definition.

(b) It will suffice to show that $I_T(\mathfrak{B}) \subseteq I_T(\mathfrak{A})$ assuming that $\mathfrak{A} \prec \mathfrak{B}$. Let $a_0 \in A - I_T(\mathfrak{A})$. Then $a_0 \notin A_0$ for some model $\mathfrak{A}_0$ of $T$ such that $\mathfrak{A}$ and $\mathfrak{A}_0$ are both contained in some model $\mathfrak{A}'$ of $T$. We will show that there are models $\mathfrak{B}_0$ and $\mathfrak{B}'$ of $T$ such that $\mathfrak{B}$ and $\mathfrak{B}_0$ are both contained in $\mathfrak{B}'$ and $A_0 \cap A = B_0 \cap B$. It then follows that $a_0 \notin B_0$, hence $a_0 \notin I_T(\mathfrak{B})$ as desired. We show the existence of such $\mathfrak{B}_0$ and $\mathfrak{B}'$ by a diagram argument, as follows. We add distinct new constants $\{e_j^1\}$ for the elements of $A$, $\{e_j^2\}$ for the elements of $A_0 - A$, and $\{e_j^3\}$ for the elements of $B - A$ (thus every element of $(B - A) \cap A_0$ corresponds to two new constants). Let $\Delta_A$ be the set of all quantifier-free sentences in $\{e_j^1\}$ and $\{e_j^2\}$ true on $\mathfrak{A}'$, and let $\Delta_B$ be the set of all quantifier-free sentences in $\{e_j^1\}$ and $\{e_j^3\}$ true on $\mathfrak{B}$. Then $T \cup \Delta_A \cup \Delta_B$ is consistent since any finite subset of it is satisfiable on $\mathfrak{A}'$, suitably interpreting the constants $\{e_j^3\}$; and any model of this set provides the desired $\mathfrak{B}'$ and $\mathfrak{B}_0$.

(c) Let $J_T(\mathfrak{A}) = \bigcap\{B\colon \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}$. Then $J(\mathfrak{A}_1) \subseteq J(\mathfrak{A}_0)$ whenever $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are models of $T$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$. We can therefore find a model $\mathfrak{A}^0$ of $T$ such that $|A^0| \leqslant |L|$ and $J_T(\mathfrak{A}^0) = J_T(\mathfrak{A})$ for every model $\mathfrak{A}$ of $T$ containing $\mathfrak{A}^0$. We claim that $J_T(\mathfrak{A}^0) = I_T(\mathfrak{A}^0)$. If not, then there is some $a_0 \in J_T(\mathfrak{A}^0) - I_T(\mathfrak{A}^0)$, hence there are models $\mathfrak{A}$ and $\mathfrak{A}'$ of $T$ such that $\mathfrak{A}$ and $\mathfrak{A}^0$ are contained in $\mathfrak{A}'$ and $a_0 \notin A$. Then $a_0 \notin J_T(\mathfrak{A}')$, hence $J_T(\mathfrak{A}') \subsetneqq J_T(\mathfrak{A}^0)$, contradicting the choice of $\mathfrak{A}^0$. Now, if $\mathfrak{A}$ is a universal model of $T$ it can be assumed to contain $\mathfrak{A}^0$, and so

$$I_T(\mathfrak{A}^0) = J_T(\mathfrak{A}^0) = J_T(\mathfrak{A}) \subseteq I_T(\mathfrak{A}),$$

therefore $J_T(\mathfrak{A}) = I_T(\mathfrak{A})$.

Our first theorem concerns the definability of the elements of $I_T(\mathfrak{A})$ within $\mathfrak{A}$, for which we introduce the following notations.

Definition. (a) $D_T^{\Sigma}(\mathfrak{A})$ is the set of all $a \in A$ such that $\mathfrak{A} \models \varphi[a]$ for some existential formula $\varphi(x)$ such that $T \models \exists^{<\omega} x\varphi$.

(b) $D_T^{\Sigma!}(\mathfrak{A})$ is the set of all $a \in A$ such that $\mathfrak{A} \models \varphi[a]$ for some existential formula $\varphi(x)$ such that $T \models \exists^{=k} x\varphi$ for some integer $k$.

Theorem 4.2. *For any model $\mathfrak{A}$ of $T$ we have*

$$C_T(\mathfrak{A}) \subseteq D_T^{\Sigma!}(\mathfrak{A}) \subseteq I_T(\mathfrak{A}) \subseteq D_T^{\Sigma}(\mathfrak{A}).$$

Proof. The first inclusion follows from Theorem 2.1, and the second inclusion is easy, using Lemma 1.1. To show the last inclusion, let $\mathfrak{A}$ be a model of $T$ and let $a \in I_T(\mathfrak{A})$. We will show that if $(\mathfrak{A}, a) \,\exists\, (\mathfrak{B}, b)$ then $b \in I_T(\mathfrak{B})$. By Lemma 4.1(b) we may assume that $|A| \leqslant |L|$ and that $\mathfrak{B}$ is sufficiently saturated. Hence there is an isomorphism $h$ of $\mathfrak{A}$ onto some $\mathfrak{A}_0 \subseteq \mathfrak{B}$ such that $h(a) = b$. Then $b \in I_T(\mathfrak{A}_0) \subseteq I_T(\mathfrak{B})$, by Lemma 4.1(a).

Since $|I_T(\mathfrak{B})| \leqslant |L|$ for every model $\mathfrak{B}$ of $T$, a standard compactness argument yields an existential formula $\varphi(x)$ such that $\mathfrak{A} \models \varphi[a]$ and $T \models \exists^{<\omega} x\varphi$, which completes the proof of the theorem.

Corollary 2.3 is an immediate consequence of this theorem. To see this, notice that if $\mathfrak{C}$ is a core model of a strongly convex theory $T$, then $\mathfrak{C} = I_T(\mathfrak{C}) = D_T^{\Sigma}(\mathfrak{C})$, by this theorem.

We can also apply Theorem 4.2 to obtain conditions under which the maximal core structure of $T$ is an intersection of models of $T$.

Theorem 4.3. *Assume that $T$ satisfies joint embedding, and let $\mathfrak{A}$ be a universal model of $T$ and*

$$\mathfrak{C} = \bigcap\{\mathfrak{B}\colon \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}.$$

*If $T$ is consistent with the set of all universal sentences true on $\mathfrak{C}$, then $\mathfrak{C}$ is a core structure for $T$.*



Proof. We first assume that $T$ is complete for existential sentences. Then, by Lemma 4.1, $C = I_T(\mathfrak{A})$ and $I_T(\mathfrak{A}) = D_T^{\Sigma!}(\mathfrak{A})$ by Theorem 4.2, since $D_T^{\Sigma}(\mathfrak{A}) = D_T^{\Sigma!}(\mathfrak{A})$ for any $T$ complete for existential sentences. Hence, if $c \in C$ then $\mathfrak{A} \models \varphi[c]$ for some existential formula $\varphi(x)$ such that $T \models \exists^{=k} x\varphi$ for some $k \in \omega$. By the hypothesis on $\mathfrak{C}$, $\mathfrak{C} \models \exists^{=k} x\varphi$ and so $\mathfrak{C} \models \varphi[c]$. Therefore condition (2) of Theorem 2.1 is satisfied and $\mathfrak{C}$ is a core structure for $T$.

For the general case, let $T_0$ be $T$ together with all the universal sentences true on $\mathfrak{C}$. Then $T_0$ is complete for existential sentences. We may assume that $\mathfrak{A}$ contains some universal model $\mathfrak{A}_0$ of $T_0$. Let

$$\mathfrak{C}_0 = \bigcap\{\mathfrak{B}\colon \mathfrak{B} \subseteq \mathfrak{A}_0 \quad \text{and} \quad \mathfrak{B} \models T_0\}.$$

Then $\mathfrak{C} \subseteq \mathfrak{C}_0$ and $\mathfrak{C}_0$ is a core structure for $T_0$ by what has already been proved. By the assumption on $\mathfrak{C}$ and the definition of $T_0$, $\mathfrak{C}$ is a model of all the existential and universal sentences true on $\mathfrak{C}_0$. Hence by Theorem 2.5(a), $\mathfrak{C}_0$ can be embedded in $\mathfrak{C}$, and so $\mathfrak{C} = \mathfrak{C}_0$. Therefore $\mathfrak{C}$ is a core structure for $T_0$.

$\mathfrak{C}$ can be embedded in every model of $T$, so to show it is a core structure for $T$ it will suffice to show that if $\mathfrak{C} \cong \mathfrak{C}'$ and $\mathfrak{C}' \subseteq \mathfrak{A}$, then $\mathfrak{C} = \mathfrak{C}'$. We will show that, given such a $\mathfrak{C}'$, there is some model $\mathfrak{A}'$ of $T$ containing $\mathfrak{A}$ and some model $\mathfrak{A}_0'$ of $T_0$ such that $\mathfrak{C}' \subseteq \mathfrak{A}_0' \subseteq \mathfrak{A}'$. Since $\mathfrak{C}$ will still be contained in every model of $T$ contained in $\mathfrak{A}'$, we will also have $\mathfrak{C} \subseteq \mathfrak{A}_0'$, and therefore $\mathfrak{C} = \mathfrak{C}'$ since $\mathfrak{C}$ is a core structure for $T_0$.

Let $L'$ be $L$ together with a new individual constant $e_c$ for every $c \in C$, and let $h$ be the isomorphism of $\mathfrak{C}$ onto $\mathfrak{C}'$. Let $\Delta_0$ be the set of all existential sentences true on $\mathfrak{A}_0$ when $e_c$ is interpreted as $c$ for every $c$, and let $\Delta_1$ be the set of all existential sentences true on $\mathfrak{A}$ when $e_c$ is interpreted as $h(c)$ for every $c$. Models $\mathfrak{A}_0'$ and $\mathfrak{A}'$ as desired are easily shown to exist, by a diagram argument, provided $\Delta_0 \cup \Delta_1 \cup T$ is consistent. And this set is consistent since any $\sigma \in \Delta_0$ is in fact true on $\mathfrak{C}$ with $e_c$ interpreted by $c$ by Theorem 2.5(d), hence is true on $\mathfrak{C}'$ with $e_c$ interpreted by $h(c)$, therefore also true on $\mathfrak{A}$ since $\sigma$ is existential, and thus $\sigma \in \Delta_1$. Therefore $\Delta_0 \cup \Delta_1 \cup T = \Delta_1 \cup T$ is satisfied on $\mathfrak{A}$.

Corollary 4.4. *Assume that $T$ satisfies joint embedding and let $\mathfrak{C}$ be a core structure for $T$. If $T$ is consistent with the set of all universal sentences true on $\mathfrak{C}$, then*

$$\mathfrak{C} = \bigcap\{\mathfrak{B}\colon \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}$$

*whenever $\mathfrak{A}$ is a universal model of $T$ containing $\mathfrak{C}$.*

Proof. Let $\mathfrak{C}_1 = \bigcap\{\mathfrak{B}\colon \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}$. Then $\mathfrak{C} \subseteq \mathfrak{C}_1$, hence $T$ is consistent with the set of all universal sentences true on $\mathfrak{C}_1$, and therefore $\mathfrak{C}_1$ is a core structure for $T$ by Theorem 4.3. Also, $\mathfrak{C}_1 \,\exists\, \mathfrak{C}$ so $\mathfrak{C}_1$ can be embedded in $\mathfrak{C}$ by Theorem 2.5(a), and so $\mathfrak{C}_1 = \mathfrak{C}$, as desired.

In the remainder of this section we continue our investigation of $I_T(\mathfrak{A})$.

Even assuming that $T$ satisfies joint embedding, Theorem 4.2 cannot be improved to more precisely determine $I_T(\mathfrak{A})$. Part of the problem, as the following example shows, is that $I_T(\mathfrak{A})$ is not independent of the choice of $\mathfrak{A}$.

EXAMPLE. There is a theory $T$, satisfying joint embedding, with models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and

$$I_T(\mathfrak{A}) = D_T^{\Sigma_1^1}(\mathfrak{A}) = D_T^{\Sigma_1^1}(\mathfrak{B}) \subsetneq I_T(\mathfrak{B}) .$$

We consider the language with a binary predicate $<$ and let $T$ be the theory given by the following:

$$\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z) ,$$

$$\forall x (\neg x < x) ,$$

$$\exists x \forall y \forall z [x \neq y \wedge x \neq z \rightarrow \neg x < y \wedge \neg y < x \wedge (y < z \vee y = z \vee z < y)] ,$$

$$\exists x \exists y\, x \neq y .$$

Then every model of $T$ consists of a chain linearly ordered by $<$ and one additional element which is not $<$-related to anything. Let $\mathfrak{B}$ be any model of $T$ with more than two elements. Then $I_T(\mathfrak{B})$ consists solely of the unique element which is not $<$-related to anything. Let $\mathfrak{A}$ be a two element model of $T$ contained in $\mathfrak{B}$. Then in $\mathfrak{A}$ neither of the element is $<$-related to anything, and therefore $\mathfrak{A}$ can be extended to models of $T$ in which either element belongs to the chain, and thus $I_T(\mathfrak{A}) = 0$. Notice that this example also shows that in general there is no model $\mathfrak{A}'$ of $T$ containing $\mathfrak{A}$ such that

$$I_T(\mathfrak{A}) = \bigcap \{B\colon\ \mathfrak{B} \subseteq \mathfrak{A}' \text{ and } \mathfrak{B} \models T\} .$$

If $T'$ is the theory $T$ together with the sentence $\exists^{\geqslant 3} x (x = x)$, then $I_{T'}(\mathfrak{A}) = D_{T'}^{\Sigma_1^1}(\mathfrak{A}) \neq 0$ and $C_{T'}(\mathfrak{A}) = 0$ for every model $\mathfrak{A}$ of $T'$, and therefore $I_{T'}(\mathfrak{A})$ is never the universe of a core structure for $T'$.

One might expect that if $I_T(\mathfrak{A})$ is independent of the choice of $\mathfrak{A}$, that is if $T$ satisfies joint embedding and $I_T(\mathfrak{A}) = I_T(\mathfrak{B})$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are models of $T$ with $\mathfrak{A} \subseteq \mathfrak{B}$, then $I_T(\mathfrak{A}) = D_T^{\Sigma_1^1}(\mathfrak{A})$ for every model $\mathfrak{A}$ of $T$. This is false, however, as more complicated examples could be devised to show.

It is true, as remarked in the proof of Theorem 4.3, that $I_T(\mathfrak{A}) = D_T^{\Sigma_1^1}(\mathfrak{A})$ for every model $\mathfrak{A}$ of a theory $T$ which is complete for existential sentences. The following theorem gives another condition under which $I_T(\mathfrak{A})$ is definable in this fashion. Recall that $T$ has the *amalgamation property* if whenever $\mathfrak{B}_1$, $\mathfrak{B}_2$, and $\mathfrak{C}$ are models of $T$ such that $\mathfrak{C} \subseteq \mathfrak{B}_1$ and $\mathfrak{C} \subseteq \mathfrak{B}_2$, then there is some model $\mathfrak{A}$ of $T$ with $\mathfrak{B}_1 \subseteq \mathfrak{A}$ and an isomorphism $h$ of $\mathfrak{B}_2$ into $\mathfrak{A}$ such that $h(c) = c$ for all $c \in C$.



THEOREM 4.5. *$I_T(\mathfrak{A}) = D_T^{\Sigma_1^1}(\mathfrak{A})$ for every model $\mathfrak{A}$ of $T$ if $T$ satisfies joint embedding and amalgamation.*

Proof. Let $\mathfrak{B}$ be a universal sufficiently saturated model of $T$. We will show that $I_T(\mathfrak{B}) = D_T^{\Sigma_1^1}(\mathfrak{B})$, which will imply the result because of Theorem 4.1. We first show

(1) Let $\mathfrak{A}$ be a model of $T$, $a \in A$, and let $\varphi(x)$ be an existential formula such that $\mathfrak{A} \models \varphi[a]$ and whenever $\mathfrak{B} \models \varphi[b]$ then $(\mathfrak{A}, a) \exists (\mathfrak{B}, b)$. Let $\psi(x)$ be existential and assume that $\exists x (\varphi \wedge \psi)$ and $\exists^{<\omega} x (\varphi \wedge \psi)$ are true on $\mathfrak{B}$. Then $\mathfrak{B} \models \forall x (\varphi \rightarrow \psi)$.

Assume, on the contrary, that there is a $b \in B$ such that $\mathfrak{B} \models (\varphi \wedge \neg \psi)[b]$. Then $(\mathfrak{A}, a) \exists (\mathfrak{B}, b)$ and so (since we may assume $|A| \leqslant |L|$) there is an isomorphism $h$ of $\mathfrak{A}$ into $\mathfrak{B}$ such that $h(a) = b$. By amalgamation one can find some model $\mathfrak{B}'$ of $T$ containing $\mathfrak{B}$ such that $\mathfrak{B}' \models \psi[b]$. Therefore $\mathfrak{B}'$ contains more elements satisfying $\varphi \wedge \psi$ than $\mathfrak{B}$ does, contradicting the universality of $\mathfrak{B}$.

Now, choose models $\mathfrak{A}_i$ of $T$ contained in $\mathfrak{B}$, for $i \in I$, such that any universal sentence consistent with $T$ is true on some $\mathfrak{A}_i$. Let $c \in I_T(\mathfrak{B})$. An easy amalgamation argument shows

(2) There are existential formulas $\varphi_i(x)$, for $i \in I$, such that $\mathfrak{A}_i \models \varphi_i[c]$, $\mathfrak{B} \models \exists^{<\omega} x \varphi_i$, and if $\mathfrak{B} \models \varphi_i[b]$ then $(\mathfrak{A}_i, c) \exists (\mathfrak{B}, b)$.

By (1), therefore, $\mathfrak{B} \models \forall x (\varphi_i \leftrightarrow \varphi_j)$ for all $i, j \in I$. So let $n \in \omega$ be such that $\mathfrak{B} \models \exists^{=n} x \varphi_i$. Now it may be true that $\mathfrak{A}_i \models \exists^{<n} x \varphi_i$, but in this case any $c_0$ satisfying $\varphi_i$ in $\mathfrak{B}$ but not in $\mathfrak{A}_i$ must satisfy some $\varphi_i'$ in $\mathfrak{A}_i$ with the properties in (2) for $c_0$. Then $\mathfrak{B} \models \forall x (\varphi_i \leftrightarrow \varphi_i \vee \varphi_i')$. Repeating this as long as necessary we see

(3) There are existential formulas $\theta_i(x)$, $i \in I$, such that $\mathfrak{B} \models \forall x (\varphi_i \leftrightarrow \theta_i)$ and $\mathfrak{A}_i \models \exists^{=n} x \theta_i$ for all $i \in I$.

By compactness we find a finite number of $\theta_i$, say $\theta_1, \ldots, \theta_m$, such that

$$T \models \exists^{=n} x \theta_1 \vee \ldots \vee \exists^{=n} x \theta_m .$$

Finally we let $\psi$ be $\bigvee_{1 \leqslant i \leqslant m} (\theta_i \wedge \exists^{\geqslant n} x \theta_i)$, and see that $\mathfrak{B} \models \psi[c]$ and $T \models \exists^{=n} x \psi$, which completes the proof.

**5. Strong substructures.** In this section we indicate how the preceding results can be generalized to apply to stronger concepts of substructure. Throughout the section, $\Gamma$ is some set of formulas containing all atomic formulas.

DEFINITION. $\mathfrak{A}$ is a *$\Gamma$-substructure* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq_\Gamma \mathfrak{B}$, if and only if $A \subseteq B$ and for every $\varphi(x_1, \ldots, x_n)$ in $\Gamma$ and every $a_1, \ldots, a_n$ in $A$,

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathfrak{B} \models \varphi[a_1, \ldots, a_n] .$$

DEFINITION. $\exists(\Gamma)$ consists of all formulas of the form

$$\exists z_1 \dots \exists z_k \varphi(\vec{z}, x_1, \dots, x_n)$$

where $\varphi$ is a boolean combination of formulas in $\Gamma$.

Let $\Gamma_0$ be the set of all atomic formulas and $\Gamma_1$ be the set of all formulas. Then $\mathfrak{A} \subseteq_{\Gamma_0} \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$, and $\mathfrak{A} \subseteq_{\Gamma_1} \mathfrak{B}$ if and only if $\mathfrak{A} \prec \mathfrak{B}$. $\exists(\Gamma_0)$ is the set of all existential formulas and $\exists(\Gamma_1) = \Gamma_1$. More generally, $\exists(\Pi_n^0) = \Sigma_{n+1}^0$.

THEOREM 5.1. *For any $T$ the following are equivalent:*

(1) *$\mathfrak{C}$ is isomorphic to exactly one $\Gamma$-substructure of every model of $T$.*

(2) *Every sentence in $\exists(\Gamma)$ true on $\mathfrak{C}$ is a consequence of $T$, and there are formulas $\varphi_i(x)$ in $\exists(\Gamma)$ and $k_i \in \omega$, for $i \in I$, such that*

$$\mathfrak{C}, T \models \exists^{=k_i} x \varphi_i \quad \textit{for all } i \in I,$$

*and*

$$\mathfrak{C} \models \forall x \bigvee_{i \in I} \varphi_i .$$

Theorem 5.1 is a generalization of Theorem 2.1 and could be proved in the same manner. More efficiently, it can be derived from Theorem 2.1 by treating each formula in $\Gamma$ as a new atomic formula in an expanded language. The details are left to the reader.

For $\Gamma = \Gamma_1$ we obtain the following

COROLLARY 5.2. *Let $T$ be a complete theory and let $\mathfrak{C}$ be a model of $T$. Then the following are equivalent:*

(1) *$\mathfrak{C}$ is isomorphic to exactly one elementary submodel of every model of $T$.*

(2) *There are formulas $\varphi_i(x)$, for $i \in I$, such that*

$$T \models \exists^{<\omega} x \varphi_i \quad \textit{for all } i \in I,$$

*and*

$$\mathfrak{C} \models \forall x \bigvee_{i \in I} \varphi_i .$$

(3) $\mathfrak{C} = \bigcap \{\mathfrak{B} : \mathfrak{B} \prec \mathfrak{A}\}$ *for every elementary extension $\mathfrak{A}$ of $\mathfrak{C}$.*

This corollary could also be proved from results in [5] (cf. Lemma 5.1 and Theorem 5.2 there). One should notice that condition (1) is stronger than just requiring that $\mathfrak{C}$ is a prime model of $T$ and has no proper elementary submodels.

Versions of most of the other results in this paper can also be given for $\Gamma$-substructures. For example, Theorem 3.2 for $\Gamma = \Gamma_1$ becomes a theorem of Park characterizing theories such that the intersection of any family of elementary submodels of a model of the theory is again an elementary submodel ([5], Theorem 5.3). Again, we leave to the reader the details of statement and derivation of the $\Gamma$-versions from the results of this paper.



## References


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