each i,

$$(*) \qquad x_i = \frac{F(x_i) - F(-\lambda_i x_i) - y_i}{1+\lambda_i}.$$

From $(*)$ it follows that the sequence x_i is also convergent. Once this is established the lemma follows by a routine argument using continuity of f and the fact that the set X is closed.

We are now in a position to prove the main result.

(3.2) THEOREM. *Let X be a closed, bounded subset of R^∞ for which the origin is in a bounded component of $R^\infty - X$ and let $f\colon X \to R^{\infty-1}$ be a compact vector field. Then there exist two points x and y in X and a positive real number λ, such that $y = -\lambda x$ and $f(x) = f(y)$.*

Proof. In view of the preceding lemma it suffices to find such x, y, λ so that $\|f(x) - f(y)\| < \delta$ where δ is a pre-assigned positive number. Given such δ we set $\varepsilon = \frac{1}{2}\delta$ and apply (2.7). This gives a finite dimensional subspace R^k of R^∞ and a compact vector field $f_\varepsilon\colon X \to R^\infty$ such that f_ε maps $X \cap R^k$ into a $(k-1)$-dimensional subspace R^{k-1} and $\|f(x) - f_\varepsilon(x)\| < \frac{1}{2}\delta$ for every $x \in X$. As observed earlier, the set $X \cap R^k$ is a compact subset of R^k for which the origin lies in a bounded component of $R^k - (X \cap R^k)$. Applying Theorem A in the introduction to the restriction map $f_\varepsilon\colon X \cap R^k \to R^{k-1}$ we get points x and y in $X \cap R^k$ and $\lambda > 0$ such that $f_\varepsilon(x) = f_\varepsilon(y)$ and $y = -\lambda x$. Since we have, $\|f(x) - f(y)\| \leqslant \|f(x) - f_\varepsilon(x)\| + \|f_\varepsilon(x) - f_\varepsilon(y)\| + \|f_\varepsilon(y) - f(y)\|$, it follows that $\|f(x) - f(y)\| < \delta$ and the theorem is proved.

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Wide tree-like spaces have a fixed point

by

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Abstract. A well-known unsolved problem is to determine whether or not a compact plane continuum which does not separate the plane has the fixed point property for continuous functions. In this paper a wide tree-like space is defined, and it is shown that the class of all wide tree-like spaces has the fixed point property for continuous functions. A characterization of a wide tree-like space is revealed. This class of tree-like spaces contains many compact plane continua, all of which do not separate the plane. The same can be said about the class of tree-like spaces, but it is not known whether or not this class has the fixed point property for continuous functions.

1. Introduction. A bounded plane continuum which does not separate the plane can be represented as the intersection of the elements of a monotonic decreasing sequence of open 2-cells. A well-known problem is to determine whether or not each such continuum has the fixed point property for continuous functions. This question has been answered in the affirmative for many special plane continua, e.g. [3]. The question of whether a tree-like space has the fixed point property for continuous functions was raised in a conversation with O. H. Hamilton. It is the purpose of this paper to answer this question for wide tree-like spaces. The class of wide tree-like spaces contains many plane continua, all of which do not separate the plane. Likewise, the class of tree-like spaces contains many plane continua, all of which do not separate the plane.

We shall use Burgess's definition [2] of a linear chain and a definition similar to his of a tree-like chain, namely: a *tree-like chain* C is a finite coherent collection of open sets such that (1) each two nonintersecting elements of C are a positive distance apart; (2) no subcollection of C consisting of more than two elements is a circular chain; and (3) no three elements have a point in common. If C is a tree-like chain and $b \in C$, then b is a *branch link* of C if and only if there exists more than two other links of C that intersect b. Also, if l is a member of a tree-like chain C, then l is called an *end link* of C if and only if there exists only one other link of C that intersects l. All spaces considered in this paper are metric spaces and (M, d) denotes the metric space with set M and metric d.

If A is a finite collection of sets, then $\|A\| = \max\{\text{diameter } A_i\colon A_i \in A\}$ and $A^* = \bigcup\{A_i\colon A_i \in A\}$. Similarly, if A is a set, then $|A| =$ diameter of A.

2. Definitions and basic theorems. The following definitions and theorems in this section are fundamental and elementary for the remainder of the results in this paper. The proofs of these theorems are not hard; and, therefore, no proofs are presented.

DEFINITION 1. If C is a tree-like chain and b a branch link of C, then A is an *arm* of b in C if and only if A is a maximal coherent subcollection of $C - \{b\}$.

DEFINITION 2. A compact connected space (M, d) is *tree-like* if and only if:

(1) $M = \bigcap_{n=1}^{\infty} C_n^*$;

(2) C_n is a tree-like chain for each n;

(3) $\overline{C}_{n+1}^* \subset C_n^*$ for each n;

(4) $\lim_{n\to\infty} \|C_n\| = 0$; and

(5) if $l \in C_{n+1}$, then there exists $l_1 \in C_n$ such that $l \subset l_1$.

The sequence $\{C_n\}$ is called a *realization for* (M, d) *with respect to the metric* d. If no ambiguity is possible, then $\{C_n\}$ is simply called a *realization for* (M, d).

DEFINITION 3. A tree-like space (M, d) is called *wide tree-like* if and only if there exists a realization $\{C_n\}$ for (M, d) with the property that given $\varepsilon > 0$, there exists N and $\delta > 0$ such that if $k \geqslant N$ and $d(x, \overline{b}) > \varepsilon$, then $d(x, A^*) \geqslant \delta$ where b is any branch link of C_k and A is any arm of b that does not contain x. Such a realization is called a *wide realization for* (M, d).

THEOREM 1. *If C is a tree-like chain, b a branch link of C, and A is an arm of b, then there exists a unique link l of C such that:*

(1) $l \cap b \neq \emptyset$ and

(2) if $\mathcal{A} = \{L\colon L \subset C,\ L$ is a linear chain with l as one end link and $b \notin L\}$, then $A = \bigcup_{L \in \mathcal{A}} L$.

THEOREM 2. *If (M, d) is tree-like and K is a subcontinuum of M, then (K, d) is tree-like.*

THEOREM 3. *If (M, d) is wide tree-like and K is a subcontinuum of M, then (K, d) is wide tree-like.*

THEOREM 4. *If (M, d) is wide tree-like and $\{C_n\}$ is a wide realization for (M, d), then any subsequence of $\{C_n\}$ is a wide realization for (M, d).*

Remark. The existence of wide tree-like spaces is easily established since every chainable continuum is wide tree-like.

3. The wide condition. The added condition of being wide assures that tree-like spaces have the desired fixed point property. This is revealed in Theorem 5 with the aid of Lemma 1 and Lemma 2.

LEMMA 1. *Let (M, d) be a tree-like space, f a continuous function from M onto M and $\varepsilon_1 > 0$ such that for each $x \in M$, $d(x, f(x)) > \varepsilon_1$. Then there do not exist a realization $\{C_n\}$ for M, natural number N, and $\|C_N\| < \varepsilon_1/10$ with linear chain $C_0 \subset C_N$ such that $C_0 = \{b_1, l_1, \ldots, l_k, b_2\}$ or $C_0 = \{b_1, b_2\}$ with the property that:*

(1) b_1 and b_2 are distinct branch links of C_N;

(2) $f(b_1)$ is contained in the arm of b_1 that contains b_2;

(3) $f(b_2)$ is contained in the arm of b_2 that contains b_1; and

(4) l_i is not a branch link for each i.

Proof. Suppose the contrary and assume the notation in the statement of this theorem. Within this proof it will be understood that a point $x \in C_0^*$ will "move up" if and only if:

$x \in b$, or

if $x \in l_i$, then there exists j such that $i < j$ and that $f(x) \in l_j \cup b_2$ or

if $x \in l_i$, then $f(x)$ is in the arm of b_1 containing b_2 but $f(x) \notin C_0^*$.

If A denotes the arm of b_1 that contains b_2, then choose $\varepsilon_2 > 0$ such that $\varepsilon_2 < d(A^*, B^*)$ where B denotes any arm of b_1 distinct from A and $\varepsilon_2 < \min\{d(l, l')|\ l, l' \in C_N\}$. Pick $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2}\min\{|l|\colon l \in C_N\}$ and $\varepsilon < \varepsilon_2$. Since f is uniformly continuous, there exists δ such that $\varepsilon > \delta > 0$ and if $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$. Let C_1 be a linear chain of open sets such that

$$C_1 = \{c_1, \ldots, c_m\}, \quad |c_i| < \delta \quad \text{for each } i = 1, 2, \ldots, m,$$
$$c_1 \cap b_1 \neq \emptyset, \quad c_m \cap b_2 \neq \emptyset,$$
$$\bigcup_{i=2}^{m} c_i \subset \Big(\bigcup_{i=1}^{k} l_i\Big) \cup b_2, \quad \text{and} \quad \Big(\bigcup_{i=2}^{m} c_i\Big) \cap b_1 = \emptyset.$$

The linear chain C_1 exists since M is connected. The supposition implies that there exists $p \in c_1 \cap b_1$ such that p "moves up". Suppose that $f(p) \in l_j$ for some $1 \leqslant j \leqslant k$ such that $j > i$ for any i such that $p \in l_i$. The largest i can be is 1. Therefore, because of the supposition and the choices of ε and δ, $j \geqslant 10$. Let $x \in c_1$. Since $p \in c_1$ and because $|c_1| < \delta$, then the largest i can be such that $x \in l_i$ is $i = 1$. The facts $d(f(x), f(p)) < \varepsilon$ and $f(p) \in l_j$, $j \geqslant 10$, imply that there exists $j_1 \geqslant 9$ such that $f(x) \in l_{j_1}$. The definition of "moves up" says that x "moves up". That is, each point of c_1 "moves up".

On the other hand, suppose $f(p)$ is a point in the arm of b_1 but $f(p) \notin C_0^*$. Theorem 1 implies the existence of a linear chain

$$C_2 = \{b_1, l_1, \ldots, l_k, b_2, l_{k+1}, \ldots, l_s\} \quad \text{or} \quad C_2 = \{b_1, b_2, l_{k+1}, \ldots, l_s\}$$

such that $f(p) \in l_s$ and no subscript smaller than s denotes a link of this linear chain that contains $f(p)$. Let $x \in c_1$ and observe that if there exists $i \leqslant k$ such that $x \in l_i$, then $i = 1$ as before, $s \geqslant k+1$, and $s-i \geqslant 10$. Because $d(f(x), f(p)) < \varepsilon$ and $f(p) \in l_s$, then $f(x)$ is contained in a link $l_t \in C_N$ that intersects l_s. This is enough to assure either $f(x) \in b_2$, $f(x) \in l_{s-1} \cup l_s$, or $C_2' = \{b_1, l_1, \ldots, l_k, b_2, l_{k+1}, \ldots, l_s, l_t\}$ (or $C_2' = \{b_1, b_2, l_{k+1}, \ldots, l_s, l_t\}$) is a linear chain with $f(x) \in l_t$.

The above two paragraphs say that in any case, each point in c_1 "moves up". Let r be the maximum natural number such that each point in c_r "moves up". Note that $r \neq m$ since $c_m \cap b_2 \neq \emptyset$ and no point of b_2 "moves up". By an argument similar to the one above, since $c_{r+1} \cap c_r \neq \emptyset$, we can prove that each point of c_{r+1} "moves up" which is a contradiction to the definition of r. Thus, the lemma is proved.

LEMMA 2. *Let (M, d) be tree-like and f a continuous function from M onto M. If there exists $\varepsilon_1 > 0$ such that for each $x \in M$, $d(x, f(x)) > \varepsilon_1$, then there does not exist a realization $\{C_n\}$ for (M, d), natural number N, and $\|C_N\| < \varepsilon_1/10$ such that C_N has at least one branch link and for each branch link b of C_N, $f(\bar{b})$ is contained in only one arm of b in C_N.*

Proof. We begin by supposing the contrary and adopting the notations introduced in the statement of this lemma. Let b_1 be a branch link of C_N and A_1 the arm of C_N such that $f(\bar{b}_1) \subset A_1^*$. If A_1 contains no branch link of C_N then A_1 is a linear chain, denoted $A_1 = \{l_1, \ldots, l_n\}$, l_n is an end link of C_N, and $l_1 \cap b_1 \neq \emptyset$. Theorem 1 implies that $\{b_1, l_1, \ldots, l_n\}$ is a linear chain and the present supposition forces $f(\bar{b}_1) \subset A_1^*$. In a way similar to that in the proof of Lemma 1, a point $x \in C_0^*$, where $C_0 = \{b_1, l_1, \ldots, l_n\}$, will be defined to "move up" if and only if:

$x \in b_1$ or

if $x \in l_i$, then there exists j such that $i < j$ and that $f(x) \in l_j$.

Here, a proof analogous to that used in the second paragraph of the proof of Lemma 1 will imply a contradiction.

If A_1 contains a branch link of C_N, then using Theorem 1 again, pick a branch link b_2 having the property that there exists a linear chain of the form $\{b_1, l_{11}, \ldots, l_{1k_1}, b_2\}$ or of the form $\{b_1, b_2\}$ such that

$$\{l_{11}, \ldots, l_{1k_1}, b_2\} \subset A_1 \quad \text{or} \quad \{b_1, b_2\} \subset A_1$$

and for each j, l_{1j} is not a branch link. In the remainder of this proof the degenerate cases of type $\{b_1, b_2\}$ will not be discussed but from the arguments presented for the cases of types $\{b_1, l_{11}, \ldots, l_{1k_1}, b_2\}$ it is clear that these can be resolved similarly. Lemma 1 assures us that $f(\bar{b}_2)$ is not contained in the arm of b_2 in C_N that contains b_1. Thus, there exists an arm A_2 of b_2 such that $f(\bar{b}_2) \subset A_2^*$ and $A_2 \subset A_1$. Assume that for natural number n, a linear chain

$$\{b_1, l_{11}, \ldots, l_{1k_1}, b_2, \ldots, b_n, l_{n1}, \ldots, l_{nk_n}, b_{n+1}\}$$

has been defined such that:

b_i, $1 \leqslant i \leqslant n+1$ is a branch link of C_N;

l_{ij} is not a branch link, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k_i$;

$f(\bar{b}_i)$ is not contained in the arm of b_i that contains b_{i-1} for each $2 \leqslant i \leqslant n+1$, and

$f(\bar{b}_i)$ is contained in the arm of b_i that contains b_{i+1} for $1 \leqslant i \leqslant n$.

Again, Lemma 1 says that $f(\bar{b}_{n+1})$ is not contained in the arm of b_{n+1} that contains b_n. Denote B as the arm of C_N containing $f(\bar{b}_{n+1})$. If B has no branch link of C_N, then B is a linear chain in C_N, denoted, $B = \{l_1, \ldots, l_k\}$ where $b_{n+1} \cap l_1 \neq \emptyset$ and l_k is an end link of C_N. Theorem 1 implies that $\{b_{n+1}, l_1, \ldots, l_k\}$ is a linear chain and our supposition says that $f(\bar{b}_{n+1}) \subset B^*$. Again, arguing similarly as was done in Lemma 1 and as was referred to in the first paragraph of this proof, a contradiction is reached.

Therefore, we must conclude that B contains a branch link of C_N. This induction argument implies that C_N has an infinite number of branch links which is a contradiction to the definition of a tree-like chain.

THEOREM 5. *If (M, d) is wide tree-like, then (M, d) has the fixed point property for continuous functions.*

Proof. Suppose that there does exist a continuous function f from M onto M that has no fixed point. It is known that if g is a continuous function from a compact connected metric space M_1 into M_1, then there exists a subcontinuum K_1 of M_1 such that $g(K_1) = K_1$. This fact, coupled with Theorem 3, assures us that no generality is lost by assuming in the proof of this theorem that f is onto M. Since f has no fixed point, there exists $\varepsilon_1 > 0$ such that for each $x \in M$, $d(x, f(x)) > \varepsilon_1$. Let $\{C_n\}$ be a realization for M satisfying Definition 3. Let N_1 and δ be the two numbers assured by Definition 3 such that if $n \geqslant N_1$, b is a branch link of C_n, $d(x, \bar{b}) > \varepsilon_1/2$, and A is any arm of b in C_n not containing x, then $d(x, A^*) \geqslant \delta$. If a realization of the type just described exists such that for each n, C_n has no branch links, then [3] proves the desired result of this theorem. Otherwise, referring to Theorem 4, we can assume that for each n, C_n has at least one branch link.

Pick $\varepsilon > 0$ such that $\varepsilon = \min\{\delta/4, \varepsilon_1/4\}$ and N_2 such that if $n \geqslant N_2$, then $\|C_n\| < \varepsilon_1/10$. Since f is uniformly continuous, there exists δ_1 such that $0 < \delta_1 < \varepsilon$ and if $d(x, y) < \delta_1$, then $d(f(x), f(y)) < \varepsilon$. Pick N_3 such that if $n \geqslant N_3$, then $\|C_n\| < \delta_1$. Let $N = \max\{N_1, N_2, N_3\}$ and b is any branch link of C_N. Suppose that there exists $x \in \bar{b}$ such that $d(f(x), \bar{b}) = r \leqslant \varepsilon_1/2$. Since $\bar{b}$ is compact, then there will exist $q \in \bar{b}$ such that

$d(f(x), \bar{b}) = d(f(x), q) = r$. Thus,

$$d(x, f(x)) \leqslant d(x, q) + d(q, f(x)) < \varepsilon + r < \varepsilon_1/4 + \varepsilon_1/2 < \varepsilon_1 .$$

This contradicts the fact that $d(x, f(x)) > \varepsilon_1$ and, therefore, $d(f(\bar{b}), \bar{b}) > \varepsilon_1/2$. Consider a particular $p \in \bar{b}$ and an arbitrary $x \in \bar{b}$. Since $N \geqslant N_3$ and $N \geqslant N_1$, then $d(f(p), f(x)) < \varepsilon < \delta/4$. This says that $|f(\bar{b})| < \delta/2$ and, thus, $f(\bar{b})$ is contained in only one arm of b. Since $N \geqslant N_2$, then $\|C_N\| < \varepsilon_1/10$, which is a contradiction to Lemma 2, and the theorem is proved.

Remark. Theorem 6 and its corollaries establish a sufficient supply of tree-like spaces that are not wide tree-like. Theorems 6 and 7 and their corollaries involve the idea of the width of a tree-like space defined similarly to the width of a continuum defined in [1].

DEFINITION. If G is any tree-like chain, then a number $\mathfrak{W}(G)$ is associated with G as follows. For each chain C in G and each element $X \in G$, there is a distance $d(X, C^*)$ from X to C^*. Let

$$\mathfrak{W}(G) = \min_{C \text{ in } G} [\max_{X \in G} d(X, C^*)] ,$$

where each maximum is obtained with C fixed. A number $\mathfrak{W}$ is called the *width of a tree-like space* (M, d) if and only if for any realization, $\{C_n\}$, for M, the sequence $\{\mathfrak{W}(C_n)\}$ converges to $\mathfrak{W}$.

Using this definition for the width of a tree-like space (M, d) it can be shown, as in [1], that each tree-like space has a width.

THEOREM 6. *Let (M, d) be tree-like with width zero. If for each realization $\{C_n\}$ for (M, d) there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and $\varepsilon_1 > 0$ such that for each natural number i, there exists a branch link $b_{n_i} \in C_{n_i}$ with at least three distinct arms A_{ij}, $j = 1, 2, 3$, and points $p_{ij} \in A^*_{ij}$, $j = 1, 2, 3$, such that $d(p_{ij}, b_{n_i}) > \varepsilon_1$, $j = 1, 2, 3$, then (M, d) is not wide tree-like.*

Proof. Suppose that (M, d) is wide tree-like. This supposition and the hypothesis assures us that there exists a wide realization $\{C_n\}$ for (M, d) and $e_1 > 0$ such that for each natural number n, there exists a branch link $b_n \in C_n$ with at least three distinct arms A_{nj}, $j = 1, 2, 3$, and points $p_{nj} \in A^*_{nj}$, $j = 1, 2, 3$, such that $d(p_{nj}, \bar{b}_n) > \varepsilon_1$, $j = 1, 2, 3$. The wide property of $\{C_n\}$ yields a natural number N_1 and $\delta > 0$ such that for each $n \geqslant N_1$ and branch link $b \in C_n$ if $d(x, \bar{b}) > \varepsilon_1$, then $d(x, A^*) \geqslant \delta$ where A is any arm of b such that $x \notin A^*$. Since the width of (M, d) is zero, then there exists a sequence of linear chains, $\{B_n\}$, $B_n \subset C_n$ with the property that $\{B_n\}$ converges to M. This allows a natural number N_2 such that for each $n \geqslant N_2$, $\mathfrak{W}(B_n) < \min\{\varepsilon_1, \delta\} = r$. Let $N = N_1 + N_2$. The definition of $\mathfrak{W}(B_N)$ implies that for each $x \in M$, $d(x, B^*_N) < r$. Since B_N is a linear chain and the branch link b_N has at least three distinct arms, then either $A_{N1} \cap B_N = \emptyset$ or $A_{N2} \cap B_N = \emptyset$ or $A_{N3} \cap B_N = \emptyset$).

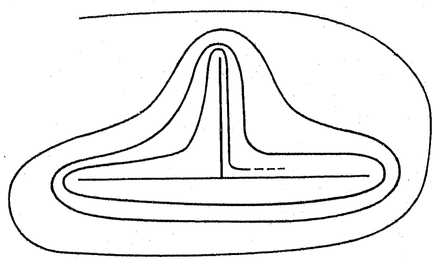
Without loss of generality suppose that $A_{N1} \cap B_N = \emptyset$. Because $d(p_{N1}, \bar{b}_N) > \varepsilon_1$, then $d(p_{N1}, B^*_N) \geqslant \delta \geqslant r$ which is a contradiction.

COROLLARY 1. *If (M, d) is tree-like with width zero and M contains a triod, then (M, d) is not wide tree-like.*

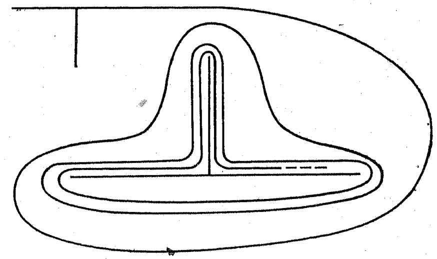
COROLLARY 2. *If (M, d) is tree-like and K is a subcontinuum of M with zero width satisfying the hypothesis of Theorem 6, then (M, d) is not wide tree-like.*

COROLLARY 3. *If (M, d) is tree-like and K is a subcontinuum of M with zero width and K contains a triod, then (M, d) is not wide tree-like.*

EXAMPLE 1. This picture indicates a tree-like subcontinuum of the plane with width zero. Corollary 1 implies that this subcontinuum is not wide tree-like.



EXAMPLE 2. This picture indicates a tree-like subcontinuum of the plane with positive width. Corollary 3 implies that this subcontinuum is not wide tree-like.



THEOREM 7. *If (M, d) is tree-like but not wide tree-like, then for each realization $\{C_n\}$ for (M, d) there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and $\varepsilon_1 > 0$ such that for each natural number i, there exists a branch link $b_{n_i} \in C_{n_i}$ with at least two distinct arms A_{ij}, $j = 1, 2$, and points $p_{ij} \in A^*_{ij}$, $j = 1, 2$, such that $d(p_{ij}, \bar{b}_{n_i}) > \varepsilon_1, j = 1, 2$.*

Proof. By supposing the contrary, there exists a realization $\{C_n\}$ for (M, d) with no subsequence satisfying all the properties mentioned in the conclusion of the statement of the theorem. For the natural number j there exists a natural number n_j such that for each branch link $b \in C_{n_j}$ and b has no more than one arm A with a point $p \in A^*$ such that $d(p, \bar{b}) > 1/i$.

Let $\varepsilon > 0$ and pick natural number $2N$ such that $1/N < \varepsilon$ and $\delta = 1/2N$. If $i \geqslant 2N$, b is a branch link of C_{n_i}, and $p \in M$ such that $d(p, \bar{b}) > \varepsilon$, then let A be any arm of b such that $p \notin A^*$. Suppose that there exists $x \in A^*$ such that $d(p, x) < 1/2N$. Since $d(p, \bar{b}) > \varepsilon > 1/N > 1/2N$, then the arm of b containing p is the only arm of b containing any point farther away from $\bar{b}$ than $1/2N$. Thus, $d(x, \bar{b}) \leqslant 1/2N$ and since $\bar{b}$ is compact pick $y \in \bar{b}$ such that $d(x, \bar{b}) = d(x, y) \leqslant 1/2N$. Therefore, $d(p, y) \leqslant d(p, x) + d(x, y) < 1/2N + 1/2N = 1/N$ which is a contradiction. This says that for each $i \geqslant 2N$ if b is a branch link of C_{n_i}, $p \in M$, $d(p, \bar{b}) > \varepsilon$, then for each arm A of b not containing p, $d(p, A^*) \geqslant 1/2N$. That is, (M, d) is wide tree-like and the hypothesis is contradicted.

4. The metric condition. If (M, d) is tree-like and $\{C_n\}$ is a realization for (M, d) such that for each n, C_n has at least one branch link, then a metric d^* can be defined for M.

Let $B = \{b\colon b$ is a branch link for C_n for some $n\}$. For each $b \in B$ define a metric d_b for M in the following way.

(i) If $x, y \in M$, $b \in C_n$, and x and y belong to distinct arms of b, then $d_b(x, y) = \inf\{d(x, p) + d(p, y)\colon p \in \bar{b}\}$.

(ii) Otherwise, if $x, y \in M$, then $d_b(x, y) = d(x, y)$.

The metric d^* is defined for M to be the sup. metric for the class of metrics, $\{d_b\colon b \in B\}$. That is, if $x, y \in M$, then

$$d^*(x, y) = \sup\{d_b(x, y)\colon b \in B\}.$$

The compactness of (M, d) makes it possible for d^* to be well defined and the above definitions are such that for each $x, y \in M$ and $b \in B$,

$$d^*(x, y) \geqslant d_b(x, y) \geqslant d(x, y).$$

If $\varepsilon > 0$, then a d-sphere about a point x with radius ε will be denoted $S_\varepsilon(x)$ and a d^*-sphere about a point x with radius ε will be denoted $S^*_\varepsilon(x)$. Assuming the metric condition, $(M, d) = (M, d^*)$, then a result is Theorem 8.

THEOREM 8. *Let (M, d) be tree-like, $\{C_n\}$ a realization for (M, d) with respect to d and d^* as defined above. If $(M, d) = (M, d^*)$, then $\{C_n\}$ is a wide realization for (M, d) with respect to d^*.*

Proof. Since $(M, d) = (M, d^*)$, then $\{C_n\}$ is a realization for (M, d) with respect to d^*. As above, $B = \{b\colon b$ is a branch link of C_n for some $n\}$. Let $L = \{x\colon$ for each open set U containing x there exists $b \in B$ such that $\bar{b} \subset U\}$ and $\varepsilon > 0$. Since $(M, d) = (M, d^*)$, then for each $p \in L$ there exists $\delta_p > 0$ such that

$$S_{\delta_p}(p) \subset S_{2\delta_p}(p) \subset S^*_{\varepsilon/4}(p) \subset S^*_\varepsilon(p).$$

Because L is closed and (M, d) is compact, then L is compact. Therefore, there exists a finite subset $\{p_1, \ldots, p_m\}$ of L such that

$$L \subset \bigcup_{i=1}^{m} S_{\delta_{p_i}}(p_i) \subset \bigcup_{i=1}^{m} S_{2\delta_{p_i}}(p_i) \subset \bigcup_{i=1}^{m} S^*_{\varepsilon/4}(p_i) \subset \bigcup_{i=1}^{m} S^*_\varepsilon(p_i).$$

The way that L is defined yields that there exists N such that for each $n \geqslant N$, if b is a branch link of C_n, then $\bar{b} \subset \bigcup_{i=1}^{m} S_{\delta_{p_i}}(p_i)$ and the d-diameter of $\bar{b}$ is less than $\frac{1}{10}\min\{\delta_{p_1}, \ldots, \delta_{p_m}\}$.

Consider a natural number $n \geqslant N$, $x \in M$, and $b \in B \cap C_n$ such that $d^*(x, \bar{b}) > \varepsilon$. The compactness of $\bar{b}$ allows a point $q \in \bar{b}$ such that $d^*(x, \bar{b}) = d^*(x, q)$. From the previous paragraph we know that there exists a natural number $1 \leqslant i \leqslant m$ such that $q \in S_{\delta_{p_i}}(p_i)$. The point x is not in $S_{2\delta_{p_i}}(p_i)$, for if so, then

$$d^*(x, q) \leqslant d^*(x, p_i) + d^*(p_i, q) < \varepsilon/4 + \varepsilon/4 < \varepsilon$$

which contradicts $d^*(x, q) > \varepsilon$. Now, consider any point y contained in an arm, A, of b in C_n that does not contain x. The definition of d^* implies

$$d^*(x, y) \geqslant d_b(x, y). \tag{1}$$

The definition of d_b implies

$$d_b(x, y) = d(x, p_{xy}) + d(p_{xy}, y) \tag{2}$$

where p_{xy} denotes a point of $\bar{b}$ such that

$$d_b(x, y) = \inf\{d(x, p) + d(p, y)\colon p \in \bar{b}\} = d(x, p_{xy}) + d(p_{xy}, y).$$

All of this is so since $\bar{b}$ is compact. Since the d-diameter of $\bar{b}$ is less than $\frac{1}{10}\delta_{p_i}$, $q \in S_{\delta_{p_i}}(p_i)$, and $x \notin S_{2\delta_{p_i}}(p_i)$, then $d(x, p_{xy}) \geqslant \frac{1}{2}\delta_{p_i}$. This fact coupled with (1) and (2) implies that $d^*(x, y) \geqslant \frac{1}{2}\delta_{p_i}$, which in turn implies, that $d^*(x, A^*) \geqslant \frac{1}{2}\delta_{p_i}$. Thus, the positive number $\delta = \frac{1}{2}\min\{\delta_{p_i}\colon i = 1, \ldots, m\}$ and the natural number N fulfills the definition for $\{C_n\}$ to be a wide realization for (M, d) with respect to d^*.

THEOREM 9. *Let (M, d) be tree-like, $\{C_n\}$ a realization for (M, d) with respect to d, and d^* as defined above. If $(M, d) = (M, d^*)$, then (M, d) has the fixed point property for continuous functions.*

Proof. Theorem 8 implies that (M, d^*) is wide tree-like. Theorem 5 says that (M, d^*) has the fixed point property for continuous functions. Since having the fixed point property is a topological property and $(M, d) = (M, d^*)$, then (M, d) has the fixed point property for continuous functions.

Theorem 10 is proven as an aid in determining a characterization of when a tree-like space is wide.

THEOREM 10. *If (M, d) is wide tree-like with wide realization $\{C_n\}$ with respect to d, then $(M, d) = (M, d^*)$ where d^* is as defined above.*

Proof. Let $x \in M$, $\varepsilon > 0$, and $S^*_\varepsilon(x)$ denote a d^*-sphere with radius ε about x. Since $\{C_n\}$ is a wide realization of (M, d) with respect to d, there exists N_1 and $0 < \delta_1 < \varepsilon/4$ with the property that for $n \geqslant N_1$, if $b \in B \cap C_n$ and $d(x, \bar{b}) > \varepsilon/4$, then $d(x, A^*) \geqslant \delta_1$ where B is as in the proof of Theorem 8 and A is an arm of b in C_n such that $x \notin A^*$. Pick N_2 such that for each $n \geqslant N_2$, $\|C_n\| < \delta_1$ and then let $N = N_1 + N_2$. Let $l \in C_N$ such that $x \in l$ and assert that $x \in l \subset S^*_\varepsilon(x)$. Let $y \in l$.

If $n \geqslant N$ and $b \in B \cup C_n$ such that $d(x, \bar{b}) > \varepsilon/4$, then x and y belong to the same arm of b in C_n since $\{x, y\} \subset l$ and $|l| < \delta_1$. Therefore, $d_b(x, y) = d(x, y) < \delta_1 < \varepsilon/4$. If b is such that $d(x, \bar{b}) \leqslant \varepsilon/4$, then pick $z \in \bar{b}$ such that $d(x, z) \leqslant \varepsilon/4$. The definition of d_b infers that

$$d_b(x, y) \leqslant d(x, z) + d(z, y) \leqslant d(x, z) + d(z, x) + d(x, y) < \varepsilon/4 + \varepsilon/4 + \delta_1 < 3\varepsilon/4 .$$

For all cases when $n \geqslant N$ and $b \in B \cap C_n$ we can now conclude that $d_b(x, y) < 3\varepsilon/4$.

If $n < N$ and $b \in B \cap C_n$, then the definition of the realization $\{C_n\}$ yields a link $l_n \in C_n$ such that $\{x, y\} \subset l \subset l_n$. By definition of d_b, then $d_b(x, y) = d(x, y)$. Since $y \in l$ and $|l| < \varepsilon/4$, then $d_b(x, y) = d(x, y) < \varepsilon/4$. Thus, this paragraph and the above paragraph convinces us that for each $b \in B$, $d_b(x, y) < 3\varepsilon/4$ which assures that $d^*(x, y) < \varepsilon$. The conclusion is, $x \in l \subset S^*_\varepsilon(x)$.

If $x \in M$ and $\varepsilon > 0$, then $x \in S^*_\varepsilon(x) \subset S_\varepsilon(x)$ since $d^*(x, y) \geqslant d(x, y)$. From this fact and the above arguments we can now conclude that $(M, d) = (M, d^*)$.

Theorem 8 and Theorem 10 imply a characterization for the wide tree-like spaces. This characterization is revealed in Theorem 11.

THEOREM 11. *If (M, d) is a tree-like space and d^* is as defined in this section, then $(M, d) = (M, d^*)$ if and only if (M, d) is wide.*

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On the fundamental dimension of approximatively 1-connected compacta

by

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Abstract. The aim of the present paper is to give a homological characterization of the fundamental dimension for approximatively 1-connected compacta and to give some applications of this characterization.

The main result is the theorem which states that for every approximatively 1-connected compactum $X \neq \emptyset$ with $\mathrm{Fd}(X) < \infty$ the fundamental dimension of X is equal to the smallest integer number $n \geqslant 0$ such that X is acyclic (in the sense of Čech cohomology) in all dimensions $\geqslant n$.

We prove also that for every movable approximatively 1-connected continuum X with infinite fundamental dimension and for every natural number n there exists a natural number $m \geqslant n$ such that m-dimensional Čech cohomology group of X with coefficients in the group of integer numbers is not trivial.

From these theorems we deduce in particular that for every $n \geqslant 3$ there exists a sequence $\{Q^n_p\}_{p=2}^\infty$ of polyhedra such that $\dim Q^n_p = \mathrm{Fd}(Q^n_p) = n$ and $\mathrm{Fd}(Q^m_q \times Q^n_p) = \max(m, n)$ for all relatively prime natural numbers p and q.

Introduction. By K we denote the Hilbert cube. The *fundamental dimension* of a compactum X (denoted by $\mathrm{Fd}(X)$) is the minimum of the dimensions of compacta Y with $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ (see [4] p. 31). We say that a pointed compactum $(X, x_0) \subset (K, x_0)$ is *approximatively n-connected* (see [3], p. 266) if for every neighborhood V of X there exists a neighborhood V_0 of X such that every map of the pointed n-sphere (S^n, a) into (V_0, x_0) is null homotopic in (V, x_0). It is known that the approximative n-connectivity of a pointed compactum $(X, x_0) \subset (K, x_0)$ depends only on the pointed shape of (X, x_0) (see [3], p. 267). Thus a pointed compactum (Y, y_0) (not necessarily lying in K) is said to be *approximatively n-connected* if there is a pointed compactum $(X, x_0) \subset (K, x_0)$ which is approximatively n-connected and homeomorphic to (Y, y_0). We say also that a compactum Y is *approximatively n-connected* if (Y, y_0) is approximatively n-connected for every $y_0 \in Y$ (see [3], p. 266).

Let $H_n(X, A; G)$ (or $H^n(X, A; G)$) denote for every pair (X, A) of compacta and every Abelian group G the n-dimensional Čech homology