On spaces which have the shape of C.W. complexes

by

Luc Demers (Ottawa)

Abstract. It is shown that a space which is a shape retract of a C.W. complex has the shape of a C.W. complex. We also give certain conditions ensuring that a finite dimensional metric space has the shape of a C.W. complex.

The aim of this note is to give certain conditions under which a space has the shape of a C.W. complex. After a preliminary section covering definitions and general results on the category of shapes, we will study the case of absolute neighborhood shape retracts, or in other words, spaces which are shape retracts of some C.W. complex. The last section will discuss the case of finite dimensional compact spaces with finitely generated Čech cohomology, including a sufficient condition for a space to have the shape of $|SX|$, the geometric realization of its singular complex.

§ 1. The category of shapes. The theory of shapes used here is essentially that given by Mardešić in [8], though our presentation is superficially different. Let $\mathcal{T}$ denote the category of topological spaces and homotopy classes of continuous maps, and let $\mathcal{P}$ be the full subcategory of $\mathcal{T}$ whose objects are spaces which have the homotopy type of a C.W. complex.

Given a space X in $\mathcal{T}$, let $\boldsymbol{X}$ denote the covariant functor from $\mathcal{P}$ to the category $\mathcal{E}$ of sets defined by $\boldsymbol{X}(P) = [X, P]$. (As usual, square brackets denote homotopy classes of continuous maps). Define a category $\mathcal{S}$ as follows: the objects of $\mathcal{S}$ are the same as those of $\mathcal{T}$; given two such objects X and Y, $\mathcal{S}(X, Y)$ is the set of natural transformations from $\boldsymbol{Y}$ to $\boldsymbol{X}$ (for the proof that this is a set, see the remark below), and composition of morphisms is given by the composition of natural transformations, i.e. if $\xi\colon X \to Y$ and $\gamma\colon Y \to Z$ are morphisms of $\mathcal{S}$, $\gamma \circ_s \xi$ is the natural transformation $\xi \circ \gamma$ from $\boldsymbol{Z}$ to $\boldsymbol{X}$.

There is a functor $S\colon \mathcal{T} \to \mathcal{S}$ defined as $S(X) = X$ for any space X and if $f\colon X \to Y$ is a homotopy class, $S(f) = f^{\#} = [f, ?]\colon \boldsymbol{Y} \to \boldsymbol{X}$.

Remark 1. The category of shapes as defined above is identical with the category of shapes in the sense of Mardešić. Indeed, they have the same objects, and a shaping ξ from X to Y is defined by Mardešić as follows: it is a function which assigns to every $f \in [Y, Q]$, Q an object of $\mathcal{P}$, a homotopy class $\xi(f)$ in $[X, Q]$, in such a way that for each Q' in $\mathcal{P}$, $f' \in [Y, Q']$ and $g \in [Q', Q]$, the equality $g \circ f' = f$ implies $g \circ \xi(f') = \xi(f)$. In other words, the following diagram is commutative:

$$\begin{tikzcd} {[X,Q]} \arrow[r, "\xi"] \arrow[d, "{[Y,g]}"'] & {[X,Q]} \arrow[d, "{[X,Q']}"] \\ {[Y,Q']} \arrow[r, "\xi"'] & {[X,Q']} \end{tikzcd}$$

It is thus seen that ξ is nothing but a natural transformation from $\boldsymbol{Y}$ to $\boldsymbol{X}$. From this identification will follow in particular that there is only a set of natural transformations from $\boldsymbol{Y}$ to $\boldsymbol{X}$ ([8], Remark 1).

We can also use directly the following result:

THEOREM 1.1 ([8], Theorem 1). *If P is an object of $\mathcal{P}$ and X is any space, S: $[X, P] \to \mathcal{S}(X, P)$ is a bijection.*

This theorem will allow us in particular to identify $\mathcal{P}$ with the full subcategory $S(\mathcal{P})$ of $\mathcal{S}$. Thus if f: $X \to P$ is a homotopy class of continuous maps, the same symbol f will be used to denote the shape morphism $S(f)$: $X \to P$.

Remark 2. Let $\mathcal{P}\mathcal{E}$ be the category of covariant functors from $\mathcal{P}$ to $\mathcal{E}$. Then by definition, $\mathcal{S}^{\mathrm{opp}}$ is isomorphic to the full subcategory of $\mathcal{P}\mathcal{E}$ whose objects are functors of the form $\boldsymbol{X}$ for X in $\mathcal{C}$.

This remark will help us to prove two theorems, the first of which is technical, but is important in the next section, and the second of which provides a nice characterization of the category $\mathcal{S}$.

THEOREM 1.2. *The functor S preserves coproducts, i.e. given a family $\{X_i: i \in I\}$ of spaces in $\mathcal{C}$, the disjoint union $Z = \bigcup \{X_i: i \in I\}$ together with the morphisms $S(j_i)$, where j_i: $X_i \to Z$ is the inclusion, is a coproduct of the family $\{X_i: i \in I\}$ in $\mathcal{S}$.*

Proof. If we identify $\mathcal{S}^{\mathrm{opp}}$ with a full-subcategory of $\mathcal{P}\mathcal{E}$, we just have to show that $\boldsymbol{Z}$, together with the natural transformations $j_i^{\#}$: $\boldsymbol{Z} \to \boldsymbol{X}_i$ is the product of the family $\{\boldsymbol{X}_i: i \in I\}$ in $\mathcal{P}\mathcal{E}$. But this is easy to check because for any Q in $\mathcal{P}$,

$$[Z, Q] = [\bigcup \{X_i: i \in I\}, Q] = \prod_i [X_i, Q]$$

and $j_i^{\#}$: $[Z, Q] \to [X_i, Q]$ is just the projection π_i: $\prod_i [X_i, Q] \to [X_i, Q]$. Q.E.D.

The next theorem characterizes $\mathcal{S}$ as the "smallest category in which each object of $\mathcal{C}$ is a limit of a diagram in $\mathcal{P}$". In order to do this, we must introduce some notation and terminology.

Given an object X of $\mathcal{C}$, we denote by $X/\mathcal{P}$ the category of objects of $\mathcal{P}$ under X, i.e. an object of $X/\mathcal{P}$ is a homotopy class f: $X \to P$ with range in $\mathcal{P}$, and a morphism from f: $X \to P$ to g: $X \to Q$ is a homotopy class h: $P \to Q$ such that $h \circ f = g$. Moreover, we denote by ϱ_X: $X/\mathcal{P} \to T$ the functor $\varrho_X(f\colon X \to P) = P$, $\varrho_X(h\colon f \to g) = h\colon P \to Q$.

THEOREM 1.3. *For each object X of $\mathcal{S}$, X, together with the family of morphisms $\{f\colon X \to P = S \circ \varrho_X(f)\colon f \in \mathrm{Ob}(X/\mathcal{P})\}$ is the limit of the functor $S \circ \varrho_x$: $X/\mathcal{P} \to \mathcal{S}$. Moreover, given a category $\mathcal{S}'$ and a functor S': $\mathcal{C} \to \mathcal{S}'$ such that for each object X of $\mathcal{C}$, $S'X$ together with the morphisms $\{S'f\colon X \to S' \circ \varrho_X(f) = P\colon f \in \mathrm{Obj}(X/\mathcal{P})\}$ is a limit of $S' \circ \varrho_X$, then there is a unique functor R: $\mathcal{S} \to \mathcal{S}'$ such that $R \circ S = S'$.*

Proof. Let $\underline{\mathcal{P}}$ be the full subcategory of $\mathcal{P}\mathcal{E}$ consisting of functors of the form $[P, ?]$ for P in $\mathcal{P}$. As we did above for $X/\mathcal{P}$, we can define the category $\boldsymbol{P}/\boldsymbol{X}$ of objects of $\underline{\mathcal{P}}$ over $\boldsymbol{X}$, where the objects are the natural transformations of the form

$$\xi\colon \boldsymbol{P} \to \boldsymbol{X}$$

and we define a functor δ_X: $\underline{\mathcal{P}}/\boldsymbol{X} \to \mathcal{P}\mathcal{E}$ as $\delta_X(\xi) = \boldsymbol{P}$.

Because of Theorem 1.1, the correspondence

$$f\colon X \to P \rightsquigarrow f^{\#}\colon \boldsymbol{P} \to \boldsymbol{X}$$

is a contravariant isomorphism between $X/\mathcal{P}$ and $\underline{\mathcal{P}}/\boldsymbol{X}$. Moreover, when we identify $\mathcal{S}^{\mathrm{opp}}$ with a full subcategory of $\mathcal{P}\mathcal{E}$ as done before, the functor $S \circ \varrho_X$: $X/\mathcal{P} \to \mathcal{S}$ is replaced by δ_X: $\underline{\mathcal{P}}/\boldsymbol{X} \to \mathcal{S}^{\mathrm{opp}} \subset \mathcal{P}\mathcal{E}$.

Thus the first part of the theorem reduces to showing that $\boldsymbol{X}$, together with the family of morphisms $\{f^{\#}\colon \boldsymbol{P} = \delta_X(f^{\#}) \to \boldsymbol{X}\}$ is a colimit of δ_X: $\mathcal{P}/\boldsymbol{X} \to \mathcal{P}\mathcal{E}$. But this is proved in [5], p. 21, 1.1.

For the second part of the theorem, we prove uniqueness first: Assume that R, R': $\mathcal{S} \to \mathcal{S}'$ are two functors such that $R \circ S = R' \circ S = S'$. Then for each object X of $\mathcal{S}$, $R(X) = R'(X) = S'X$. It remains to show that R and R' coincide on morphisms. First, let X be an object of $\mathcal{S}$, and P an object of $\mathcal{P}$. Then, since S: $[X, P] \to \mathcal{S}(X, P)$ is a bijection by Theorem 1.1, R and R' must coincide on $\mathcal{S}(X, P)$.

Now let X and Y be two arbitrary objects of $\mathcal{S}$, and consider the collection $\{f\colon Y \to P\colon f \in \mathrm{Ob}(Y/\mathcal{P})\}$. Let ξ: $X \to Y$ be a morphism in $\mathcal{S}$, and let ξ_f: $X \to P$ be $\xi_f = f \circ \xi$. Since Y is the limit of $S \circ \varrho_Y$, ξ is the unique morphism such that $f \circ \xi = \xi_f$ for all $f \in \mathrm{Obj}(X/\mathcal{P})$. Similarly, since $S'Y$ is the limit of $S' \circ \varrho_Y$, $R(\xi)$ is the unique morphism such that $R(f) \circ R(\xi) = R(\xi_f)$ for all $f \in \mathrm{Ob}(Y/\mathcal{P})$. Since R and R' coincide on morphisms of range in $\mathcal{P}$, it follows that $R'(\xi)$ also satisfies the equations $R(f) \circ R'(\xi) = R(\xi_f)$ for all f in $X/\mathcal{P}$. Thus $R = R'$.

As for existence, if f: $X \to P$ is a morphism with range in $\mathcal{P}$ define $R(f) = S'(f)$. If ξ: $X \to Y$ is a morphism between two arbitrary objects of $\mathcal{S}$, let $R(\xi)$ be the unique morphism $S'X \to S'Y$ such that $S'(f) \circ R(\xi) = S'(\xi_f)$ for all $f \in \mathrm{Ob}(Y/\mathcal{T})$.

It is easy to check that R is a functor and that $R \circ S = S'$. Q.E.D.

§ 2. Spaces which are shape retracts of C.W. complexes. In analogy with ANR's, a class of spaces called *absolute neighborhood shape retracts* (ANSR's) has been defined. They can be characterized as those spaces which are shape retracts of some ANR ([1], [7]). In this direction, we prove the following result:

THEOREM 2.1. *A connected topological space is a shape retract of some C.W. complex if and only if it has the shape of a C.W. complex.*

Proof. Let X be a topological space, Q a C.W. complex, and $f\colon X \to Q$, $g\colon Q \to X$ shape morphisms such that $g \circ f = \mathrm{id}(X)$. Define a functor $H\colon \mathcal{T}^{\mathrm{opp}} \to \mathcal{E}$ as $H(P) = \mathcal{S}(P, X)$ (Recall that $\mathcal{T}$ is considered as a full subcategory of $\mathcal{S}$). We will show below that H is a representable functor, i.e. there exists a C.W. complex $\bar{X}$ and a natural transformation $T\colon [?, \bar{X}] \to \mathcal{S}(?, X)$ which is an equivalence of functors on $\mathcal{T}$.

Granted this result, let us show that $\bar{X}$ has the shape of X. Let $h = T(X)\big(\mathrm{id}(\bar{X})\big)\colon \bar{X} \to X$. Then, if P is an object of $\mathcal{T}$ and $\varphi\colon P \to X$ is a morphism of $\mathcal{S}$, let $\bar{\varphi}\colon P \to \bar{X}$ be the unique homotopy class of maps such that $T(P)(\bar{\varphi}) = \varphi$. Then we have a commutative diagram

$$\begin{tikzcd}
{[\bar{X}, \bar{X}]} \arrow[r, "T(X)"] \arrow[d, "{[\bar{\varphi}, \bar{X}]}"'] & \mathcal{S}(\bar{X}, X) \arrow[d, "{\mathcal{S}(\bar{\varphi}, X)}"] \\
{[P, \bar{X}]} \arrow[r, "T(P)"'] & \mathcal{S}(P, X)
\end{tikzcd}$$

i.e. $T(P) \circ [\bar{\varphi}, \bar{X}]\big(\mathrm{id}(\bar{X})\big) = \mathcal{S}(\bar{\varphi}, X) \circ T(\bar{X})\big(\mathrm{id}(\bar{X})\big)$ which means that $\varphi = h \circ \bar{\varphi}$.

Let then $\bar{g}\colon Q \to \bar{X}$ be $T(Q)^{-1}(g)$, and let $j\colon X \to \bar{X}$ be $j = \bar{g} \circ f$. Then, as above, $h \circ \bar{g} = g$, so that $h \circ j = h \circ \bar{g} \circ f = \mathrm{id}(X)$. On the other hand,

$$h \circ j \circ h = h \circ \bar{g} \circ f \circ h = h = h \circ \mathrm{id}(\bar{X}) .$$

Thus $T(\bar{X})(j \circ h) = T(\bar{X})\big(\mathrm{id}(\bar{X})\big)$ which implies that $j \circ h = \mathrm{id}(\bar{X})$. Hence h is a shape isomorphism.

To show that H is representable, we use the criterion of E. H. Brown [3]. First H has to transform a coproduct into a product. This is true by Theorem 1.1.

Next, let $f_i\colon A \to Y_i$, $i = 1, 2$ be two continuous maps between C.W. complexes, let Z be the double mapping cylinder of (f_1, f_2), and let $g_i\colon Y_i \to Z$, $i = 1, 2$ be the two inclusions. We have to show that if $u_i \in H(Y_i)$, $i = 1, 2$, satisfy $H(f_1)(u_1) = H(f_2)(u_2)$, then there exists a $v \in H(Z)$ such that $H(g_i)(v) = u_i$, $i = 1, 2$.

Since $H(Y_i) = \mathcal{S}(Y_i, X)$, $H(f_1)(u_1) = H(f_2)(u_2)$ means $u_1 \circ [f_1] = u_2 \circ [f_2]$, where the square brackets denote the homotopy class of a map. Then

$$f \circ u_1 \circ [f_1] = f \circ u_2 \circ [f_2] \in \mathcal{S}(A, Q) = [A, Q] ,$$

and moreover, $f \circ u_i \in [Y_i, Q]$. By the properties of the double mapping cylinder, there is $w \in [Z, Q]$ such that

$$w \circ [g_1] = f \circ u_1 , \quad w \circ [g_2] = f \circ u_2 .$$

Let then $v = g \circ w$. We obtain

$$H(g_1)(v) = v \circ [g_1] = g \circ w \circ [g_1] = g \circ f \circ u_1 = u_1 ,$$
$$H(g_2)(v) = v \circ [g_2] = g \circ w \circ [g_2] = g \circ f \circ u_2 = u_2 .$$

Now to be able to apply Theorem 2.8 of [3], it remains to be proved that $H(\{0\})$ is a singleton. Let $\mathcal{T}_1$ be the full subcategory of $\mathcal{T}$ consisting of the connected spaces in $\mathcal{T}$, let $X/\mathcal{T}_1$ be the corresponding subcategory of $X/\mathcal{T}$, and let $\varrho_1\colon X/\mathcal{T}_1 \to \varrho | X/\mathcal{T}_1$.

LEMMA 2.2. *If X is a connected space, $X = \lim S \circ \varrho_1$.*

Proof. This follows easily from Theorem 1.3, using the fact that $X/\mathcal{T}_1$ is cofinal in $X/\mathcal{T}$. Indeed, given any object $f\colon X \to P$ of $X/\mathcal{T}_1$ let P_1 be the component of P containing $f(X)$, and let $i_1\colon P_1 \to P$ be the homotopy class of the inclusion. Then $f = i_1 \circ f_1$, where $f_1\colon H \to P_1$ is an object of X/P_1. Q.E.D.

With the help of this lemma, we complete the proof of 2.1:

$$H(\{0\}) = \mathcal{S}(\{0\}, \lim P_1) = \lim [\{0\}, P_1] = * \quad \text{Q.E.D.}$$

§ 3. Finite dimensional spaces with local properties. In this section, we will use the pointed shape category $\mathcal{S}^{\boldsymbol{\cdot}}$. It is obtained from the pointed homotopy category $\mathcal{C}^{\boldsymbol{\cdot}}$ in a similar way as $\mathcal{S}$ is obtained from $\mathcal{C}$.

THEOREM 3.1. *A compact connected pointed metric space X which is locally pathwise connected, simply connected, finite dimensional, with finitely generated Čech cohomology, has the pointed shape of a finite C.W. complex.*

Proof. Let $\mathcal{T}_1^{\boldsymbol{\cdot}}$ be the category of pointed pathwise connected spaces which have the homotopy type of C.W. complexes, and let $\mathcal{T}_0^{\boldsymbol{\cdot}}$ be the full subcategory of $\mathcal{T}_1^{\boldsymbol{\cdot}}$ whose objects have only a finite number of non trivial homotopy groups, and for which π_1 operates trivially on π_n, $n \geqslant 1$.

Let $\pi\colon \mathcal{T}_1^{\boldsymbol{\cdot}} \to E$ be the functor $\pi(P) = [X, P]$. Then π satisfies axioms h^*, e^* and c^* of [4], so that there exists an object $\bar{X}$ of $P_1^{\boldsymbol{\cdot}}$, unique up to homotopy type, and a natural transformation

$$T\colon [\bar{X}, ?] | \mathcal{T}_0^{\boldsymbol{\cdot}} \to [X, ?] | \mathcal{T}_0^{\boldsymbol{\cdot}}$$

which is an equivalence of functors. We will show that T extends to an equivalence of functors on $\mathcal{T}_1^{\cdot}$.

LEMMA 3.2. *$\bar{X}$ is simply connected.*

Proof. $\bar{X}$ is pathwise connected since $\bar{X}$ is an object of $\mathcal{T}_1^{\cdot}$. Moreover, it follows from its construction that $\pi_1(\bar{X})$ is abelian. Indeed, $\bar{X}$ is constructed by a sequence of approximations as follows: for each positive integer n, let $\{g_a^n\}$ be a set (finite) of generators of $[X, K(Z, n)]$, and let $X_1 = \prod K(Z, n)_a$.

Since $[X, K(Z, 1)] = 0$, because X is simply connected and locally pathwise connected, X_1 is simply connected.

Since $[X, X_1] = [X, \prod K(Z, n)_a] = \prod [X, K(Z, n)_a]$, let $u_1\colon X \to X_1$ be the homotopy class whose projections are the $g_{a'}^n$'s.

Let $T(u_1) = u_1^{\#}\colon [X_1, K(Z, 2)] \to [X, K(Z, 2)]$. Let $\{f_\beta\}$ be a set of generators of $\operatorname{Ker}(u_1)$, and let $Y_2 = \prod K(Z, 2)_\beta$. Let $f\colon X_1 \to Y_2$ be the homotopy class whose projections are the f_β, and let $p_2\colon X_2 \to X_1$ be the fibre of this map.

Then we have an exact sequence

$$\to \pi_2(X_1) \to \pi_2(Y_2) \to \pi_1(X_2) \to \pi_1(X_1) = 0 .$$

Hence $\pi_1(X_2)$ is a quotient of an abelian group. Moreover, when we repeat this procedure, to construct $X_3, X_4, \ldots$, we do not change the fundamental group, so that $\pi_1(\bar{X})$ remains an abelian group.

Hence $K(\pi_1(\bar{X}), 1)$ is an object in $\mathcal{T}_0^{\cdot}$, and thus in particular

$$[\bar{X}, K(\pi_1(\bar{X}), 1)] \simeq \operatorname{Hom}(\pi_1(\bar{X}), \pi_1(\bar{X}))$$

on the one hand, and on the other, $[\bar{X}, K(\pi_1(\bar{X}), 1)] \simeq [X, K(\pi_1(\bar{X}), 1)]$, and this last set has a unique element, since X is locally pathwise connected and simply connected. Hence $\pi_1(\bar{X}) = \{1\}$. Q.E.D.

LEMMA 3.3. *$\bar{X}$ has the homotopy type of a finite* C.W. *complex.*

Proof. By the preceding lemma, $\bar{X}$ is simply connected. Moreover, since $\check{H}^*(X) = H^*(\bar{X})$ is finitely generated by assumption, $H^i(\bar{X}) = 0$ for i large enough. Thus $\bar{X}$ admits a finite homology decomposition ([6], p. 53)

$$X_2 \subset X_3 \subset \ldots \subset X_n \simeq \bar{X} .$$

Here X_2 is a Moore space of type $K'(H_2(\bar{X}), 2)$, and X_{k+1} is the mapping cone of a map $K'(H_{k+1}(\bar{X}), k) \to X_k$. Since the groups $H_i(\bar{X})$ are all finitely generated, it suffices to show that if A is a finitely generated abelian group, $K'(A, k)$ has the homotopy type of a finite C.W. complex. Moreover, $K'(A \oplus B, k) \simeq K'(A, k) \vee K'(B, k)$, so that it suffices to consider the cases $A = Z$ or $A = Z/pZ$.

But $K'(Z, k) \simeq S^k$, and $K'(Z/pZ, k)$ is the mapping cone of a map $S^k \to S^k$ of degree p. Q.E.D.

LEMMA 3.4. *Let $Y \xrightarrow{q_{n+1}} Y_{n+1} \xrightarrow{p_n} Y_n \to \ldots \to Y_1$ be a Moore-Postnikov factorization of a* C.W. *complex Y. Then $[X, Y] \cong [\bar{X}, Y]$.*

Proof. We will deal with the case of $\bar{X}$ first.

It is well-known that the natural map

$$[\bar{X}, Y] \to \lim [\bar{X}, Y_n]$$

is a surjection. Moreover, we have fibrations

$$Y_{n+1} \to Y_n \to K(\pi_{n+1}(Y), n+2) .$$

Hence, when n is large enough,

$$[\bar{X}, Y_{n+1}] \cong [\bar{X}, Y_n] ,$$

since $\bar{X}$ has the homotopy type of a finite C.W. complex. Thus

$$\underleftarrow{\lim} [\bar{X}, Y_n] \cong [\bar{X}, Y_n] \quad \text{for large } n .$$

Now the fibration $q_n\colon Y \to Y_n$ has a fibre F_n which is $(n-1)$-connected, so that by standard obstruction arguments,

$$[\bar{X}, Y] \cong [\bar{X}, Y_n] \quad \text{for } n \text{ large enough} .$$

For the case of X, we can show, as above, that

$$\underleftarrow{\lim} [X, Y_n] \cong [X, Y_n] \quad \text{for large } n .$$

Take then n large, for instance $n \geqslant 2\dim(X)$, and consider the fibration $F_n \to Y \xrightarrow{q_n} Y_n$.

Let $\{\mathfrak{U}_i\}$, $i = 1, 2, \ldots$ be a sequence of coverings of X whose nerves N_i are finite complexes of $\dim \leqslant 2\dim(X)+1$.

Then by [10] ((2.1) and (2.2), p. 340), there is a map $\psi_i\colon N_i \to Y_n$ for some i, such that $\psi_i \circ \varphi_i \sim f$, where $\varphi_i\colon X \to N_i$ is the canonical map.

Since $\dim N_i < n-1$ for n large, and F_n is $(n-1)$-connected, ψ_i has a lift $\varrho_i\colon N_i \to Y$. Let $g = \varrho_i \circ \psi_i$. Then $q_n \circ g \sim f$, and hence $[X, Y] \to [X, Y_n]$ is onto.

Similarly, assume $f, g\colon X \to Y$ are maps such that q_n of $\sim q_n \circ g$. Again by [10] ((2.1) and (2.2)), there is an i and maps $\psi_i\colon N_i \to Y$, $\psi_i'\colon N_i \to Y$ such that $\psi_i' \circ \varphi_i \sim g$ and $\psi_i \circ \varphi_i \sim f$.

Then $q_n \circ \psi_i \circ \varphi_i \sim q_n$ of $\sim q_n \circ g \sim q_n \circ \psi_i' \circ \varphi_i$. Hence $q_n \circ \psi_i$ and $q_n \circ \psi_i'$ are two bridge mappings for the same map $q_n \circ f \sim q_n \circ g$. Then by [10] ((2.4), Bridge homotopy theorem), there is a $k > i$ and a bridge map $\varrho_k\colon N_k \to Y_n$ for $q_n \circ f$, such that

$$q_n \circ \psi_k \sim \varrho_k \sim q_n \circ \psi_k'$$

where $\psi_k = \psi_i \circ \nu_{ki}$, $\psi_k' = \psi_i' \circ \nu_{ki}$ and $\nu_{ki}\colon N_k \to N_i$ is the nerve projection.

Since $\dim N_k < n-1$, this implies that ψ_k and ψ'_k are homotopic, and hence f and g are homotopic. Thus $[X, Y] \cong [X, Y_n]$ for n large enough. Q.E.D.

End of the proof of 3.1. Let Y be an object of $\mathfrak{T}_1^{\cdot}$, and let p: $\tilde{Y} \to Y$ be a universal covering of Y. Since $\bar{X}$ and X are simply connected and locally pathwise connected, p_*: $[X, \tilde{Y}] \to [X, Y]$ and p_*: $[\bar{X}, \tilde{Y}] \to [\bar{X}, Y]$ are bijections.

We define $T(Y)$: $[\bar{X}, Y] \to [X, Y]$ as follows: let

$$\tilde{Y} \xrightarrow{\tilde{q}_{n+1}} Y_{n+1} \to \ldots \to \tilde{Y}_1$$

be a Moore-Postnikov Factorization of $\tilde{Y}$, and consider the following sequence of maps:

$$[\bar{X}, Y] \xleftarrow{p_*} [\bar{X}, Y] \xrightarrow{q_n^*} [\bar{X}, Y_n] \xrightarrow{T(Y_n)} [X, Y_n] \xleftarrow{q_n^*} [X, Y] \xrightarrow{p_*} [X, Y].$$

All arrows are bijections, if n is large so that we can define

$$T(Y) = p_* \circ (q_{n_*})^{-1} \circ T(Y_n) \circ q_{n^*} \circ (p_*)^{-1}.$$

It is easy to check that T becomes a natural equivalence of functors on $\mathfrak{T}_1^{\cdot}$. But since X and $\bar{X}$ are connected, T is also an equivalence of functors on $\mathfrak{T}^{\cdot}$, so that X and $\bar{X}$ have the same pointed shape. Q.E.D.

THEOREM 3.4. *If in addition to the hypotheses of* 3.1, X *is also homologically simply connected, it has the shape of* $|SX|$, *the geometric realization of its singular complex.*

Proof. Let ξ: $X \to \bar{X}$ be the map $T(\bar{X})(\mathrm{id}(\bar{X}))$. Then for any f: $\bar{X} \to Y$, $T(Y)(f) = f \circ \xi$. Let ω: $|SX| \to X$ be the canonical map. Then ω induces isomorphisms $H^i(X) \to H^i(|SX|) = H^i(X)$ for all i, by [9], p. 340.

But ξ also induces isomorphisms $H^i(\bar{X}) \to H^i(X)$ for all i. Hence $\xi \circ \omega$: $|SX| \to \bar{X}$ induces isomorphisms $H^i(\bar{X}) \to H^i(|SX|)$ for all i. Since $|\bar{X}|$ and $|SX|$ are simply connected, this implies that $\xi \circ \omega$ is a homotopy equivalence. Q.E.D.

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